ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 41, Number 6, 2011

ANALYSIS OF ONE PILE MISÉRE NIM FOR TWO ALLIANCES

ANNELA R. KELLY

The traditional game of Nim is an impartial game for two players that plays a central role in combinatorial game theory. The analysis of the two-player game relies on binary representations of numbers, see [**1, 2**]. However, the game for three or more players has not been studied extensively. In this paper, we will consider a one-pile Nim misére version for more than two players. Typically, the strategy for the game of one-pile Nim with three or more players cannot be completely determined without considering alliances. The paper [**3**] establishes the basic terminology and certain fundamental results for the game with three or more players. The current paper expands it to a general result.

Definition 1. Consider the game for three or more players that follows the rules of misére Nim for each player. Suppose that the players form two alliances and that each player is in exactly one alliance. Also assume that each player will support his alliance's interests as long as it benefits his own interests. This game will be called *Survivor Nim*.

We call the game Survivor Nim as it models aspects of the popular television show. The end of a game can vary slightly from the standard two-player misére Nim. In Survivor Nim, if there remain fewer counters than an alliance's combined minimum move, the alliance must take these and lose. In the following we will assume (unless stated otherwise) that the larger alliance will start the game.

Definition 2. Suppose we have a game with k players. The set of turns, where we start with player one, and all k players have their turn, is called a *cycle*.

Hence each game can have several cycles and it typically ends with an incomplete cycle.

Received by the editors on September 29, 2008, and in revised form on March 30, 2009.

DOI:10.1216/RMJ-2011-41-6-1895

In this paper, we consider the game for $k = 2n + 1$ players. Suppose each player is allowed to remove $1, \ldots, m$ ($m > 1$, $m \in \mathbf{N}$) counters on their turn. We assume that a set of $n + 1$ players (not necessarily consecutive) form one alliance and the remaining n players form another alliance.

Proposition 3. *If there are exactly two consecutive players in one alliance and no consecutive players in the other alliance, then there are two games with j and $j+1$ counters* (*for some $j \in \mathbf{N}$*), *that the alliance with consecutive players loses.*

Proof. We can look at this game as a two-player game, where we combine the possible consecutive moves for players in the alliance because each player supports his alliance. Hence, this game is reducible to a two player game, where each player removes some counters at his turn. Suppose the two consecutive players in one alliance are p and $p + 1$. Up to player p's turn, the game follows standard two-player game strategy (as in [**3**, Proposition 2]). Hence, there exists a game where at player p's turn there is one counter left and player p will lose. Let j denote the number of counters at the start of this game. Furthermore, if the game starts with $j + 1$ counters, then at player p's turn there are two counters left, and player p or player $p + 1$ will lose.

In general, a comparable version of the proposition holds for most possible partitions of the players into alliances. However in Proposition 3, we assumed that there are no consecutive players in the other alliance to avoid certain extreme situations.

The following useful definition is motivated by Proposition 3.

Definition 4. A collection of consecutive losses for an alliance is called a *round.*

Proposition 5. *Suppose any $n+1$ players form one alliance* (*alliance* 1) *and the remaining n players form another alliance* (*alliance* 2). *If both alliances play wisely, then alliance* 1 *can take $m-1$ more counters than alliance* 2 *in the first cycle* (*unless the game ends during the first cycle*).

Proof. The game is essentially a two-player game, where we combine all the possible moves for players in a given alliance. Hence in this game, alliance 1 can take from $n+1$ up to $mn+m$ counters in one cycle and alliance 2 can take from n up to mn counters in one cycle. Since both alliances play wisely, then one of them will take away the largest possible number of counters, the other one the smallest possible number of counters [**3**, Proof of Theorem 8]. Thus,

$$\begin{aligned}(1)\quad \max(\text{Alliance 1}) + \min(\text{Alliance 2}) &= mn+n+m \\ > \min(\text{Alliance 1}) + \max(\text{Alliance 2}) &= mn+n+1\end{aligned}$$

in the first cycle. □

Therefore, alliance 1's options in using up to $m-1$ additional moves allow them to force alliance 2 to lose more quickly in the following cycles.

