ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 41, Number 6, 2011

BOUNDS USING THE INDEX OF NAGAMI

MARÍA MUÑOZ

ABSTRACT. We consider the cardinal invariant *index of Nagami*, $Nag(X)$, which measures how the topological space X is determined by its compact subsets via upper semi-continuous compact valued maps defined on arbitrary topological spaces, to establish relationships between cardinal functions of X and $C_p(X)$, where $C_p(X)$ is the space of continuous functions with the pointwise convergence topology. Applications to Lindelöf Σ-spaces are given.

1. Introduction. Let X be a topological space and $C_p(X)$ the space of continuous functions with the pointwise convergence topology; a natural question that arises is how the topological properties of both spaces are related. A way to "measure" topological properties is using cardinal inequalities in which cardinal functions are involved. A large list of well-known results about these questions can be found, see for example [**1, 9**].

The aim of this paper is to give new topological cardinal inequalities for a topological space. For that we use the index of Nagami, $Nag(X)$ which measures how the topological space X is determined by its compact subsets via upper semi-continuous compact valued maps defined on arbitrary topological spaces.

In Section 3 we use the index of Nagami to establish a bound of the network weight of a topological space using continuous bijections. These results will be used in Section 4 to establish inequalities between the cardinal functions of a topological space X and the space of all real valued continuous functions on X in the topology of pointwise convergence, $C_p(X)$. Among other results, we prove that for a subspace $H \subset C_p(X)$, $nw(H) \leq \max\{Nag(X), d(H)\}$. Monolithic properties are also studied. Finally, the last section is dedicated to the hereditarily Lindelöf number and the spread. Applications to the class of Lindelöf

2010 AMS *Mathematics subject classification.* Primary 54A25, Secondary 54C60.
Keywords and phrases. Usco, topology of pointwise convergence, network weight, density, spread, Lindelöf number, index of Nagami.
Received by the editors on July 26, 2007, and in revised form on April 15, 2009.
DOI:10.1216/RMJ-2011-41-6-1977

Σ-spaces are given. The class of Lindelöf Σ-spaces was introduced by Nagami [**15**], see also [**13**]. A topological space X is said to be a Lindelöf Σ-space if it is the image of some subset of $\mathbf{N}^{\mathbf{N}}$ under an upper semicontinuous compact-valued map. The class of Lindelöf Σ-spaces is the minimal class which contains all second countable spaces and all compact spaces and is closed with respect to finite products, closed subspaces and continuous images.

2. Terminology and definitions. We denote by X, Y, Z topological spaces. All our topological spaces are assumed to be Tychonoff. Our basic references are [**1, 7, 9,11**]. A *cardinal number* $\mathfrak{m}$ is the set of all ordinals which precede it. In particular, $\mathfrak{m}$ is a set of cardinality $\mathfrak{m}$. Thus, $\alpha < \mathfrak{m}$ and $\alpha \in \mathfrak{m}$ are the same. The following set-theoretic notation is adopted: $\mathfrak{m}$, $\mathfrak{n}$ and $\mathfrak{t}$ are infinite cardinal numbers. ω is the smallest infinite cardinality. The cardinal number assigned to the set of all real numbers is denoted by $\mathfrak{c}$. A *cardinal function* ϕ is a function from the class of all topological spaces (or some precisely defined subclass) into the class of all infinite cardinalities such that $\phi(X) = \phi(Y)$ whenever X and Y are homeomorphic. The requirement that cardinal functions take on only infinite cardinal numbers as values simplifies statements of theorems and places the emphasis on infinite cardinal arithmetic. We use well-known cardinal functions. The *weight* of X, $w(X)$, is the minimal cardinality of a base for the topology of X. A *network* in a space X is a family $\mathcal{F}$ of subsets of X such that for any point $x \in X$ and any open neighborhood U of x there is $F \in \mathcal{F}$ such that $x \in F \subset U$. The *network weight* of X, $nw(X)$, is the minimal cardinality of a network in X. The *Lindelöf degree of* X, denoted by $\ell(X)$, is defined as the smallest infinite cardinality $\mathfrak{m}$ such that every open cover of X has a subcollection of cardinality $\leq \mathfrak{m}$ which covers X. The *hereditarily Lindelöf degree of* X, $h\ell(X)$ is defined by $\sup\{\ell(Y) : Y \subseteq X\}$. It is obvious that $\ell(X) \leq h\ell(X)$. The *density* of X, $d(X)$, is the minimal cardinality of an everywhere dense set in X. The *hereditarily density of* X, $hd(X)$, is $\sup\{d(Y) : Y \subseteq X\}$.

