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THE FIRST COHOMOLOGY GROUP OF MODULE EXTENSION BANACH ALGEBRAS

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ABSTRACT. Let A be a Banach algebra and X a Banach A-bimodule. Then $\mathcal{S} = A \oplus X$, the l_1-direct sum of A and X becomes a module extension Banach algebra when equipped with the algebra product $(a,x).(a',x') = (aa', ax' + xa')$. In this paper we compute the first cohomology group $H^1(\mathcal{S},\mathcal{S})$ for module extension Banach algebras $\mathcal{S}$. Also we obtain results on n-weak amenability of commutative module extension Banach algebras. We have shown that there are many different examples of non-n-weakly amenable Banach algebras.

1. Introduction. Let A be a Banach algebra and X a Banach A-bimodule. A derivation from A into X is a bounded linear map satisfying

$$D(ab) = a.(Db) + (Da).b \quad (a, b \in A).$$

For each $x \in X$ we denote by ad_x the derivation $D(a) = a.x - x.a$, for all $a \in A$, called an inner derivation. We denote by $Z^1(A,X)$ the space of all derivations from A into X, and by $B^1(A,X)$ the space of all inner derivations from A into X. The first cohomology group of A with coefficients in X, denoted by $H^1(A,X)$, is the quotient space $Z^1(A,X)/B^1(A,X)$. This first cohomology group of a Banach algebra gives vast information about the structure of A. If X is a Banach A-bimodule, X^* (the dual space of X) is an A-bimodule as usual. Let $n \in \mathbf{N}$, the set of non-negative integers. A Banach algebra A is called amenable if $H^1(A,X^*) = 0$ for every A-bimodule X. A Banach algebra A is called n-weakly amenable (weakly amenable in case $n = 1$) if $H^1(A, A^{(n)}) = 0$, where $A^{(n)}$ is the n-th dual space of A and $A^{(0)} = A$ (cf. [**3**]). In [**5, 6**] the authors have calculated the first cohomology group of a class of Banach algebras which they called *triangular Banach algebras*.

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Motivated by these earlier investigations, in this paper we shall focus on a special kind of Banach algebras, called *module extension Banach algebras*. If A is a Banach algebra and X is a Banach A-bimodule, then the module extension Banach algebra corresponding to A and X is $\mathcal{S} = A \oplus X$, the l_1-direct sum of A and X with the algebra product defined as follows:

$$(a, x).(a', x') = (aa', ax' + xa'), \quad (a, a' \in A, x, x' \in X).$$

As has been discussed in [**11**], every triangular Banach algebra is isometrically isomorphic to a module extension Banach algebra. Therefore, module extension Banach algebras are generalized forms of triangular Banach algebras.

In Section 2, we compute the first cohomology group $H^1(\mathcal{S}, \mathcal{S})$ of module extension Banach algebras $\mathcal{S} = A \oplus X$. As a consequence, we show that $H^1(\mathcal{S}, \mathcal{S}) \neq 0$ if $\mathcal{S} = A \oplus A$ (a known result of [**11**]). We prove that if A is a commutative Banach algebra, X is a non-zero symmetric A-bimodule (i.e., $ax = xa$ for all $a \in A$ and $x \in X$) and $\mathcal{S} = A \oplus X$, then $H^1(\mathcal{S}, \mathcal{S}) \neq 0$ and so $\mathcal{S}$ cannot be n-weakly amenable for any $n \in \mathbf{N}$. Furthermore, we compute the first cohomology group $H^1(\mathcal{S}, Y)$ for $\mathcal{S} = A \oplus X$ and an A-bimodule Y which is an $\mathcal{S}$-bimodule in the canonical fashion.

In Section 3, we compute the first cohomology group $H^1(\mathcal{S}, \mathcal{S})$ and $H^1(\mathcal{S}, Y)$ for many concrete examples of module extension Banach algebras $\mathcal{S}$ and $\mathcal{S}$-bimodules Y.

2. The first cohomology group $H^1(\mathcal{S}, \mathcal{S})$. Our aim in this section is to calculate $H^1(\mathcal{S}, \mathcal{S})$ and $H^1(\mathcal{S}, Y)$ and then apply this group to characterize n-weak amenability of $\mathcal{S}$. Then we obtain some interesting non-n-weakly amenable Banach algebras.

Notation 2.1. If A is a Banach algebra and X is a Banach A-bimodule, we denote by

(i) $\mathrm{Z}(A)$ the algebraic center of A,

(ii) $C_A(X, X)$ the set $\{\mathrm{ad}_a : X \to X \mid a \in \mathrm{Z}(A)\}$,

(iii) $\mathrm{Hom}_A(X, X)$ the set of all bounded A-bimodule homomorphisms from X into X.

The following proposition and corollary characterize derivations on $\mathcal{S}$. Proofs are basically the proofs of [**11**, Theorem 2.2 and Lemma 3.2], respectively, when $m = 0$. So we omit them.