Remark 6. If we extend the discussion to the case $m = 1$, then inequality (1) is in fact an equation, i.e.,

$$\begin{aligned}\max(\text{Alliance 1}) &+ \min(\text{Alliance 2}) \\ &= \min(\text{Alliance 1}) + \max(\text{Alliance 2}) \\ &= 2n+1.\end{aligned}$$

In some situations, alliance 1 could have the first and the last round in each cycle. If this is the case, then the first alliance's last turn in a cycle merges into his first turn in the next cycle. From now on, we assume without loss of generality that alliance 2 has the last turn in each cycle. Before proving the technical results, the next example illustrates the terminology and the notation required in the propositions that follow. As in Theorem 8 in [**3**], for an odd second index j, let $N_{i\,j}$ denote the largest value in round $(j+1)/2$ of losses for alliance 1 in a cycle i. For an even second index j, let $N_{i\,j}$ denote the largest value in round $j/2$ of losses for alliance 2 in a cycle i.

Example 7. Consider the game with 7 players. Suppose each player can take 1 or 2 counters; alliance 1 is formed by players 1, 2, 5, 6 and alliance 2 is formed by players 3, 4, 7.

For clarity, consider this game to be a two-player game, where we combine all the possible moves for players in each alliance. Since alliance 1 is formed by two groups of two consecutive players, alliance 1 can take 2, 3 or 4 counters at his every turn. However, as alliance 2 is formed by a group of two consecutive players and one more player, alliance 2 will alternate in their turns taking 2, 3 or 4 counters first and then 1 or 2 counters. In this particular partition, alliance 1 loses all the games with 2 counters or less. Denote the largest value of alliance 1 losses in the first round by $N_{11} = 2$ (the first index 1 of N denotes the first cycle, the second index 1 of N denotes that we are considering the first round of losses in the cycle). If the game has more than 2 counters but no more than 4+2=6 counters to start with, alliance 1 can force alliance 2 to lose by leaving at most 2 counters after his move. Alliance 1 will win the games from 3 through 6 counters, i.e., $N_{12} = 4 + 2 = 6$. In the following work, it helps to notice that the indices N_{ij} can be written as sums of the minimum and maximum number of counters the alliances can take in each cycle. Next, alliance 1 will lose the games with $N_{12} + 1 = 7$ up to 2+4+2=8 counters. Thus, $N_{13} = 8$. Alliance 2 loses the games from $N_{13}+1 = 9$ up to $N_{14} = 4+2+4+1 = 11$ counters. Following through with wise play, alliance 2 can force alliance 1 to lose the game in the second cycle with

$$N_{21} = 2 + 4 + 2 + 2 + 2 = 12 \text{ counters}.$$

Alliance 2 loses the games with $N_{21} + 1 = 13$ up to $N_{22} = 4 + 2 + 4 + 1 + 4 + 2 = 17$ counters; alliance 1 loses the game with

$$N_{22} + 1 = N_{23} = 2 + 4 + 2 + 2 + 2 + 4 + 2 = 18 \text{ counters}.$$

Next, alliance 2 loses the games with $N_{23} + 1 = 19$ up to $N_{24} = 4 + 2 + 4 + 1 + 4 + 2 + 4 + 1 = 22$ counters. Notice that the lengths of the rounds in the second cycle for alliance 1 have reduced from 2 to 1. A similar pattern applies to the third cycle and one can easily verify that

$$N_{31} = 2 + 4 + 2 + 2 + 2 + 4 + 2 + 2 + 2 = 22 = N_{24};$$

and

$$N_{32} = 4 + 2 + 4 + 1 + 4 + 2 + 4 + 1 + 4 + 2 = 28;$$
$$N_{33} = 2 + 4 + 2 + 2 + 2 + 4 + 2 + 2 + 2 + 4 + 2 = 28 = N_{32},$$

i.e., these rounds coincide. Hence, alliance 2 loses all the games with more than $N_{23} + 1 = 19$ counters.

