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REGULARITY AND EXACT CONTROLLABILITY FOR THE TIMOSHENKO BEAM WITH PIEZOELECTRIC ACTUATOR

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ABSTRACT. In this paper, we study an initial, boundary value problem of the Timoshenko beam with attached piezoelectric actuators. We establish the regularity of the solution of the Timoshenko beam equation. The main results concern the dependence of the space of exactly controllable initial data on the location of the actuator. Our approach is based on the Hilbert uniqueness method combined with some results from the theory of diophantine approximation.

1. Introduction. In recent years, there has been much interest in the problems of the stability or the controllability for an elastic beam (see [**9**–**12**] and references therein), but little attention has been paid to the case of a Timoshenko beam with piezoelectric actuator. Tucsnak studied the regularity and exact controllability of an Euler-Bernoulli beam with a piezoelectric actuator in [**9**]. Zhang investigates boundary feedback stabilization of the undamped Timoshenko beam with both ends free in [**10**]. In this paper, we study the regularity and exact controllability of a Timoshenko beam with piezoelectric actuators. More precisely, we consider the initial and boundary value problem of the piezoelectric actuators which are attached to the simply supported Timoshenko beam:

$$(1.1)\quad w_{tt}(x,t) - k_1 w_{xx}(x,t) + k_1 \varphi_x(x,t) = u_1(t)\frac{d}{dx}[\delta_{\eta_1}(x) - \delta_{\xi_1}(x)], \quad 0 < x < \pi,\ t > 0,$$

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$$\varphi_{tt}(x,t) - k_2\varphi_{xx}(x,t) - k_1 w_x(x,t) + k_1\varphi(x,t) = u_2(t)\frac{d}{dx}[\delta_{\eta_2}(x) - \delta_{\xi_2}(x)], \quad 0 < x < \pi,\ t > 0,$$

$$w(0,t) = w(\pi,t) = \varphi_x(0,t) = \varphi_x(\pi,t) = 0, \quad t > 0,$$

$$w(x,0) = w_0(x), w_t(x,0) = w_1(x), \varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x), \quad 0 < x < \pi.$$

Here, a beam of length π moves in the xt-plane, $w(x,t)$ is the deflection of the beam from its equilibrium, and $\varphi(x,t)$ is the total rotatory angle of the beam at x. Two wave speeds $k_j > 0$ are not equal, $\xi_j \in (0,\pi), \eta_j \in (0,\pi)$, $j = 1,2$, stand for the ends of the actuators, and δ_y is the Dirac mass at the point y. The boundary condition means that both ends of the beam are simply supported. The controls are given by the functions $u_1, u_2 : [0,T] \to \mathbf{R}$ representing the time variation of voltage applied to the actuators.

In this paper we are interested in the following exact controllability problem: Under what conditions on the actuators is the system (1.1) exactly controllable for the given initial data? We will adapt the Hilbert uniqueness method developed in [**7**] to the exact controllability of system (1.1). The main difficulty in this approach is establishing the observability inequality. We deal with this difficulty by using Fourier series, asymptotic expansions and Diophantine approximation theory.

First, we give the following exact controllability definition.

Definition 1.1. We say that the initial data $w_0, w_1, \varphi_0, \varphi_1$ are "exactly L^2-controllable in (ξ_1,η_1), (ξ_2,η_2) at time T" if $u_1, u_2 \in L^2(0,T)$ exist such that the solution (w,φ) of system (1.1) satisfies the condition

$$w(x,T) = w_t(x,T) = \varphi(x,T) = \varphi_t(x,T) = 0, \quad 0 < x < \pi. \tag{1.2}$$

The plan of this paper is as follows. In Section 2 we state our main results. In Section 3 we show the existence and regularity results for the system. The proof of the main result is given in Section 4.

2. Preliminaries and main results. Before stating preliminaries and the main results of this paper, the relevant Sobolev space notations and other definitions are briefly surveyed (see [**1**]).

To study the well-posedness and controllability of the system in (1.1), we need the following crucial proposition.

Let us consider the homogenous initial and boundary value problem:

(2.1)
$$\begin{gathered} v_{tt}(x,t) - k_1 v_{xx}(x,t) + k_1 \phi_x(x,t) = 0, \quad 0 < x < \pi,\ t \in (0,T), \\ \phi_{tt}(x,t) - k_2\phi_{xx}(x,t) - k_1 v_x(x,t) + k_1\phi(x,t) = 0,\ 0 < x < \pi,\ t \in (0,T), \\ v(0,t) = v(\pi,t) = \phi_x(0,t) = \phi_x(\pi,t) = 0, \quad t \in (0,T), \\ v(x,0) = v_0(x), v_t(x,0) = v_1(x), \phi(x,0) = \phi_0(x), \\ \phi_t(x,0) = \phi_1(x), \quad 0 < x < \pi. \end{gathered}$$

It is well known that the initial and boundary value problem (2.1) is well posed in $H^\alpha(0,\pi) \times H^{\alpha-1}(0,\pi) \times H^\alpha(0,\pi) \times H^{\alpha-1}(0,\pi)$ for all $\alpha \geq 0$; moreover, as a consequence of the Hilbert uniqueness method (HUM), introduced in [**7**], the following result holds.