Proposition 2.2. *Let $\mathcal{S} = A \oplus X$. Then $D \in Z^1(\mathcal{S}, \mathcal{S})$ if and only if*

$$(*) \qquad D(a,x) = (D_A(a) + T_1(x),\, D_X(a) + T_2(x)) \quad (a \in A, x \in X)$$

such that

(i) $D_A \in Z^1(A, A)$,

(ii) $D_X \in Z^1(A, X)$,

(iii) *$T_1 : X \to A$ is an A-bimodule homomorphism such that $T_1(x)y + xT_1(y) = 0$ for all $x, y \in X$,*

(iv) *$T_2 : X \to X$ is a bounded linear map such that*

$$T_2(ax) = aT_2(x) + D_A(a)x,\ T_2(xa) = T_2(x)a + xD_A(a)\ (a \in A, x \in X).$$

Moreover, D is inner if and only if D_A and D_X are inner, $T_1 = 0$ and if $D_A = \mathrm{ad}_a$, then $T_2 = \mathrm{ad}_a$.

Corollary 2.3. *Let $\varphi \in \mathrm{Hom}_A(X, X)$, and let $D_\varphi : \mathcal{S} \to \mathcal{S}$ be defined by $D_\varphi(a, x) = (0, \varphi(x))$. Then D_φ is a derivation on $\mathcal{S}$ and D_φ is inner if and only if $\varphi \in C_A(X, X)$.*

Proposition 2.4. *If $\mathcal{S} = A \oplus A$, then every derivation $D \in Z^1(A, A)$ gives rise to a derivation $\widetilde{D} \in Z^1(\mathcal{S}, \mathcal{S})$ with $\widetilde{D}(a, b) = (D(a), D(b))$. Moreover, $\widetilde{D}$ is inner if and only if D is inner.*

Proof. It is routine to check that $\widetilde{D}$ is a bounded derivation. Moreover, if $D = \mathrm{ad}_a$ is inner, then $\widetilde{D} = \mathrm{ad}_{(a,0)}$ is inner. Conversely, if $\widetilde{D} = \mathrm{ad}_{(a,b)}$ for some $a, b \in A$, then for $c, d \in A$ we have

$$(D(c), D(d)) = \widetilde{D}(c, d) = \mathrm{ad}_{(a,b)}(c, d) = (\mathrm{ad}_a(c), \mathrm{ad}_a(d) + \mathrm{ad}_b(c)).$$

Thus $D = \mathrm{ad}_a$ and $b \in Z(A)$. Hence D is also inner. $\square$

Now we are ready to state and prove one of the main theorems of this paper.

Theorem 2.5. *Let A be a Banach algebra and X a Banach A-bimodule. Let $H^1(A,A)=0$ and the only A-bimodule homomorphism $T:X\to A$ such that $T(x)y+xT(y)=0$ for all $x,y\in X$, be $T=0$. Then for $\mathcal{S}=A\oplus X$,*

$$H^1(\mathcal{S},\mathcal{S})\cong H^1(A,X)\oplus\frac{\mathrm{Hom}_A(X,X)}{C_A(X,X)},$$

where $\cong$ denotes the vector space isomorphism.

Proof. First note that $C_A(X,X)\subseteq \mathrm{Hom}_A(X,X)$, because for $a\in Z(A)$, $b,c\in A$ and $x\in X$ we have

$$\mathrm{ad}_a(bxc)=a(bxc)-(bxc)a=baxc-bxac=b(ax-xa)c=b\,\mathrm{ad}_a(x)c.$$

Now let $\Psi: Z^1(A,X)\oplus \mathrm{Hom}_A(X,X)\to H^1(\mathcal{S},\mathcal{S})$ be defined by $\Psi(D,T)=[\delta_{D,T}]$, where $\delta_{D,T}(a,x)=(0,\,D(a)+T(x))$ and $[\delta_{D,T}]$ represents the equivalence class of $\delta_{D,T}$ in $H^1(\mathcal{S},\mathcal{S})$. Clearly Ψ is linear and surjective because of Proposition 2.2, every derivation $\delta\in Z^1(\mathcal{S},\mathcal{S})$ is of the form $\delta(a,x)=(D_A(a),\,D_X(a)+T_2(x))$ such that D_A, D_X and T_2 satisfy in conditions (i), (ii) and (iv) of Proposition 2.2, respectively (note that $T_1=0$). Since $H^1(A,A)=0$, we have $D_A=\mathrm{ad}_b$ for some $b\in A$. Let $T=T_2-\mathrm{ad}_b$. Then $T\in \mathrm{Hom}_A(X,X)$ and $\delta(a,x)-\delta_{D_X,T}(a,x)=(\mathrm{ad}_b(a),\mathrm{ad}_b(x))$. Thus $\delta-\delta_{D_X,T}$ is inner and so $\Psi(D,T)=[\delta_{D_X,T}]=[\delta]$. Also we have

$$\begin{aligned}\ker\Psi &=\{(D,T)\in Z^1(A,X)\oplus\mathrm{Hom}_A(X,X)|\,\delta_{D_X,T}\text{ is inner}\}\\ &=\{(D,\mathrm{ad}_a)|\,D\in B^1(A,X),\,\mathrm{ad}_a:X\to X\text{ for some }a\in Z(A)\},\\ &\qquad\qquad\text{(by Corollary 2.3)}\\ &=B^1(A,X)\oplus C_A(X,X).\end{aligned}$$

Hence

$$H^1(\mathcal{S},\mathcal{S})\cong H^1(A,X)\oplus\frac{\mathrm{Hom}_A(X,X)}{C_A(X,X)}.\qquad\square$$

Corollary 2.6. (i) *Let $\mathcal{S}=A\oplus A$. Then by Proposition* 2.4 *and Theorem* 2.5, *$H^1(\mathcal{S},\mathcal{S})\neq 0$. This is the special case* (*when $m=0$*) *of the known result of* [**11**, page 4142].