The notion of the usco map has been studied in [**4, 6**] among others. The definition is the following.

Definition 2.1. Let X and Y be topological spaces. A multivalued map $\phi : X \to 2^Y$ is said to be:

(i) *upper semicontinuous at* $x_0 \in X$ if $\phi(x_0)$ is not empty and for each open set V in Y with $\phi(x_0) \subset V$ there exists an open neighborhood U of x_0 in X such that $\phi(U) \subset V$.

(ii) *upper semicontinuous* if it is upper semicontinuous at each point $x \in X$.

(iii) *usco* if ϕ is upper semicontinuous and the set $\phi(x)$ is compact for each $x \in X$.

A usco map can be characterized in terms of nets, see [**12, 14**]. The following implication will be used throughout this paper.

Proposition 2.2. *Let X and Y be topological spaces and $\phi : X \to 2^Y$ a usco map. If $(x_j)_{j\in J} \subset X$ is a net which converges to $x \in X$ and $(y_j)_{j\in J} \subset Y$ is a net such that $y_j \in \phi(x_j)$ for every $j \in J$, then $(y_j)_{j\in J}$ has a cluster point y which belongs to $\phi(x)$.*

The notion of the *Nagami index of* a topological space can be found in [**3**].

Definition 2.3. Let X be a topological space. The *index of Nagami* of X, $Nag(X)$ is the smallest infinite cardinality $\mathfrak{m}$ such that there exist a topological space Y, with $w(Y) \le \mathfrak{m}$ and a usco map $\phi : Y \to 2^X$ such that $X = \bigcup\{\phi(y) : y \in Y\}$.

When X is a Lindelöf Σ-space then $Nag(X)$ is countable.

3. Continuous bijections. The following proposition will allow us to prove the main theorem of this section.

Proposition 3.1. *Let X, Y and Z be topological spaces, and let $\phi : X \to 2^Y$ be a usco map such that $Y = \bigcup_{x\in X} \phi(x)$ and $f : Y \to Z$ a continuous bijection. Then Y is the image under a continuous map of a closed subspace of the space $X \times Z$. In particular,*

$$nw(Y) \le \max\{w(X), w(Z)\}.$$

Proof. We define $T := \{(x,y) : y \in \phi(x)\} \subset X \times Y$. Then T is closed in $X \times Y$ in the product topology, see [**10**, Proposition 2.22]. Now, we consider the subspace T of $X \times Y$ with the inherited topology and the map $\Phi : T \to X \times Z$ defined by $\Phi(x,y) := (x, f(y))$ for every $(x,y) \in T$. We will prove that $\Phi(T)$ is closed in $X \times Z$ and that $\Phi : T \to \Phi(T)$ is a continuous closed bijection. To prove that $\Phi(T)$ is closed in $X \times Z$ we consider a net $(x_j, z_j)_{j \in J} \subset \Phi(T)$ convergent to (x,z) in $X \times Z$. Since f is a one-to-one map, there exist unique y, y_j such that $f(y) = z$ and $f(y_j) = z_j$ for every $j \in J$. Now $(x_j, y_j)_{j \in J} \subset T$; thus, $y_j \in \phi(x_j)$ for every $j \in J$. Since $(x_j)_{j \in J}$ is a convergent net to x in X we have, again for ϕ being a usco, that $(y_j)_{j \in J}$ has a cluster point $y^* \in \phi(x)$. We also have that f is continuous; hence, $f(y^*)$ is a cluster point of $(f(y_j))_{j \in J}$. But $(z_j)_{j \in J} = (f(y_j))_{j \in J}$ is convergent to z; hence, $z = f(y^*)$. But $y^* = y$ because f is one-to-one, and $(x,z) = \Phi(x,y)$ where $(x,y) \in T$; and we have proved that $\Phi(T)$ is closed in $X \times Z$. The same proof allows us to affirm that Φ is a closed map. To see that Φ is continuous we consider a convergent net $(x_j, y_j)_{j \in J}$ to (x,y) in T and we will prove that $(\Phi(x_j, y_j))_{j \in J}$ converges. The net $(\Phi(x_j, y_j))_{j \in J}$ is equal to $(x_j, f(y_j))_{j \in J}$ and now it is obvious that it converges to $(x, f(y))$. Finally, the proof that Φ is one-to-one is trivial. Thus, $\Phi(T)$ is homeomorphic to T, and since $\Phi(T)$ is a subspace of $X \times Z$, then $w(\Phi(T)) \leq \max\{w(X), w(Z)\}$, [**7**, Proposition 2.3.13, page 81]; and hence $w(T) \leq \max\{w(X), w(Z)\}$. On the other hand, Y is the continuous image of T by the projection $\pi : T \to Y$, where $\pi(x,y) = y$ for every $(x,y) \in T$. Now, we have that $nw(Y) \leq w(T)$ because if $\mathcal{O} = \{O_i : i \in I\}$ is a base for the topology of T, then the family $\mathcal{N} = \{f(O_i) : \subset \in I\}$ is a network for Y. Finally, we obtain $nw(Y) \leq \max\{w(X), w(Z)\}$ and the proof is over. □

