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ALMOST WARFIELD GROUPS

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ABSTRACT. In this paper, we introduce the class of almost Warfield groups, a generalization of the previously studied class of almost totally projective p-groups. It is shown that almost Warfield groups satisfy mixed versions of several properties enjoyed by almost totally projective p-groups. For example, an almost Warfield group is an almost strongly separable subgroup of any group in which it appears as an isotype subgroup. Also, as our main result, we demonstrate that any isotype Knice subgroup of a global Warfield group is almost Warfield.

1. Introduction. Throughout, all groups considered are additively written abelian groups, and G will always denote such a group. Of central importance in this paper is the notion of a Knice subgroup in the global (mixed) context as defined by Hill and Megibben [**2**]. The definition of a Knice subgroup depends upon the auxiliary concepts of *primitive element* and $*$-*valuated coproduct*. The details can be found in the paper cited above and will not be reviewed here.

Definition 1.1. A subgroup N of G is a *Knice* subgroup if the following two conditions are satisfied.

(a) N is a *nice* subgroup of G; that is, for all primes p and ordinals α, the cokernel of the inclusion map

$$(p^\alpha G + N)/N \rightarrowtail p^\alpha(G/N)$$

contains no element of order p.

(b) To each finite subset S of G there corresponds a finite (possibly empty) set of primitive elements $\{x_1, x_2, \dots, x_n\}$ such that

$$N \oplus \langle x_1\rangle \oplus \langle x_2\rangle \oplus \cdots \oplus \langle x_n\rangle$$

is a $*$-valuated coproduct that contains some positive multiple of $\langle S\rangle$.

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Nice subgroups were introduced by Hill [**1**] in connection with Axiom 3. Recall that a collection $\mathcal{C}$ of subgroups of G is an *Axiom* 3 *system* for G if

(0) $\mathcal{C}$ contains the trivial subgroup 0,

(1) $\{N_\alpha\}_{\alpha\in A} \subseteq \mathcal{C}$ implies that $\sum_{\alpha\in A} N_\alpha \in \mathcal{C}$, and

(2) each countable subgroup of G is contained in a countable member of $\mathcal{C}$.

In [**1**] it was shown that a p-primary group is totally projective if and only if it has an Axiom 3 system consisting entirely of nice subgroups. Later, Hill and Megibben [**3**] proved the corresponding result for mixed groups: a group G is a global Warfield group (that is, a summand of a simply presented mixed group) if and only if G has an Axiom 3 system of Knice subgroups.

Motivated by a problem concerning units of commutative modular group algebras, in [**5, 6**] Hill and the author considered weak Axiom 3 systems and the class of almost totally projective p-groups. By a *weak Axiom* 3 *system*, we mean a collection $\mathcal{C}$ of subgroups of G that satisfies conditions (0) and (2) above, with condition (1) replaced by the weaker condition

(1$'$) $\mathcal{C}$ is closed under unions of ascending chains.

A p-primary abelian group is then called *almost totally projective* if it has a weak Axiom 3 system of nice subgroups. The purpose of this note is to introduce the class of almost Warfield groups.

Definition 1.2. An abelian group G is an *almost Warfield group* if it has a weak Axiom 3 system consisting entirely of Knice subgroups.

Observe that a p-primary abelian group is almost totally projective if and only if it is almost Warfield. However, in general, an almost Warfield group can be a nonsplit mixed group with nontrivial p-torsion for infinitely many primes p. Nevertheless, we show that almost Warfield groups satisfy mixed versions of several properties enjoyed by almost totally projective p-groups. For example, in the next section we show that an almost Warfield group is an almost strongly separable subgroup of any group in which it appears as an isotype subgroup. In the third and final section, we obtain our main result: any isotype

Knice subgroup of a global Warfield group is almost Warfield. This generalizes a result in [**5**] which states that any balanced subgroup of a totally projective p-group is almost totally projective.

In the sequel, there will be several occasions to deal with the class of *k-groups*, those groups in which the trivial subgroup 0 is a Knice subgroup. The class of k-groups is quite extensive. For example, the class of k-groups contains (but is not restricted to) the classes of all torsion groups, torsion free separable groups (in the sense of Baer), and almost Warfield groups. In connection with k-groups, the notion of a k-subgroup will prove to be useful.