(ii) *Let $n \in \mathbf{N}$ and $A^{(n)}$ be the n-th dual space of A which is a Banach A-bimodule as usual. Let $H^1(A,A) = 0$ and $\mathcal{S} = A \oplus A^{(n)}$. Then $H^1(\mathcal{S},\mathcal{S}) \neq 0$.*

Let A be a commutative Banach algebra such that $H^1(A,A) \neq 0$. As it is discussed in [**3**, page 23], A cannot be n-weakly amenable for any $n \in \mathbf{N}$. In particular, $H^1(A, A^{(2n)}) = 0$ (for some $n \in \mathbf{N}$) implies $H^1(A,A) = 0$ by [**3**, Proposition 1.2]. Also, if $H^1(A, A^{(2n+1)}) = 0$ (for some $n \in \mathbf{N}$) then $H^1(A, A^*) = 0$ again by [**3**, Proposition 1.2]. Therefore $H^1(A, A^{(n)}) = 0$ for each $n \in \mathbf{N}$, by [**2**, Theorem 2.8.63], which is a contradiction. Now, in the following corollary we extend [**11**, Proposition 5.2] when $H^1(A,A) = 0$:

Corollary 2.7. *Let A be a commutative Banach algebra, X a non-zero symmetric A-bimodule, $\mathcal{S} = A \oplus X$ and $H^1(A,A) = 0$. Then $\mathcal{S}$ cannot be n-weakly amenable for any $n \in \mathbf{N}$.*

Proof. We have $C_A(X,X) = 0$ and $\mathrm{Hom}_A(X,X) \neq 0$. Thus by Theorem 2.5, $H^1(\mathcal{S},\mathcal{S}) \neq 0$ and so $\mathcal{S}$ cannot be n-weakly amenable for any $n \in \mathbf{N}$. □

Example 2.1. Let X be a Banach space. It is clear that $H^1(\mathbf{C},\mathbf{C}) = 0$ and $H^1(\mathbf{C}, X) = 0$. Let $T : X \to \mathbf{C}$ be a continuous linear functional such that $T(x)y + xT(y) = 0$ for all $x, y \in X$. Then $T(x)T(y) = 0$ for all $x, y \in X$, and so $T = 0$. Clearly $\mathrm{Hom}_{\mathbf{C}}(X,X) = \mathcal{B}(X)$ (the space of all bounded linear operators from X into X) and $C_{\mathbf{C}}(X,X) = 0$, hence by Theorem 2.5, $H^1(\mathbf{C} \oplus X, \mathbf{C} \oplus X) = \mathcal{B}(X)$.

Corollary 2.8. *Let $\mathcal{S} = A \oplus A$, $H^1(A,A) = 0$ and the only A-bimodule homomorphism $T : A \to A$ that satisfies $T|_{A^2} = 0$, be $T = 0$. Then $H^1(\mathcal{S},\mathcal{S}) \cong \mathrm{Hom}_A(A,A)$.*

Remark 2.1. Note that there are many classes of Banach algebras A satisfied in the conditions of Corollary 2.8. For example:

(i) If A is a von-Neumann algebra, a commutative C^*-algebra, a W^*-algebra or a simple unital C^*-algebra (i.e., A has no proper closed two-sided ideal), then $H^1(A,A) = 0$ [**9**].

(ii) If A is a semi-simple commutative Banach algebra, then $H^1(A,A) = 0$ [10].

(iii) If $\overline{A^2} = A$, especially if A is weakly amenable or if A has a one sided approximate identity, then the only A-bimodule homomorphism $T : A \to A$ satisfying $T|_{A^2} = 0$, is $T = 0$.

Example 2.2. Let A be one of the Banach algebras listed in Remark 2.1 (i). Then for $\mathcal{S} = A \oplus A$, we have $H^1(\mathcal{S}, \mathcal{S}) \cong \mathrm{Hom}_A(A, A)$.

It is easy to see that if A is a unital Banach algebra, then $\mathrm{Hom}_A(A, A) \cong \mathrm{Z}(A)$. Therefore, if A is a unital Banach algebra and $H^1(A, A) = 0$, then for $\mathcal{S} = A \oplus A$ we have $H^1(\mathcal{S}, \mathcal{S}) \cong \mathrm{Z}(A)$ which is non-zero.