Theorem 3.2. *Let X and Y be topological spaces and $f : X \to Y$ a continuous bijection. Then*

$$nw(X) \leq \max\{Nag(X), w(Y)\}. \tag{3.1}$$

Proof. Given X, there exist a topological space Z with $w(Z) = Nag(X)$ and a usco map $\phi : Z \to 2^X$ such that $X = \bigcup_{z \in Z} \phi(z)$. Now we use Proposition 3.1 to conclude that $nw(X) \leq \max\{Nag(X), w(Y)\}$. □

We cannot remove $Nag(X)$ from inequality (3.1). The space $(\widetilde{\mathbf{R}}, \tau_d)$, of all real numbers with the discrete topology gives us an example. The identity map from this space onto $\mathbf{R}$ with the usual topology is a continuous bijection, $w(\mathbf{R}) = \omega$ and $nw(\widetilde{\mathbf{R}}) = \mathfrak{c}$; thus, $nw(\widetilde{\mathbf{R}}) > w(\mathbf{R})$.

Equation (3.1) holds for every topological space Y which is an image of X under a continuous bijection. Now, it makes sense to have the following definition, see [**1**, page 5].

Definition 3.3. Let X be a topological space; we call $iw(X)$ the smallest cardinal $\mathfrak{m}$ for which there exist a topological space Y with $w(Y) \leq \mathfrak{m}$ and a continuous bijection $f : X \to Y$.

Theorem 3.2 can be reformulated in these terms as:

Theorem 3.4. *Let X be a topological space. Then*

$$nw(X) \leq \max\{Nag(X), iw(X)\}.$$

4. Network weight and density. In this section we establish bounds of the network weight of a subspace.

Theorem 4.1. *Let X be a topological space, $G \subset X$ a subspace of X and $H \subset C(X)$ a subspace of $C_p(X)$. Then:*

(i) $nw(\overline{H}^{\tau_p}) \leq \max\{Nag(X), |H|\}$.

(ii) $nw(H) \leq \max\{Nag(X), d(H)\}$.

(iii) $nw(G) \leq \max\{Nag(C_p(X)), d(G)\}$.

Proof. (i) We consider a subspace $H \subset C_p(X)$. Let $f : X \to \mathbf{R}^H$ be the map defined by $f(x) = (h(x))_{h \in H}$ for every $x \in X$. We define $T := \{(h(x))_{h \in H} \in \mathbf{R}^H : x \in X\}$. Let τ be the product topology on $\mathbf{R}^H$. Now $w(\mathbf{R}^H, \tau) \leq |H|$ and hence $w(T, \tau) \leq |H|$. Let $\widetilde{\tau}$ be the quotient topology in T defined as the largest topology such that $f : X \to T$ is continuous. Also, this topology $\widetilde{\tau}$ is the topology on T such that for all functions $g \in \mathbf{R}^T$, $g \circ f \in C(X)$ [**11**, Theorem 9, page