Proposition 2.1. *For all $\alpha \geq 0$, problem* (2.1) *of all initial data in $H^\alpha(0,\pi) \times H^{\alpha-1}(0,\pi) \times H^\alpha(0,\pi) \times H^{\alpha-1}(0,\pi)$ is "exactly L^2-controllable in (ξ_1,η_1), (ξ_2,η_2) at time T" if and only if there exists a positive constant $C > 0$ such that*

(2.2)
$$\int_0^T \{[v_x(\xi_1,t) - v_x(\eta_1,t)]^2 + [\phi_x(\xi_2,t) - \phi_x(\eta_2,t)]^2\}\,dt$$
$$\geq C[\|v_0\|^2_{H^{-\alpha}(0,\pi)} + \|v_1\|^2_{H^{-\alpha-1}(0,\pi)} + \|\phi_0\|^2_{H^{-\alpha}(0,\pi)} + \|\phi_1\|^2_{H^{-\alpha-1}(0,\pi)}],$$

for all $(v_0,v_1,\phi_0,\phi_1) \in H^1(0,\pi) \times L^2(0,\pi) \times H^1(0,\pi) \times L^2(0,\pi)$.

We shall also need some results from the theory of Diophantine approximation.

Denote by $\mathbf{Q}$ the set of all rational numbers. Let us also denote by $\mathbf{S}$ the set of all numbers $\rho \in (0,\pi)$ such that $\rho/\pi \notin \mathbf{Q}$ and if $[0,a_1,a_2,\dots,a_n,\dots]$ is the expansion of ρ/π as a continued fraction, then (a_n) is bounded. Let us notice that $\mathbf{S}$ is obviously uncountable and, by a classical result on Diophantine approximation (see [**3**, page

120]), its Lebesgue measure is equal to zero. In particular, by the Euler-Lagrange theorem (see [**5**, page 57]) $\mathbf{S}$ contains all $\rho \in (0,\pi)$ such that ρ/π is an irrational quadratic number. According to a classical result (see [**9**]), if $\rho \in \mathbf{S}$, then there exists a constant C_ρ such that

$$|\sin(n\rho)| \geq \frac{C_\rho}{n} \text{ for all } n \geq 1. \tag{2.3}$$

Let us denote by $\mathbf{B}$ the set of all numbers $\xi \in (0,\pi)$, where ξ is an irrational number with coprime factorization

$$\frac{\xi}{\pi} = \frac{p}{q}, \text{ where } q \text{ is odd.}$$

Then there exists a constant C_ξ such that

$$|\cos(n\xi)| \geq C_\xi \text{ for all } n \geq 1. \tag{2.4}$$

We make use of the result of Proposition 2.4 in [**9**] given as follows.

For any $\varepsilon > 0$, there exists a set $\mathbf{B}_\varepsilon \subset [(0,\pi)\backslash\pi\mathbf{Q}]$, the Lebesque measure of $\mathbf{B}_\varepsilon$ being equal to π, such that for all $\rho \in \mathbf{B}_\varepsilon$ we have

$$|\sin(n\rho)| \geq \frac{C_\rho}{n^{1+\varepsilon}} \text{ for all } n \geq 1. \tag{2.5}$$

Now we can state our main results as follows.

Theorem 2.1. *Suppose that $(\xi_1+\eta_1)/2$, $(\xi_1-\eta_1)/2$ and $(\xi_2-\eta_2)/2$ belong to $\mathbf{S}$, and that $(\xi_2+\eta_2)/2$ belongs to $\mathbf{B}$. Then all initial data in $H^1(0,\pi)\times L^2(0,\pi)\times H^1(0,\pi)\times L^2(0,\pi)$ are exactly L^2-controllable in (ξ_1,η_1), (ξ_2,η_2) at time T, for any $T>0$.*

Theorem 2.2. *For any $\varepsilon > 0$, suppose that $(\xi_1+\eta_1)/2$, $(\xi_1-\eta_1)/2$ and $(\xi_2-\eta_2)/2$ belong to $\mathbf{B}_\varepsilon$, and that $(\xi_2+\eta_2)/2$ belongs to $\mathbf{B}$. Then all initial data in $H^{1+\varepsilon}(0,\pi)\times H^{\varepsilon}(0,\pi)\times H^{1+\varepsilon}(0,\pi)\times H^{\varepsilon}(0,\pi)$ are exactly L^2-controllable in (ξ_1,η_1), (ξ_2,η_2) at time T, for any $T>0$.*

3. Existence and regularity of solutions. In this section, we give existence and regularity results for the system in (1.1).