Example 2.3. Let $A = c_0$ with pointwise multiplication, and $\mathcal{S} = c_0 \oplus c_0$. Then $\mathrm{Hom}_{c_0}(c_0, c_0)$ can be identified with l_∞, viewed as multiplication operators [**5**]. Also, since c_0 is weakly amenable, $H^1(c_0, c_0) = 0$ by [**2**, Theorem 2.8.63]. It follows that

$$H^1(c_0 \oplus c_0, c_0 \oplus c_0) \cong l_\infty.$$

If Y is a Banach A-bimodule, then it is also a Banach $\mathcal{S}$-bimodule by the canonical projection $\mathcal{S} \to A$, $(a, x) \mapsto a$ for $\mathcal{S} = A \oplus X$. With this action we can compute the first cohomology group $H^1(\mathcal{S}, Y)$ of $\mathcal{S} = A \oplus X$ with coefficients in Y.

Theorem 2.9. *Let X and Y be Banach A-bimodules and $\mathcal{S} = A \oplus X$. Then by the above discussion, we have*

$$H^1(\mathcal{S}, Y) \cong H^1(A, Y) \oplus \mathrm{Hom}_A(X, Y),$$

as vector spaces. Specifically,

$$H^1(\mathcal{S}, A) \cong H^1(A, A) \oplus \mathrm{Hom}_A(X, A).$$

Proof. Let $\Psi : Z^1(A, Y) \oplus \mathrm{Hom}_A(X, Y) \to H^1(\mathcal{S}, Y)$ be defined by $\Psi(D, T) = [\delta_{D,T}]$, where $\delta_{D,T}(a, x) = D(a) + T(x)$ and $[\delta_{D,T}]$ represents the equivalence class of $\delta_{D,T}$ in $H^1(\mathcal{S}, Y)$. Clearly Ψ is a well-defined linear map. It is then routinely checked that $\delta_{D,T}$ is inner if and only if

D is inner and $T = 0$. Therefore, $\ker \Psi = B^1(A, Y) \oplus 0$. For surjectivity, let $\delta \in Z^1(\mathcal{S}, Y)$ be a derivation. Then $\delta(a, x) = \delta o i_1(a) + \delta o i_2(x)$, where $i_1 : A \to \mathcal{S}$ and $i_2 : X \to \mathcal{S}$ are inclusion maps. Set $D = \delta o i_1$ and $T = \delta o i_2$. Then $D \in Z^1(A, Y)$, $T \in \mathrm{Hom}_A(X, Y)$ and $\delta = \delta_{D,T}$. Hence $H^1(\mathcal{S}, Y) \cong H^1(A, Y) \oplus \mathrm{Hom}_A(X, Y)$. □

Example 2.4. Let X be a Banach space and $\mathcal{S} = \mathbf{C} \oplus X$. Then we have $H^1(\mathbf{C} \oplus X, \mathbf{C}) \cong X^*$.

3. Examples. In this section we present a number of examples of module extension Banach algebras $\mathcal{S}$, and we explicitly determine $H^1(\mathcal{S}, \mathcal{S})$ and $H^1(\mathcal{S}, Y)$. Furthermore, we give many different examples of non-n-weakly amenable Banach algebras.

Example 3.1. Let $\mathbf{D}_n$ denote the $n \times n$ diagonal matrices over the field $\mathbf{C}$, called the diagonal algebra, and let $\mathbf{M}_n$ denote the set of all $n \times n$ matrices over the field $\mathbf{C}$.

(i) Let $A = X = \mathbf{D}_n$ with usual matrix actions. It is clear that $H^1(\mathbf{D}_n, \mathbf{D}_n) = 0$. Therefore, by Corollary 2.8 we have

$$H^1(\mathbf{D}_n \oplus \mathbf{D}_n, \mathbf{D}_n \oplus \mathbf{D}_n) \cong \mathbf{D}_n.$$

(ii) Let $A = \mathbf{D}_n$ and $X = \mathbf{M}_n$. Then X is an A-bimodule in the obvious way. We have $H^1(\mathbf{D}_n, \mathbf{D}_n) = 0$ and $H^1(\mathbf{D}_n, \mathbf{M}_n) = 0$. If $T : \mathbf{M}_n \to \mathbf{D}_n$ is a $\mathbf{D}_n$-bimodule homomorphism such that $AT(B) + T(A)B = 0$ for all $A, B \in \mathbf{M}_n$, then $T(A)T(B) = 0$ for all $A, B \in \mathbf{M}_n$. Let $A = B$; then $T(A)^2 = 0$. Since $T(A) \in \mathbf{D}_n$, $T(A) = 0$, and so $T = 0$. Also by [**5**, Example 4.1], $\mathrm{Hom}_{\mathbf{D}_n}(\mathbf{M}_n, \mathbf{M}_n) \cong \mathbf{M}_n$. It is easy to see that

$$C_{\mathbf{D}_n}(\mathbf{M}_n, \mathbf{M}_n) \cong \{[d_i - d_j] : 1 \le i, j \le n, d_1, \dots, d_n \in \mathbf{C}\},$$

which is of dimension $n - 1$. Hence by Theorem 2.5 we have

$$H^1(\mathbf{D}_n \oplus \mathbf{M}_n, \mathbf{D}_n \oplus \mathbf{M}_n) \cong \frac{\mathbf{M}_n}{\{[d_i - d_j] : 1 \le i, j \le n, d_1, \dots, d_n \in \mathbf{C}\}}.$$

Therefore, $\dim H^1(\mathbf{D}_n \oplus \mathbf{M}_n, \mathbf{D}_n \oplus \mathbf{M}_n) = n^2 - n + 1$ (vector space dimension).