95]. Now we have

$$X \xrightarrow{\widetilde{f}} (T, \widetilde{\tau}),$$

where $\widetilde{f}$ is a continuous map onto, being $\widetilde{f}(x) := f(x)$ for every $x \in X$. By Theorem 3.2 we have that $nw(T, \widetilde{\tau}) \leq \max\{Nag(T, \widetilde{\tau}), w(T, \tau)\}$. The index of Nagami does not increase by continuous image, because the composition of a usco and a continuous map onto is again usco, see [**3, 6**]; thus, $Nag(T, \widetilde{\tau}) \leq Nag(X)$, and

$$nw(T, \widetilde{\tau}) \leq \max\{Nag(X), |H|\}.$$

In fact, as $nw(C_p(T, \widetilde{\tau})) = nw(T, \widetilde{\tau})$, see [**1**, Theorem I.1.3, page 26], we have the inequality $nw(C_p(T, \widetilde{\tau})) \leq \max\{Nag(X), |H|\}$. The space $C_p(T, \widetilde{\tau})$ is homeomorphic to the subspace $F = \{g \circ \widetilde{f} : g \in C_p(T, \widetilde{\tau})\}$ of $C_p(X)$. We show that F is closed in $C_p(X)$. To this end, let $s \in \overline{F}^{\tau_p}$ be a map in $C_p(X)$. Then there is a net $(s_i)_{i \in I}$ in F such that $(s_i) \xrightarrow{\tau_p} s$. For each $i \in I$, choose $g_i \in C(T, \widetilde{\tau})$ such that $s_i = g_i \circ \widetilde{f}$. Note that, for each $t \in T$, all s_i's are constant on $\widetilde{f}^{-1}(t)$. Since $s_i \xrightarrow{\tau_p} s$, we conclude that s is constant on $\widetilde{f}^{-1}(t)$ as well. Thus, we can define a map $g : (T, \widetilde{\tau}) \to \mathbf{R}$ such that for each $t \in T$, $g(t) = s(x)$ for any $x \in \widetilde{f}^{-1}(t)$. This implies that $s = g \circ \widetilde{f}$, and the continuity of g follows from the fact of $s \in C(X)$ and the choice of $\widetilde{\tau}$. Therefore, $s \in F$, which verifies that F is closed in $C_p(X)$. Fix $h \in H$. Then $h = \pi_h \circ i \circ \widetilde{f}$, where $\pi_h : \mathbf{R}^H \to \mathbf{R}$ is the projection in the h-coordinate and $i : (T, \widetilde{\tau}) \to (T, \tau)$ is the identity map. The map $\pi_h \circ i : (T, \widetilde{\tau}) \to \mathbf{R}$, again because of $h \in C(X)$ and the choice of $\widetilde{\tau}$ verifies that $\pi_h \circ i \in C_p(T, \widetilde{\tau})$. Thus, $h \in F$. It follows that $\overline{H}^{\tau_p} \subset \overline{F}^{\tau_p} = F$ and $nw(\overline{H}^{\tau_p}) \leq nw(F) = nw(T, \widetilde{\tau}) \leq \max\{Nag(X), |H|\}$, and the proof is finished.

(ii) follows immediately from (i). Let $H' \subset H$ be a set with the minimal cardinality such that $H \subseteq \overline{H'}^{\tau_p}$; then using (i) we have

$$nw(H) = nw(\overline{H'}^{\tau_p}) \leq \max\{Nag(X), |H'|\} \leq \max\{Nag(X), d(H)\}.$$

(iii) Using (ii) we have that for every $H \subset C_p(C_p(X))$ we obtain the following inequality

$$nw(H) \leq \max\{Nag(C_p(X)), d(H)\}.$$

To finish the proof, it is enough to observe that the network weight is a hereditarily invariant cardinal function and $X \subset C_p(C_p(X))$ is a subspace. The inclusion is given by $i : X \to C_p(C_p(X))$, where $i(x) : C_p(X) \to \mathbf{R}$ is defined as $i(x)(h) = h(x)$ for every $h \in C_p(X)$. □

The following definition can be found in [**1**, page 83].