The following result, regarding the first cycle, will hold assuming the groups of moves satisfy certain technical restrictions. The general case is proved in Proposition 10.

Proposition 8. *Suppose alliance* 1 *can take*

$$\begin{aligned} &l_{11}, \dots ., l_{1n_1} \text{ counters on his first turn;} \\ &l_{21}, \dots , l_{2n_2} \text{ counters on his second turn;} \\ &\quad\vdots \\ &l_{j1}, \dots , l_{jn_j} \text{ counters on his } j\text{-th turn} \end{aligned}$$

in one cycle. Also assume alliance 2 *can take*

$$\begin{aligned} &k_{11}, \dots , k_{1m_1} \text{ counters on his first turn;} \\ &k_{21}, \dots , k_{2m_2} \text{ counters on his second turn;} \\ &\quad\vdots \\ &k_{j1}, \dots , k_{jm_j} \text{ counters on his } j\text{-th turn} \end{aligned}$$

in one cycle. If

$$\begin{aligned} (2) \quad l_{11} < l_{1n_1} + k_{11} < l_{11} + k_{1m_1} + l_{21} < \cdots \\ < l_{1n_1} + k_{11} + \cdots + l_{jn_1} + k_{j1}, \end{aligned}$$

then the maximum number of counters in each round for alliances in the first cycle can be calculated using the following formulas:

$$\begin{aligned} N_{11} &= l_{11} \\ N_{12} &= l_{1n_1} + k_{11} \\ N_{13} &= l_{11} + k_{1m_1} + l_{21} \\ N_{14} &= l_{1n_1} + k_{11} + l_{2n_1} + k_{21} \\ &\vdots \\ N_{1(2j-1)} &= l_{11} + k_{1m_1} + l_{21} + \cdots + k_{j-1,m_{j-1}} + l_{j1} \\ N_{1(2j)} &= l_{1n_1} + k_{11} + l_{2n_2} + \cdots + l_{jn_1} + k_{j1}, \end{aligned} \tag{3}$$

Proof. As before, consider this game to be a two-player game, where we combine all the possible moves for players in a given alliance. Alliance 1 can take from l_{11} up to l_{1n_1} counters to start the game. Hence, if the game starts with no more than l_{11} counters, alliance 1 will lose, i.e., $N_{11} = l_{11}$. If the game has more than l_{11} counters to start with (but no more than $l_{1n_1} + k_{11}$ counters), alliance 1 can force alliance 2 to lose by leaving less than k_{11} counters after his move. Alliance 1 can do this for the games with up to $l_{1n_1} + k_{11}$ counters; hence, $N_{12} = l_{1n_1} + k_{11}$ (and we have $N_{12} > N_{11}$ from (2)). Consider the game with $N_{12} + 1$ counters. Since we assumed that (2) holds, then $l_{1n_1} + k_{11} < l_{11} + k_{1m_1} + l_{21}$, and no matter how many counters alliance 1 takes, alliance 2 can leave fewer than l_{21} counters and force alliance 1 to lose. Alliance 2 can do this for games up to $l_{11}+k_{1m_1}+l_{21}$ counters; hence, $N_{13} = l_{11} + k_{1m_1} + l_{21}$. The similar process repeats up to the last round $2j$ in the first cycle; hence (3) holds. □

However, in several situations the values in (3) would indicate rounds that coincide. The following example shows that the formulas in Proposition 8 need to be modified in that case to skip overlapping values as appropriate.

Example 9. Consider the game with 11 players. Suppose each player can take 1 or 2 counters; alliance 1 is formed by players 1, 5, 7, 8, 9, 10 and alliance 2 is formed by players 2, 3, 4, 6, 1.