Definition 1.3. A subgroup N of a group G is called a *k-subgroup* if to each finite subset S of N there corresponds a finite (possibly empty) subset $\{x_1, x_2, \ldots, x_n\} \subseteq N$ such that each x_i is a primitive element of G, and $\langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \cdots \oplus \langle x_n \rangle$ is a $*$-valuated coproduct in G that contains some positive multiple of $\langle S \rangle$.

With this terminology, Theorem 2.1 of [**7**] shows that every Knice subgroup of a k-group is a k-subgroup. For later use, we make the following observation.

Proposition 1.4. *A group G is a k-group if and only if the set of all k-subgroups of G is a weak Axiom 3 system for G.*

Proof. Clearly 0 is a k-subgroup of any group G. Also, it is not difficult to see that the collection of all k-subgroups of any group G is closed under unions of ascending chains. Moreover, in a k-group, every countable subgroup is contained in a countable k-subgroup by [**8**, Proposition 2.2 (4)].

Conversely, if G has a weak Axiom 3 system of k-subgroups, then any finite subset of G is contained in a k-subgroup. Thus, if S is a finite subset of G, there exists a $*$-valuated coproduct $M = \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \cdots \oplus \langle x_n \rangle$ in G such that each x_i is primitive in G and M contains some positive multiple of $\langle S \rangle$. Since 0 is a nice subgroup of any group, it follows that 0 is a Knice subgroup of G. □

2. Separability of almost Warfield groups. In this section, we apply Theorem 21.2.3 of [**10**] to prove that an almost Warfield group satisfies a mixed version of a separability property enjoyed by any almost totally projective p-group. For the convenience of the reader, we include the relevant definitions below. Here, and in the sequel, we write $|x|_p^G$ for the p-height of x in the containing group G, and we use the notation $\|x\|^G$ for the height matrix of x in G.

One property of an almost totally projective p-group is that it is a strongly separable subgroup of any group in which it appears as an isotype subgroup (for example, see [**10**, Theorem 21.1.3]). Recall that a subgroup H of G is *isotype* if $H \cap p^\alpha G = p^\alpha H$ for all primes p and ordinals α, and is a *strongly separable subgroup* if to each $g \in G$ there is a corresponding countable subset $A \subseteq H$ such that, for each $h \in H$, there is an $a \in A$ with $\|g+h\|^G \leq \|g+a\|^G$. However, in [**4**] it was shown that a global Warfield group need not be strongly separable in a group in which it appears as an isotype subgroup. Thus, to extend the separability property of almost totally projective p-groups to almost Warfield groups, we need the following weaker version of separability.

Definition 2.1. Call a subgroup H of G *almost strongly separable* if to each $g \in G$ there is a corresponding countable subset $A \subseteq H$ with the following property: for each $h \in H$ and prime p, there are $a, b \in A$ and a positive integer m such that $|g+h|_p^G \leq |g+a|_p^G$ and $\|m(g+h)\|^G \leq \|m(g+b)\|^G$.

Certainly every strongly separable subgroup is almost strongly separable. Thus, an almost totally projective p-group is almost strongly separable in any group in which it appears as an isotype subgroup. For an indication of the importance of separability and its various generalizations, we direct the reader to the introductory section of [**10**]. We require two more definitions and a proposition.

Definition 2.2. A subgroup N of G is called *almost balanced* if it satisfies the following two conditions.

(i) N is a nice subgroup of G.

(ii) To each $g \in G$ there corresponds a positive integer m such that coset $mg + N$ contains an element x with $\|x\|^G = \|mg+N\|^{G/N}$.

It follows from [**3**, Proposition 1.7] that every Knice subgroup is almost balanced, while an almost balanced subgroup N of G is Knice if and only if G/N is a k-group.

Definition 2.3. A collection $\mathcal{C}$ of countable subgroups of G is called an $\aleph_0$-*cover* for G if $\mathcal{C}$ satisfies conditions (0) and (2) of Section 1 and

(1$''$) $\mathcal{C}$ is closed under unions of ascending chains of countable length.

We can now state a special case of Theorem 21.2.3 of [**10**].

Proposition 2.4 [**10**, Theorem 21.2.3]. *If G has an $\aleph_0$-cover of almost balanced pure subgroups, then G is almost strongly separable in any group in which it appears as an isotype subgroup.*

As an application of Proposition 2.4, we obtain a mixed version of the separability property of almost totally projective p-groups.