(iii) Let $A = \mathbf{D}_n$ and $X = \mathbf{M}_n$ with the following action:

$$[\lambda_i].[x_{ij}] = [x_{ij}].[\lambda_i] := [\lambda_1 x_{ij}] \quad ([\lambda_i] \in \mathbf{D}_n, [x_{ij}] \in \mathbf{M}_n).$$

Then as in Example 2.1 we find that $H^1(\mathbf{D}_n \oplus \mathbf{M}_n, \mathbf{D}_n \oplus \mathbf{M}_n) = \mathcal{B}(\mathbf{M}_n)$, which is of dimension n^4.

Example 3.2. Let H be an infinite dimensional Hilbert space, let $\mathcal{B}(H)$, $\mathcal{K}(H)$ and $\mathcal{Q}(H)$ be the spaces of all bounded operators from H into H, compact operators from H into H and the Calkin algebra of H, respectively. Since $H^1(\mathcal{B}(H), \mathcal{B}(H)) = 0$, every derivation $D : \mathcal{B}(H) \to \mathcal{K}(H)$ is of the form $D = \mathrm{ad}_T$ for some $T \in \mathcal{B}(H)$. Thus the linear map $\Psi : Z^1(\mathcal{B}(H), \mathcal{K}(H)) \to \frac{\mathcal{B}(H)}{\mathbf{C}I \oplus \mathcal{K}(H)}$ with $\Psi(\mathrm{ad}_T) = [T]$ is well defined with kernel $\ker \Psi = B^1(\mathcal{B}(H), \mathcal{K}(H))$. Thus $H^1(\mathcal{B}(H), \mathcal{K}(H)) \cong \frac{\mathcal{B}_0(H)}{\mathbf{C}I \oplus \mathcal{K}(H)} \cong \frac{\mathrm{Z}(\mathcal{Q}(H))}{\mathbf{C}}$, where $\mathcal{B}_0(H) = \{T \in \mathcal{B}(H) | TS - ST \in \mathcal{K}(H)$, for all $S \in \mathcal{B}(H)\}$ is a subspace of $\mathcal{B}(H)$ containing $\mathbf{C}I \oplus \mathcal{K}(H)$. It is a routine verification that $\mathrm{Hom}_{\mathcal{B}(H)}(\mathcal{K}(H), \mathcal{K}(H)) \cong \mathbf{C}$. Therefore, by Theorem 2.5 we have

$$H^1(\mathcal{B}(H) \oplus \mathcal{K}(H), \mathcal{B}(H) \oplus \mathcal{K}(H)) \cong \frac{\mathrm{Z}(\mathcal{Q}(H))}{\mathbf{C}} \oplus \mathbf{C} \cong \mathrm{Z}(\mathcal{Q}(H)).$$

(Note that the only $\mathcal{B}(H)$-bimodule homomorphism $\varphi : \mathcal{K}(H) \to \mathcal{B}(H)$ such that $\varphi(T)S + T\varphi(S) = 0$ for all $T, S \in \mathcal{K}(H)$ is $\varphi = 0$ because $\mathcal{K}(H)$ has an approximate identity.) Furthermore, by Theorem 2.9,

$$H^1(\mathcal{B}(H) \oplus \mathcal{K}(H), \mathcal{K}(H)) \cong \frac{\mathrm{Z}(\mathcal{Q}(H))}{\mathbf{C}} \oplus \mathbf{C} \cong \mathrm{Z}(\mathcal{Q}(H)).$$

Also we have $\mathrm{Hom}_{\mathcal{B}(H)}(\mathcal{K}(H), \mathcal{B}(H)) \cong \mathbf{C}$, and so by Theorem 2.9,

$$H^1(\mathcal{B}(H) \oplus \mathcal{K}(H), \mathcal{B}(H)) \cong \mathbf{C}.$$

Example 3.3. (i) Let G be a discrete group. By [**7**], $H^1(l^1(G), l^1(G)) = 0$. Also we have $\mathrm{Hom}_{l^1(G)}(l^1(G), l^1(G)) \cong \mathrm{Z}(l^1(G))$. Hence by Corollary 2.8, $H^1(l^1(G) \oplus l^1(G), l^1(G) \oplus l^1(G)) \cong \mathrm{Z}(l^1(G))$.