In this game

$$N_{11} = 1 < N_{12} = 5 < N_{13} = 8 = N_{14}.$$

In the given game, alliance 1 loses games with up to $N_{11} = 1$ counter, alliance 2 loses games with up to $N_{12} = 5$ counters and since $N_{14} = N_{13}$, then alliance 1 loses games starting with $N_{12} + 1$ up to $N_{15} = l_{11} + k_{13} + l_{21} + k_{22} + l_{31} = 1 + 6 + 1 + 2 + 4 = 14$ counters. Inequality (2) is not satisfied; therefore, several rounds coincide in this example. The following proposition generalizes the results in Proposition 8 in case inequality (2) does not hold.

Proposition 10. *Suppose alliance* 1 *can take*

$$\begin{aligned}
&l_{11}, \dots, l_{1n_1} \text{ counters on his first turn;}\\
&l_{21}, \dots, l_{2n_2} \text{ counters on his second turn;}\\
&\quad\vdots\\
&l_{j1}, \dots, l_{jn_j} \text{ counters on his } j\text{-th turn}
\end{aligned}$$

in one cycle. Also assume alliance 2 *can take*

$$\begin{aligned}
&k_{11}, \dots, k_{1m_1} \text{ counters on his first turn;}\\
&k_{21}, \dots, k_{2m_2} \text{ counters on his second turn;}\\
&\quad\vdots\\
&k_{j1}, \dots, k_{jm_j} \text{ counters on his } j\text{-th turn}
\end{aligned}$$

in one cycle. Then the maximum number of counters in each round for alliances in the first cycle can be calculated using the formulas (3)*, where we skip the round* $i+1$*, if* $N_{1\,i} \geq N_{1\,i+1}$.

Proof. If the conditions in (2) are satisfied, the desired result is obtained in Proposition 8. Suppose the conditions (2) are not satisfied. Consider the smallest index i, such that $N_{1\,i} \geq N_{1\,i+1}$. If i is odd, alliance 1 will be losing games with up to $N_{1\,i+2}$ counters, i.e., the round $i+1$ is absorbed into round i. If i is even, alliance 2 will be losing games with up to $N_{1\,i+2}$ counters, i.e., round $i+1$ is absorbed into round i. Hence, if $N_{1\,i} \geq N_{1\,i+1}$, one will skip round $i+1$. □

Next, we prove that the formulas similar to the ones in Proposition 10 also hold for later cycles.

Proposition 11. *Suppose alliance* 1 *can take*

$$\begin{aligned}
&l_{11}, \dots, l_{1n_1} \text{ counters on his first turn;}\\
&l_{21}, \dots, l_{2n_2} \text{ counters on his second turn;}\\
&\quad\vdots\\
&l_{j1}, \dots, l_{jn_j} \text{ counters on his } j\text{-th turn}
\end{aligned}$$

in one cycle. Also assume alliance 2 can take

$$k_{11}, \dots, k_{1m_1} \text{ counters on his first turn;}$$
$$k_{21}, \dots, k_{2m_2} \text{ counters on his second turn;}$$
$$\vdots$$
$$k_{j1}, \dots, k_{jm_j} \text{ counters on his } j\text{-th turn}$$

in one cycle. Then the largest number of counters in each round for alliances can be calculated using the following formulas, where r denotes the cycle:

$$
\begin{aligned}
N_{r1} &= (r-1)(mn+n+1) + l_{11} \\
N_{r2} &= (r-1)(mn+m+n) + l_{1n_1} + k_{11} \\
N_{r3} &= (r-1)(mn+n+1) + l_{11} + k_{1m_1} + l_{21} \\
N_{r4} &= (r-1)(mn+m+n) + l_{1n_1} + k_{11} + l_{2n_1} + k_{21} \\
&\vdots \\
N_{r(2j-1)} &= (r-1)(mn+n+1) + l_{11} + k_{1m_1} + \cdots + l_{j1} \\
N_{r(2j)} &= (r-1)(mn+m+n) + l_{1n_1} + k_{11} + \cdots + l_{jn_1} + k_{j1},
\end{aligned} \tag{4}
$$

where we skip the round if the previous value is larger than or equal to the following one.