Definition 4.2. A topological space X is called $\mathfrak{m}$-*monolithic* if $nw(\overline{A}) \leq \mathfrak{m}$ for every $A \subset X$ such that $|A| \leq \mathfrak{m}$. A space X is called *strongly* $\mathfrak{m}$-*monolithic* if for every $Y \subset X$ with $|Y| \leq \mathfrak{m}$, the weight of the space $\overline{Y}$ does not exceed $\mathfrak{m}$.

We obtain the following result, which was proved in [**3**] using properties of the index of Nagami.

Corollary 4.3. *Let X be a topological space and $H \subset C(X)$ τ_p-compact. Then H is strongly $Nag(X)$-monolithic.*

Proof. We consider $H' \subset H$ with $|H'| \leq Nag(X)$. Then, after Theorem 4.1 (i) and since, for compact space, the network weight and the weight coincides [**7**, Theorem 3.1.19, page 127], $w(\overline{H'}) = nw(\overline{H'}) \leq \max\{Nag(X), |H'|\} = Nag(X)$. □

Corollary 4.4. *Let X be a topological space and $H \subset X$ compact. Then H is strongly $Nag(C_p(X))$-monolithic.*

Proof. We consider $H' \subset H$ with $|H'| \leq Nag(C_p(X))$. Then, by Theorem 4.1 (iii),

$$w(\overline{H'}) = nw(\overline{H'}) \leq \max\{Nag(C_p(X), |H'|\} = Nag(C_p(X)). \qquad \square$$

The following corollaries can be found in [**1**, Theorem II.6.9].

Corollary 4.5. *Let X be a Lindelöf Σ-space and $H \subset C_p(X)$. Then*

$$nw(H) = d(H).$$

In particular, if H is τ_p-compact subspace then H is metrizable, see [**5**].

Proof. By Theorem 4.1 (ii) we have that $nw(H) \leq d(H)$. Another inequality is always true, see [**8**, page 14]. For H a compact space $w(H) = nw(H)$; hence, $w(H) = nw(H) = d(H) = \omega$ and the proof is over. □

Corollary 4.6. *Let X be a topological space such that $C_p(X)$ is a Lindelöf Σ-space. Then for every subspace H of X we have that $nw(H) = d(H)$.*

5. Hereditarily Lindelöf number and spread. The *spread* of X, $s(X)$, is the smallest infinite cardinal $\mathfrak{m}$ such that the cardinality of every discrete subspace of X does not exceed $\mathfrak{m}$.

The following result is due to Šapirovskiĭ [**16**].

Proposition 5.1. *Let $\mathcal{U}$ be an open cover of a topological space X, and let $s(X) \leq \mathfrak{m}$. Then there exist a subset A of X with $|A| \leq \mathfrak{m}$ and a subfamily $\mathcal{V}$ of $\mathcal{U}$ with $|\mathcal{V}| \leq \mathfrak{m}$ such that $X = \overline{A} \cup (\cup \mathcal{V})$.*

Theorem 5.2. *Let X be a topological space. Then*:

(i) $h\ell(X) \leq \max\{Nag(C_p(X)), s(X)\}$.

(ii) $h\ell(C_p(X)) \leq \max\{Nag(X), s(C_p(X))\}$.

Proof. (i) Let $\mathcal{U}$ be any family of open subsets of X. It is enough to show that there is a subfamily $\mathcal{W}^*$ with $|\mathcal{W}^*| \leq \max\{Nag(X), s(X)\}$, such that $\cup \mathcal{U} = \cup \mathcal{W}^*$. By Proposition 5.1, we can find a subset A of X with $|A| \leq s(X)$ and a subfamily $\mathcal{V}$ of $\mathcal{U}$ with $|\mathcal{V}| \leq s(X)$ such that $\cup \mathcal{U} = \overline{A} \cup (\cup \mathcal{V})$. Theorem 4.1 (iii) implies that $nw(\overline{A}) \leq \max\{Nag(C_p(X)), |A|\}$. Hence, since $\ell(\overline{A}) \leq nw(\overline{A})$, [**7**, Theorem 3.8.12, page 193], we have that there exists a subfamily $\mathcal{W}$ such that

$$|\mathcal{W}| \leq \max\{Nag(C_p(X)), s(X)\}$$

and $\overline{A} \subseteq \cup \mathcal{W}$. Finally $\cup \mathcal{U} = (\cup \mathcal{W}) \cup (\cup \mathcal{V})$. The family $\mathcal{W}^* := \mathcal{V} \cup \mathcal{W}$ has cardinality $|\mathcal{W}^*| \leq \max\{Nag(X), s(X)\}$ and the proof is over.