(ii) Let G be a discrete abelian group. For $1 < p < \infty$ let $A = l^1(G)$, $X_p = l^p(G)$ and $\mathcal{S}_p = l^1(G) \oplus l^p(G)$ where $l^1(G)$ acts on $l^p(G)$ by convolution. Since G is amenable, $H^1(l^1(G), l^p(G)) = 0$. Let $T : l^p(G) \to l^1(G)$ be an $l^1(G)$-bimodule homomorphism such that $f * T(g) + T(f) * g = 0$ for all $f, g \in l^p(G)$. Then, for any $x \in G$, $\delta_x * T(\delta_e) + T(\delta_x) * \delta_e = 0$ where e is the unit of G. But $\delta_x * T(\delta_e) = T(\delta_x) * \delta_e = T(\delta_x * \delta_e) = T(\delta_x)$, thus $T(\delta_x) = 0$ and so $T = 0$. Now by Theorem 2.5,

$$H^1(\mathcal{S}_p, \mathcal{S}_p) \cong \frac{\mathrm{Hom}_{l^1(G)}(l^p(G), l^p(G))}{C_{l^1(G)}(l^p(G), l^p(G))}.$$

It is easy to see that $C_{l^1(G)}(l^p(G), l^p(G)) = 0$. Finally $\mathrm{Hom}_{l^1(G)}(l^p(G), l^p(G))$ is the Banach algebra $PM_p(G)$ of p-pseudomeasures on G [**5**, Example 4.2]. Therefore $H^1(\mathcal{S}_p, \mathcal{S}_p) \cong PM_p(G)$.

For example, when $p = 2$, $H^1(\mathcal{S}_2, \mathcal{S}_2) \cong PM_2(G) = VN(G)$ is the von-Neumann algebra of G. Note that $H^1(\mathcal{S}_p, \mathcal{S}_p)$ is never zero and so $\mathcal{S}_p$ cannot be n-weakly amenable for any $n \in \mathbf{N}$ by Corollary 2.7.

(iii) Let G be a discrete abelian group, $A = l^1(G)$ and $X = l^\infty(G)$. Then by [**1**], $\mathrm{Hom}_{l^1(G)}(l^\infty(G), l^\infty(G))$ is isometrically isomorphic with $l^\infty(G)^*$ as Banach algebras. As in (ii), $C_{l^1(G)}(l^\infty(G), l^\infty(G)) = 0$ and so $H^1(\mathcal{S}_\infty, \mathcal{S}_\infty) \cong l^\infty(G)^*$. Hence $\mathcal{S}_\infty$ is not n-weakly amenable for any $n \in \mathbf{N}$.

Example 3.4. Let G be an abelian locally compact group. We can identify $l^1(G)$ with $M_d(G)$, the space of discrete measures on G, so $l^1(G)$ acts on $L^1(G)$ and $M(G)$ (the measure algebra of G) by convolution. It is obvious that $L^1(G)$ and $M(G)$ are symmetric $l^1(G)$-bimodules, and so $H^1(l^1(G), L^1(G)) = 0$ and $H^1(l^1(G), M(G)) = 0$. Since $l^1(G)$ is strictly dense in $M(G)$, we have $\mathrm{Hom}_{l^1(G)}(L^1(G), L^1(G)) \cong M(G)$ and $\mathrm{Hom}_{l^1(G)}(M(G), M(G)) \cong M(G)$. Therefore by Theorem 2.5,

(i) $H^1(l^1(G) \oplus L^1(G), l^1(G) \oplus L^1(G)) \cong M(G)$.

(ii) $H^1(l^1(G) \oplus M(G), l^1(G) \oplus M(G)) \cong M(G)$.

Hence $l^1(G) \oplus L^1(G)$ and $l^1(G) \oplus M(G)$ cannot be n-weakly amenable for any $n \in \mathbf{N}$.

Example 3.5. Let G be a locally compact group. Since $L^1(G)$ is an ideal of $M(G)$, the Banach algebra $L^1(G)$ is an $M(G)$-bimodule as usual. By [**7**] we have $H^1(M(G), M(G)) = 0$. It is an easy application of Wendel's theorem [**2**, Theorem 3.3.40] that $\mathrm{Hom}_{M(G)}(L^1(G), L^1(G)) \cong \mathrm{Z}(M(G))$ (note that $\mu \in M(G)$ commutes with every $f \in L^1(G)$ if and only if $\mu \in \mathrm{Z}(M(G))$). For calculating the first cohomology group $H^1(M(G), L^1(G))$, we define the linear map $\Phi : Z^1(M(G), L^1(G)) \to \frac{M(G)}{\mathrm{Z}(M(G))+L^1(G)}$ by $\Phi(D) = [\mu]$ where $D = \mathrm{ad}_\mu$ for some $\mu \in M(G)$ by noting that $H^1(M(G), M(G)) = 0$, and $[\mu]$ denotes the equivalence class of $\mu \in M(G)$ in the quotient space $\frac{M(G)}{\mathrm{Z}(M(G))+L^1(G)}$. It is easy to see that $\ker\Phi = B^1(M(G), L^1(G))$. Thus $H^1(M(G), L^1(G)) \cong \frac{M_0(G)}{\mathrm{Z}(M(G))+L^1(G)}$, where $M_0(G) = \{\nu \in M(G) | \lambda * \nu - \nu * \lambda \in L^1(G)$, for all $\lambda \in M(G)\}$ is a subspace of $M(G)$, containing $\mathrm{Z}(M(G)) + L^1(G)$. In fact, if $P : M(G) \to \frac{M(G)}{L^1(G)}$ is the canonical map, then $M_0(G) = P^{-1}(\mathrm{Z}(\frac{M(G)}{L^1(G)}))$. It is obvious that $C_{M(G)}(L^1(G), L^1(G)) = 0$. Hence by Theorem 2.5,

$$H^1(M(G) \oplus L^1(G), M(G) \oplus L^1(G)) \cong \frac{M_0(G)}{\mathrm{Z}(M(G))+L^1(G)} \oplus \mathrm{Z}(M(G)).$$

Finally, by Theorem 2.9 we can obtain:

(i) $H^1(M(G) \oplus L^1(G), L^1(G)) \cong \frac{M_0(G)}{\mathrm{Z}(M(G))+L^1(G)} \oplus \mathrm{Z}(M(G))$.