Theorem 2.5. *An almost Warfield group G is almost strongly separable in any group in which it appears as an isotype subgroup.*

Proof. Since the collection of all pure subgroups of G is a weak Axiom 3 system, and because the intersection of two weak Axiom 3 systems is again one, G has a weak Axiom 3 system $\mathcal{C}$ consisting of pure Knice subgroups. Due to the fact that every Knice subgroup of G is almost balanced, it follows easily that

$$\{N \in \mathcal{C} : |N| \leq \aleph_0\}$$

is an $\aleph_0$-cover of almost balanced pure subgroups for G. An application of Proposition 2.4 completes the proof. □

It is easy to modify the above to obtain a stronger result. First observe that if $\mathcal{C}$ is a weak Axiom 3 system of subgroups for some group G, then for any infinite cardinal κ and any subgroup $H \subseteq G$ of cardinality not exceeding κ, there is an $N \in \mathcal{C}$ with $H \subseteq N$ and $|N| \leq \kappa$. Thus, using the terminology of [**10**], if $\mathcal{C}$ is a weak Axiom 3 system of pure Knice subgroups of G, then $\{N \in \mathcal{C} : |N| \leq \kappa\}$ is a κ-cover for G consisting of almost balanced pure subgroups. Therefore, by applying the full force of Theorem 21.2.3 in [**10**], we obtain

Corollary 2.6. *For every infinite cardinal κ, an almost Warfield group is an almost strongly κ-separable subgroup of any group in which it appears as an isotype subgroup.*

3. Isotype Knice subgroups of Warfield groups. In this final section, we prove our main result; that is, an isotype Knice subgroup of a global Warfield group is an almost Warfield group. We conclude with some remarks regarding cardinality and dimension of almost Warfield groups. One last definition is required.

Definition 3.1. Two subgroups H and N of G are called *almost strongly compatible* if for each pair $(h, x) \in H \times N$ and prime p there are corresponding $y, z \in H \cap N$ and a positive integer m such that $|h + x|_p^G \leq |h + y|_p^G$ and $\|m(h + x)\|^G \leq \|mh + z\|^G$.

To indicate that the subgroups H and N are almost strongly compatible, we write $H \parallel N$. Observe that almost strong compatibility is a symmetric relation, and is inductive in the sense that if $\{N_\alpha\}_{\alpha<\lambda}$ is an ascending chain of subgroups of G with $H \parallel N_\alpha$ for all α, then $H \parallel (\bigcup_{\alpha<\lambda} N_\alpha)$. We can now state

Proposition 3.2 [**9**, Theorems 2.1 and 2.4]. *Let H and N be Knice subgroups of G with H isotype in G, N pure in G, and $H \parallel N$. Then, $H \cap N$ is a Knice subgroup of G if and only if $(H + N)/H$ is a k-subgroup of G/H. Furthermore, if $H \cap N$ is pure in H, then $H \cap N$ is Knice in H if and only if it is Knice in G.*

Our application of Proposition 3.2 requires several lemmas.

Lemma 3.3. *If H is a subgroup of G, and if $\mathcal{C}$ is the set of all subgroups N of G such that $H \cap N$ is pure in H, then $\mathcal{C}$ is a weak Axiom 3 system for G.*

Proof. Clearly $0 \in \mathcal{C}$, and $\mathcal{C}$ is closed under unions of ascending chains because purity is an inductive property. Finally, if K is a countable subgroup of G, select a countable pure subgroup P of H that contains $H \cap K$. Then certainly $N = K + P$ is a countable subgroup of G that contains K. Moreover, $N \in \mathcal{C}$ because $H \cap N = H \cap (K + P) = (H \cap K) + P = P$ is pure in H. □

Lemma 3.4. *If H is an almost strongly separable subgroup of G, and if $\mathcal{C}$ is the set of all subgroups N of G such that $H \parallel N$, then $\mathcal{C}$ is a weak Axiom 3 system for G.*

Proof. Certainly $0 \in \mathcal{C}$ and, as remarked above, $\mathcal{C}$ is closed under unions of ascending chains. That each countable subgroup of G is contained in a countable member of $\mathcal{C}$ follows from [**7**, Proposition 4.6]. □

Lemma 3.5. *Suppose that H is a subgroup of G and G/H is a k-group. If $\mathcal{C}$ is the set of all subgroups N of G such that $(H + N)/H$ is a k-subgroup of G/H, then $\mathcal{C}$ is a weak Axiom 3 system for G.*

Proof. By Proposition 1.4, the set $\mathcal{C}_{G/H}$ of all k-subgroups of G/H is a weak Axiom 3 system for G/H. Thus, $\mathcal{C}$ is the collection of all subgroups N of G satisfying the condition that $(H + N)/H \in \mathcal{C}_{G/H}$. We intend to show that $\mathcal{C}$ is a weak Axiom 3 system for G.