Proof. We compute N_{21} by extending the formulas in Proposition 10. Hence, to get N_{21}, we add l to the smallest number of counters that alliance 1 can remove in the first cycle and the largest number of counters that alliance 2 can remove in the first cycle, i.e.,

$$N_{21} = l_{11} + n + 1 + nm = l_{11} + mn + n + 1.$$

However, if $N_{1(2j)} > N_{21}$, we skip the round N_{21}. Similarly, to compute N_{22}, we add $l_{1n_1} + k_{11}$ to the largest number of counters that alliance 1 can remove in the first cycle and the smallest number of counters that alliance 2 can remove in the first cycle, i.e.,

$$N_{22} = l_{1n_1} + k_{11} + (n+1)m + n = l_{1n_1} + k_{11} + mn + m + n.$$

In case $N_{21} \geq N_{22}$, we skip round two in the second cycle.

Similarly, in each cycle, $N_{r\,k}$ can be computed using the following rules.

1) If k is odd, add the sum of the number of counters that alliance 1 can remove in the previous cycles and the largest number of counters that alliance 2 can remove in the previous cycles to N_{1k}, i.e.,

$$N_{rk} = N_{1k} + (r-1)(mn+n+1).$$

2) If k is even, add the sum of the largest number of counters that alliance 1 can remove in the previous cycles and the number of counters that alliance 2 can remove in the previous cycles to N_{1k}, i.e.,

$$N_{rk} = N_{1k} + (r-1)(mn+m+n).$$

In these formulas we skip the round $k+1$ in a cycle r, if $N_{rk} > N_{r(k+1)}$, $1 < k < 2j-1$ or round 1 in cycle r if $N_{r1} > N_{(r-1)(2j)}$, $r > 1$. In larger cycles we might need to skip several rounds in a row. □

Proposition 12. *Suppose alliance* 1 *can take*

$$\begin{aligned} &l_{11}, \dots, l_{1n_1} \text{ counters on his first turn;}\\ &l_{21}, \dots, l_{2n_2} \text{ counters on his second turn;}\\ &\quad\vdots\\ &l_{j1}, \dots, l_{jn_j} \text{ counters on his } j\text{-th turn} \end{aligned}$$

in one cycle. Also assume alliance 2 *can take*

$$\begin{aligned} &k_{11}, \dots, k_{1m_1} \text{ counters on his first turn;}\\ &k_{21}, \dots, k_{2m_2} \text{ counters on his second turn;}\\ &\quad\vdots\\ &k_{j1}, \dots, k_{jm_j} \text{ counters on his } j\text{-th turn} \end{aligned}$$

in one cycle. The length of round j of alliance 1 *losses reduces by $m-1$ in each cycle.*

Proof. We notice from formula (1) and from formulas (4) in Proposition 11 that in each cycle the sum of the smallest number of counters that alliance 1 can remove in a cycle added to the largest number of counters that alliance 2 can remove in a cycle is $m-1$ more than the sum of the largest number of counters that alliance 1 can remove in a cycle added to the smallest number of counters that alliance 2 can remove in a cycle. The proposition follows. □

Proposition 13. *Suppose the largest round of alliance* 1 *losses in the first cycle has length* k. *Then alliance* 1 *wins all the games starting from cycle* r, *where* r *satisfies*

$$r \geq \frac{k}{m-1} + 1. \tag{5}$$

Proof. Given the formulas (4) in Proposition 11, for any particular game we can compute the largest number of consecutive losses for alliance 1 in the first cycle. Let's denote this value by k. By Proposition 12, alliance 1 losses are decreasing in each cycle by $m-1$. In cycle r, every round for alliance 1 has reduced by $(r-1)(m-1)$. Hence alliance 1 will not lose in cycle r, where r satisfies

$$k-(r-1)(m-1) \leq m-1.$$

Thus, (5) follows. □

Corollary 14. If any $n+1$ players form alliance 1 and the remaining n players form alliance 2, then for a sufficiently large game, the larger alliance will win.