(ii) $H^1(M(G) \oplus M(G), L^1(G)) \cong \frac{M_0(G)}{\mathrm{Z}(M(G))+L^1(G)} \oplus \mathrm{Z}(L^1(G))$.

(iii) $H^1(M(G) \oplus L^1(G), M(G)) \cong H^1(M(G) \oplus M(G), M(G)) \cong \mathrm{Z}(M(G))$.

Note that $\mathrm{Hom}_{M(G)}(M(G), L^1(G)) \cong \mathrm{Z}(L^1(G))$, $\mathrm{Hom}_{M(G)}(L^1(G), M(G)) \cong \mathrm{Z}(M(G))$ and $\mathrm{Hom}_{M(G)}(M(G), M(G)) \cong \mathrm{Z}(M(G))$.

Example 3.6. Let G be a locally compact group such that $H^1(L^1(G), L^1(G)) = 0$. For example G can be a locally compact abelian group or a discrete group.

(i) Let $A = X = L^1(G)$. Then by Wendel's theorem we have $\mathrm{Hom}_{L^1(G)}(L^1(G), L^1(G)) \cong Z(M(G))$. So by Corollary 2.8, $H^1(L^1(G) \oplus L^1(G), L^1(G) \oplus L^1(G)) \cong \mathrm{Z}(M(G))$. Thus $H^1(L^1(G) \oplus L^1(G), L^1(G) \oplus L^1(G))$ is never zero and so if G is abelian, $L^1(G) \oplus L^1(G)$ cannot be n-weakly amenable for any $n \in \mathbf{N}$.

(ii) Let $A = L^1(G)$ and $X = M(G)$. By [**7**] we have $H^1(L^1(G), M(G)) = 0$. As in Example 3.5 we have $\mathrm{Hom}_{L^1(G)}(M(G), M(G)) \cong \mathrm{Z}(M(G))$. Therefore, by Theorem 2.5 we have $H^1(L^1(G) \oplus M(G),\ L^1(G) \oplus$

$M(G)) \cong Z(M(G))$ and so if G is abelian, $L^1(G) \oplus M(G)$ cannot be n-weakly amenable for any $n \in \mathbf{N}$.

By Theorem 2.9 we have the following results:

(iii) $H^1(L^1(G)\oplus L^1(G), L^1(G)) \cong \mathrm{Hom}_{L^1(G)}(L^1(G), L^1(G)) \cong Z(M(G))$.

(iv) $H^1(L^1(G)\oplus L^1(G), M(G)) \cong \mathrm{Hom}_{L^1(G)}(L^1(G), M(G)) \cong Z(M(G))$.

(v) $H^1(L^1(G)\oplus M(G), L^1(G)) \cong \mathrm{Hom}_{L^1(G)}(M(G), L^1(G)) \cong Z(L^1(G))$.

(vi) $H^1(L^1(G)\oplus M(G), M(G)) \cong \mathrm{Hom}_{L^1(G)}(M(G), M(G)) \cong Z(M(G))$.

Example 3.7. Let G be a compact group, and let $A(G)$ be the Fourier algebra of G defined by Eymard in [**4**]. Then $A(G)$ is a unital commutative semi-simple Banach algebra and so by [**10**], $H^1(A(G), A(G)) = 0$. Therefore, by Corollary 2.8,

$$H^1(A(G) \oplus A(G), A(G) \oplus A(G)) \cong A(G).$$

Hence $A(G) \oplus A(G)$ cannot be n-weakly amenable for any $n \in \mathbf{N}$.

Example 3.8. Let G be an abelian locally compact group. Then, for $1 < p < \infty$, $L^p(G)$ is an $L^1(G)$-bimodule by convolution. We have $H^1(L^1(G), L^1(G)) = 0$. We show that the only $L^1(G)$-bimodule homomorphism $T : L^p(G) \to L^1(G)$ such that $T(f) * g + f * T(g) = 0$ for all $f, g \in L^p(G)$ is $T = 0$. Let $(e_\alpha)_{\alpha\in\Gamma} \subseteq \mathbf{C}_c(G)$ be a bounded approximate identity for $L^1(G)$ and $L^p(G)$ (as an $L^1(G)$-bimodule). Then for such T we have $T(e_\beta) * e_\alpha + e_\beta * T(e_\alpha) = 0$ for all $\alpha, \beta \in \Gamma$. But $T(e_\beta * e_\alpha) = T(e_\beta) * e_\alpha = e_\beta * T(e_\alpha)$, and so $T(e_\beta) * e_\alpha = 0$. Since $T(e_\beta) * e_\alpha \to T(e_\beta)$ for all $\beta \in \Gamma$, $T(e_\beta) = 0$ for all $\beta \in \Gamma$. Now let $f \in L^p(G)$; then $T(f) * e_\alpha + f * T(e_\alpha) = 0$. Thus $T(f) * e_\alpha = 0$ and so $T = 0$ (note that $C_{L^1(G)}(L^p(G), L^p(G)) = 0$ because $C_c(G)$ is dense in both $L^p(G)$ and $L^1(G)$). Hence by Theorem 2.5 we have