Certainly the trivial subgroup 0 of G is contained in $\mathcal{C}$. To see that $\mathcal{C}$ is closed under unions of ascending chains, suppose that

$$N_0 \subseteq N_1 \subseteq \cdots \subseteq N_\alpha \subseteq \cdots \quad (\alpha < \lambda)$$

is an ascending chain indexed by some ordinal λ with $N_\alpha \in \mathcal{C}$ for all $\alpha < \lambda$. Then,

$$(H + N_0)/H \subseteq (H + N_1)/H \subseteq \cdots \subseteq (H + N_\alpha)/H \subseteq \cdots \quad (\alpha < \lambda)$$

is an ascending chain in $\mathcal{C}_{G/H}$ with union $(H + \bigcup_{\alpha<\lambda} N_\alpha)/H \in \mathcal{C}_{G/H}$. Therefore, $\bigcup_{\alpha<\lambda} N_\alpha \in \mathcal{C}$.

Finally, suppose that K is a countable subgroup of G. Then, $(H + K)/H$ is a countable subgroup of G/H, and so there exists a countable subgroup $L/H \in \mathcal{C}_{G/H}$ that contains $(H+K)/H$. Write L as $L = H + N$ where N is a countable subgroup of G that contains K. Since $(H+N)/H = L/H \in \mathcal{C}_{G/H}$, $N \in \mathcal{C}$ and the proof is complete. □

We now have the necessary ingredients to prove our main result.

Theorem 3.6. *An isotype Knice subgroup of a global Warfield group is an almost Warfield group.*

Proof. Let G be a global Warfield group, and suppose that H is an isotype Knice subgroup of G. By [**10**, Theorem 21.3.4], G has an Axiom 3 system $\mathcal{C}_1$ of pure Knice subgroups that is closed under arbitrary intersections. Next apply Lemma 3.3 to obtain a weak Axiom 3 system $\mathcal{C}_2$ for G such that $N \in \mathcal{C}_2$ if and only if $H \cap N$ is pure in H. Because every pure Knice subgroup of G is almost strongly separable in G by [**9**, Remark 3.2], H is almost strongly separable in G. Hence, Lemma 3.4 shows that G has a weak Axiom 3 system $\mathcal{C}_3$ such that $N \in \mathcal{C}_3$ if and only if $H \parallel N$. Since H is Knice in G, G/H is a k-group by [**3**, Proposition 1.7]. Therefore, Lemma 3.5 applies to show that G has a weak Axiom 3 system $\mathcal{C}_4$ such that $N \in \mathcal{C}_4$ if and only if $(H+N)/H$ is a k-subgroup of G/H.

Set $\mathcal{C} = \bigcap_{i=1}^{4} \mathcal{C}_i$. Then each $N \in \mathcal{C}$ has all the subgroup properties mentioned in the previous paragraph, and $\mathcal{C}$ is a weak Axiom 3 system for G. In particular, for each $N \in \mathcal{C}$, N is a pure Knice subgroup of G, $H \cap N$ is pure in H, $H \parallel N$, and $(H+N)/H$ is a k-subgroup of G/H. Since H is an isotype Knice subgroup of G, Proposition 3.2 implies that the set $\mathcal{C}_H = \{H \cap N : N \in \mathcal{C}\}$ consists of pure Knice subgroups of H. So, the proof will be complete once we have shown that $\mathcal{C}_H$ is a weak Axiom 3 system for H. It is clear that $\mathcal{C}_H$ satisfies conditions (0) and (2) of Section 1. Indeed, $0 \in \mathcal{C}_H$ since $0 \in \mathcal{C}$. Moreover, if K is a countable subgroup of H, select a countable subgroup $N \in \mathcal{C}$ that contains K. Then, $H \cap N$ is a countable subgroup in $\mathcal{C}_H$ that contains K.