Corollary 15. *If the largest round of alliance* 1 *losses in the first cycle has length* k, *then alliance* 1 *wins all the games that start with more than* $r \cdot \max\{N_{1\,2j-1}, N_{1\,2j}\}$ *counters, where* r *is defined in* (5).

Example 16. Consider the game from Example 9. Suppose each player can take 1 or 2 counters; alliance 1 is formed by players 1, 5, 7, 8, 9, 10 and alliance 2 is formed by players 2, 3, 4, 6, 11.

Since $N_{11} = 1$, $N_{12} = 5$, $N_{13} = N_{14} = 8$, , $N_{15} = 14$, and $N_{16} = 17$, then $k = 9$. We get that $r > 10$ by Proposition 13. Therefore alliance 1 can win all the games more than $r \cdot \max\{N_{15}, N_{16}\} = 170$ counters. This number is high due to the large second round for alliance 1 in the first cycle.

We can generalize these results to any misére Nim game with two alliances. First, let's state a proposition for alliances with equal numbers of players.

Proposition 17. *If any n players form alliance 1 and the remaining n players form alliance 2, then this game is reducible to two-player misere Nim, where each player can take $n, \dots, mn$ counters.*

Proof. The proof follows from [**3**, Proposition 4] and from the formula

$$\begin{aligned}\max(\text{Alliance 1}) + \min(\text{Alliance 2}) &= mn + n \\ &= \min(\text{Alliance 1}) + \max(\text{Alliance 2}).\end{aligned}$$ □

In this situation, each alliance has an unbounded set of games they can win. The key for us in our main work is that we can quantify the size of a game where the larger alliance's advantage is impossible for the other alliance to overcome.

Main Theorem. *If any $n + j$ $(j \geq 1)$ players form alliance 1 and the remaining n players form alliance 2, then for a sufficiently large game, the larger alliance will win.*

Proof. Alliance 1 can take $n + j$ up to $mn + mj$ counters in one cycle and alliance 2 can take n up to mn counters in one cycle. Since both alliances play wisely, then one of them will take away the largest possible number of counters, the other one the smallest possible number of counters [**3**, proof of Theorem 8]. Modifying formulas (1) in the proof of Proposition 5, in each cycle,

$$\begin{aligned}\max(\text{Alliance 1}) + \min(\text{Alliance 2}) &= mn + n + mj \\ &> \min(\text{Alliance 1}) + \max(\text{Alliance 2}) = mn + n + j.\end{aligned} \tag{6}$$

Substituting (6) in the proofs of Proposition 12 and Proposition 13, the Main Theorem follows. □

The final result shows that if the game starts with a large enough number of counters, the larger alliance will win the game no matter how the alliance members are distributed. Furthermore, given the distribution of the players in alliance and the number of counters to be removed, one can exactly determine (by modifying formula (5)) how big of a game is necessary.

REFERENCES

1. E. Berlekamp, J.Conway and R.Guy, *Winning ways for your mathematical plays*, Vol. 1, A.K. Peters, Natick, Massachusetts, 2001.

2. Arthur Holshouser, Harold Reiter and James Rudzinski, *Dynamic one-pile Nim*, Fibonacci Quart. **41** (2003), 253–262.

3. Annela Kelly, *One-pile misére Nim for three or more players*, IJMMS **40796** (2006).

Department of Mathematics and Computer Science, Bridgewater State University, Bridgewater, MA 02320
Email address: annela.kelly@bridgew.edu