$$\begin{aligned} H^1(L^1(G) \oplus L^p(G), L^1(G) \oplus L^p(G)) &\cong \mathrm{Hom}_{L^1(G)}(L^p(G), L^p(G)) \\ &\cong PM_p(G). \end{aligned}$$

Therefore, $H^1(L^1(G) \oplus L^p(G), L^1(G) \oplus L^p(G)) \neq 0$ and so $L^1(G) \oplus L^p(G)$ cannot be n-weakly amenable for any $n \in \mathbf{N}$.

Example 3.9. Let G be an abelian compact group with normalized Harr measure. Then $L^p(G)$ for $1 \leq p < \infty$ is a commutative Banach algebra with the convolution product. It is well known that $L^p(G)$ is semi-simple, but we give its proof. Let $\pi : L^1(G) \to \mathbf{C}$ be an irreducible representation of $L^1(G)$. Then it is easy to see that $\widetilde{\pi} = \pi|_{L^p(G)} : L^p(G) \to \mathbf{C}$ is an irreducible representation of $L^p(G)$. Thus $rad(L^p(G)) \subseteq rad(L^1(G)) = \{0\}$ and so $H^1(L^p(G), L^p(G)) = 0$ by [**10**]. As in Example 3.8, we have $C_{L^p(G)}(L^1(G), L^1(G)) = 0$ and the only $L^1(G)$-bimodule homomorphism $T : L^1(G) \to L^p(G)$ such that $T(f) * g + f * T(g) = 0$ for all $f, g \in L^1(G)$ is $T = 0$. Since $C_c(G)$ is dense in both $L^p(G)$ and $L^1(G)$, by Wendel's theorem we have $\mathrm{Hom}_{L^p(G)}(L^1(G), L^1(G)) \cong M(G)$. Therefore,

$$H^1(L^p(G) \oplus L^1(G), L^p(G) \oplus L^1(G)) \cong M(G).$$

Thus $L^p(G) \oplus L^1(G)$ cannot be n-weakly amenable for any $n \in \mathbf{N}$.

Example 3.10. Let G be a locally compact group such that $H^1(L^1(G), L^1(G)) = 0$. Then for $1 < p < \infty$, $L^p(G)$ is an $L^1(G)$-bimodule by the following module actions:

$$f.g = f * g, \ g.f = 0 \quad (f \in L^1(G), g \in L^p(G)).$$

We denote by $L^p(G)_0$ the space $L^p(G)$ as an $L^1(G)$-bimodule with the above actions. Since $L^q(G)^* = L^p(G)$, $\frac{1}{p} + \frac{1}{q} = 1$, is an $L^1(G)$-bimodule with zero left action, we have $H^1(L^1(G), L^p(G)_0) = 0$ by [**8**, Proposition 2.1.3]. Let $T : L^p(G)_0 \to L^1(G)$ be an $L^1(G)$-bimodule homomorphism such that $T(f) * g = 0$ for all $f, g \in L^p(G)_0$. Then $T = 0$ because we can use a bounded approximate identity instead of g. Since $L^1(G)$ has a bounded left approximate identity for $L^p(G)_0$, it is easy to see that $C_{L^1(G)}(L^p(G)_0, L^p(G)_0) \cong \mathrm{Z}(L^1(G))$. Therefore by Theorem 2.5 we have

$$\begin{aligned} H^1(L^1(G) \oplus L^p(G)_0, L^1(G) \oplus L^p(G)_0) \\ &\cong \frac{\mathrm{Hom}_{L^1(G)}(L^p(G)_0, L^p(G)_0)}{\mathrm{Z}(L^1(G))} \\ &\supseteq \frac{\mathrm{Z}(M(G))}{\mathrm{Z}(L^1(G))}. \end{aligned}$$

Note that for every $\mu \in Z(M(G))$ the map $T_\mu : L^p(G) \to L^p(G)$ defined by $T_\mu(f) = \mu * f$ (for all $f \in L^p(G)$) is in $\mathrm{Hom}_{L^1(G)}(L^p(G)_0, L^p(G)_0)$.

When $p = 1$, by Wendel's theorem, $\mathrm{Hom}_{L^1(G)}(L^1(G)_0, L^1(G)_0) \cong M(G)$ and so

$$H^1(L^1(G) \oplus L^1(G)_0, L^1(G) \oplus L^1(G)_0) \cong \frac{M(G)}{\mathrm{Z}(L^1(G))}.$$

Thus, if G is discrete, then $H^1(l^1(G) \oplus l^1(G)_0, l^1(G) \oplus l^1(G)_0) \cong l^1(G)/\mathrm{Z}(l^1(G))$.

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