(ii) The proof mimics the steps used in (i), now using Theorem 4.1 (iii). □

The following theorem follows using that $hd(X^2) \leq h\ell(C_p(X))$, see [**1**, Corollary II.5.27, page 73] and Theorem 5.2 (ii).

Theorem 5.3. *Let X be a topological space. Then*

$$hd(X^2) \leq \max\{Nag(X), s(C_p(X))\}.$$

The following corollary can be found in [**2**].

Corollary 5.4. *If X is a Lindelöf Σ-space and the spread of $C_p(X)$ is countable, then $C_p(X)$ is hereditarily Lindelöf, and $X \times X$ is hereditarily separable.*

Proof. After Theorem 5.2 (ii) we have that $h\ell(C_p(X)) \leq s(C_p(X))$, and by Theorem 5.3, a Lindelöf Σ-space X verifies that $hd(X^2) \leq s(C_p(X))$. $\square$

REFERENCES

1. A.V. Arkhangel'skiĭ, *Topological function spaces*, Math. Appl. **78**, Kluwer Academic Publishers Group, Dordrecht, 1989, translated from the Russian by R.A.M. Hoksbergen.

2. ———, *On Lindelöf property and spread in C_p-theory*, Topol. Appl. **74** (1996), 83–90.

3. B. Cascales, M. Muñoz and J. Orihuela, *Index of K-determination of topological spaces*, to appear.

4. B. Cascales and L. Oncina, *Compactoid filters and USCO maps*, J. Math. Anal. Appl. **282** (2003), 826–845.

5. B. Cascales and J. Orihuela, *On pointwise and weak compactness in spaces of continuous functions*, Bull. Soc. Math. Belg. **40** (1998), 331–352.

6. L. Drewnowski and I. Labuda, *On minimal upper semicontinuous compact-valued maps*, Rocky Mountain J. Math. **20** (1990), 737–752.

7. R. Engelking, *General topology*, PWN-Polish Scientific Publishers, Warsaw, 1977, translated from the Polish by the author, Monograf. Matemat. **60**.

8. R. Hödel, *On a theorem of Arkhangel'skiĭ concerning Lindelöf P-spaces*, Canadian J. Math. **27** (1975), 459–468.

9. ———, *Cardinal functions* I, in *Handbook of set-theoretic topology*, Elsevier Science Publishers, Amsterdam, 1984.

10. S. Hu and N.S. Papageorgiou, *Handbook of multivalued analysis*, Volume 1, Kluwer Academic Publishers, Dordrecht, 1997.

11. J.L. Kelley, *General topology*, Grad. Texts Math. **27**, Springer-Verlag, New York, 1975, reprint of the 1955 edition (Van Nostrand, Toronto, Ont.).

12. K. Kuratowski, *Topology*, vol. I, PWN, Polish Scientific Publishers, Warszawa, 1966.

13. E. Michael, *On Nagami's* Σ*-spaces and some related matters*, Proc. Washington State University Conference on General Topology, 1970, 13–19.

14. M. Muñoz, *Indice de* K*-determinación y* σ*-fragmentabilidad de aplicaciones*, Master's thesis, Universidad de Murcia, 2003.

15. K. Nagami, Σ*-spaces*, Fund. Math. **65** (1965), 162–192.

16. B. Šapirovskiĭ, *On discrete subspaces of topological spaces*; *Weight, tightness and Suslin number*, Soviet. Math. Dokl. **13** (1972), 215–219.

17. M. Talagrand, *A new countably determined Banach space*, Israel J. Math. **47** (1984), 75–80.

Departamento de Matemática Aplicada y Estadística, Universidad Politécnica de Cartagena, (Murcia), Spain
Email address: maria.mg@upct.es