Thus, it remains to show that $\mathcal{C}_H$ is closed under unions of ascending chains. To this end, suppose that

$$H \cap N_0 \subseteq H \cap N_1 \subseteq \cdots \subseteq H \cap N_\alpha \subseteq \cdots \quad (\alpha < \lambda)$$

is an ascending chain in $\mathcal{C}_H$ indexed by some ordinal λ with all $N_\alpha \in \mathcal{C}$. For each $\alpha < \lambda$, set $M_\alpha = \bigcap_{\alpha \le \beta < \lambda} N_\beta$. We claim that $M_\alpha \in \mathcal{C}$ for all α. With α now temporarily fixed, note that $\{N_\beta : \alpha \le \beta < \lambda\} \subseteq \mathcal{C}_1$. Since $\mathcal{C}_1$ is closed under intersections, we have that $M_\alpha \in \mathcal{C}_1$ (and M_α is a pure Knice subgroup of G). Now observe that $H \cap M_\alpha = H \cap N_\alpha$. Then, $H \cap M_\alpha$ is a pure Knice subgroup of H. In particular, $M_\alpha \in \mathcal{C}_2$. Next, $M_\alpha \subseteq N_\alpha$, $H \cap M_\alpha = H \cap N_\alpha$, and $H \parallel N_\alpha$ imply that $H \parallel M_\alpha$. Hence, $M_\alpha \in \mathcal{C}_3$. To see that $M_\alpha \in \mathcal{C}_4$, recall that H and M_α are Knice subgroups of G with H isotype in G, M_α pure in G, and $H \parallel M_\alpha$. Then, the fact that $H \cap M_\alpha$ is a pure Knice subgroup of H and Proposition 3.2 imply that $(H + M_\alpha)/H$ is a k-subgroup of G/H. So, $M_\alpha \in \mathcal{C}_4$. We conclude that $M_\alpha \in \mathcal{C}$ for all α, as claimed.

Finally, observe that

$$M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\alpha \subseteq \cdots \quad (\alpha < \lambda)$$

is an ascending chain in G with each $M_\alpha \in \mathcal{C}$. Since $\mathcal{C}$ is a weak Axiom 3 system for G, $M = \bigcup_{\alpha < \lambda} M_\alpha \in \mathcal{C}$. Therefore, the relation

$$\bigcup_{\alpha<\lambda} (H \cap N_\alpha) = \bigcup_{\alpha<\lambda} (H \cap M_\alpha) = H \cap M \in \mathcal{C}_H$$

completes the proof. □

It is known that, even in the p-primary case, an almost totally projective p-group need not appear as an isotype subgroup of a totally projective group (see [**5**, Section 3]). Thus, an almost Warfield group is not necessarily an isotype Knice subgroup of a global Warfield group. As a result, the following proposition can be viewed as a slight generalization of the remark in [**8**] which states that if G is an isotype Knice subgroup of a global Warfield group with $|G| \le \aleph_1$, then G is itself a global Warfield group.

Proposition 3.7. *If G is an almost Warfield group of cardinality not exceeding $\aleph_1$, then G is a global Warfield group.*

Proof. If $\mathcal{C}$ is a weak Axiom 3 system of Knice subgroups for G, then $|G| \le \aleph_1$ implies that we can extract from $\mathcal{C}$ an ascending chain of countable pure Knice subgroups with union G. Therefore, G is a global Warfield group by [**7**, Theorem 2.5]. □

By [**5**, Proposition 12], for every nonnegative integer n, there exists an almost totally projective p-group of balanced projective dimension n. Since the notions of balanced projective dimension and sequentially pure projective dimension coincide for primary groups, it follows that there are almost Warfield groups of arbitrarily large (but finite) sequentially pure projective dimension. Thus, because a group is a global Warfield group if and only if it has sequentially pure projective dimension 0, there is an abundance of almost Warfield groups that are not global Warfield groups.

Our final result complements Proposition 3.7. Here we write $\dim(G)$ for the sequentially pure projective dimension of G.

Proposition 3.8. *If G is an almost Warfield group of cardinality $\aleph_n$ for some positive integer $n \geq 2$, then* $\dim(G) \leq n$. *Moreover, if G also appears as an isotype Knice subgroup of a global Warfield group, then* $\dim(G) \leq n-1$.

Proof. Since an almost Warfield group is a k-group, Theorem 3.7 of [**8**] implies that $\dim(G) \leq n$. The inequality $\dim(G) \leq n-1$ in the case when G is an isotype Knice subgroup of a global Warfield group follows from the last paragraph of the proof of Theorem 3.7 of [**8**]. □

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