Theorem 3.1. *Suppose that* $w_0, \varphi_0 \in H^1(0,\pi)$ *and* $w_1, \varphi_1 \in L^2(0,\pi)$. *Then the initial and boundary value problem* (1.1) *admits a unique solution having regularity*

$$(w,\varphi) \in C([0,T]; L^2(0,\pi)\times L^2(0,\pi)) \cap C^1([0,T]; H^{-1}(0,\pi)\times H^{-1}(0,\pi)). \tag{3.1}$$

Consider $\tau \in [0,T]$. *We introduce the homogenous initial and boundary value problem:*

$$\begin{gathered} v_{tt}(x,t) - k_1 v_{xx}(x,t) + k_1\phi_x(x,t) = 0, \quad 0 < x < \pi, \quad t \in (0,\tau), \\ \phi_{tt}(x,t) - k_2\phi_{xx}(x,t) - k_1 v_x(x,t) + k_1\phi(x,t) = 0, \ 0 < x < \pi, \ t \in (0,\tau), \\ v(0,t) = v(\pi,t) = \phi_x(0,t) = \phi_x(\pi,t) = 0, \quad t \in (0,\tau), \\ v(x,\tau) = 0, v_t(x,\tau) = f(x), \phi(x,\tau) = 0, \phi_t(x,\tau) = g(x), \quad 0 < x < \pi. \end{gathered} \tag{3.2}$$

To prove Theorem 3.1, we need the following lemma.

Lemma 3.2. *For any* $(f,g) \in L^2(0,\pi) \times L^2(0,\pi)$, *the initial and boundary value problem* (3.2) *admits a unique solution having the regularity*

$$(v,\phi) \in C([0,T]; H^1(0,\pi) \times H^1(0,\pi)) \cap C^1([0,T]; L^2(0,\pi) \times L^2(0,\pi)). \tag{3.3}$$

Moreover, for any $\rho \in (0,\pi)$ *the function* $v_x(\rho,\cdot)$ *and* $\phi_x(\rho,\cdot)$ *are in* $L^2(0,T)$ *and there exists a constant* $C > 0$ *such that*

$$\|v_x(\rho,\cdot)\|^2_{L^2(0,T)} + \|\phi_x(\rho,\cdot)\|^2_{L^2(0,T)} \le C[\|f\|^2_{L^2(0,\pi)} + \|g\|^2_{L^2(0,\pi)}]. \tag{3.4}$$

Proof. For simplicity, we will denote $\partial v/\partial x$ by v'. Let $H = H^1(0,\pi) \times L^2(0,\pi) \times H^1(0,\pi) \times L^2(0,\pi)$ with the norm

$$\|(v,y,\phi,\psi)\|_H = \left(\int_0^\pi [k_1|\phi - v'|^2 + |y|^2 + k_2|\phi'|^2 + |\psi|^2]\, dx \right)^{1/2}.$$

We define a linear operator A_0 as follows:

$$\begin{aligned} D(A_0) = \{(v,y,\phi,\psi) | v,\phi \in H^2(0,\pi), y,\psi \in H^1(0,\pi), v(0) = v(\pi) \\ = \phi'(0) = \phi'(\pi) = y(0) = y(\pi) = 0\}, \end{aligned}$$

$$A_0 : D(A_0) \longrightarrow H,$$
$$A_0(v, y, \phi, \psi) = (y, [k_1 v'' - k_1 \phi'], \psi, [k_2 \phi'' + k_1 v' - k_1 \phi]),$$
$$(v, y, \phi, \psi) \in D(A_0).$$

Then problem (3.2) can be formulated as the following Cauchy problem on H:

$$\frac{dZ}{dt} = A_0 Z, \quad Z(0) = Z_0,$$

where $Z = (v, y, \phi, \psi)$ and $Z_0 = (v_0, y_0, \phi_0, \psi_0)$.

One can easily check that A_0 is skew-adjoint. So, by Stone's theorem, it generates a semi-group of isometries in H. This implies that problem (3.2) admits a unique solution (v, ϕ) satisfying (3.3).

To prove (3.4), we need to compute the eigenvalues of A_0. Now we solve the characteristic equation

$$A_0(v, y, \phi, \psi) = \lambda(v, y, \phi, \psi), \quad (v, y, \phi, \psi) \in D(A_0).$$

Eliminating the unknown y, ϕ, ψ, we obtain

$$\begin{cases} k_1 k_2 v_{xxxx} - (k_1 + k_2)\lambda^2 v_{xx} + \lambda^2 (k_1 + \lambda^2) v = 0, \\ v(0) = v(\pi) = v_{xx}(0) = v_{xx}(\pi) = 0. \end{cases}$$

Eliminating the unknown v, y, ψ, we obtain

$$\begin{cases} k_1 k_2 \phi_{xxxx} - (k_1 + k_2)\lambda^2 \phi_{xx} + \lambda^2 (k_1 + \lambda^2) \phi = 0, \\ \phi_x(0) = \phi_x(\pi) = \phi_{xxx}(0) = \phi_{xxx}(\pi) = 0. \end{cases}$$

Eigenvalues of these two characteristic equations have asymptotic expansions:

i) $k_1 > k_2$; $\lambda_{n,1}^2 = -k_2 n^2 (1 + O(1/n^2))$, $\lambda_{n,2}^2 = -k_1 n^2 (1 + O(1/n^2))$;

ii) $k_1 < k_2$; $\lambda_{n,1}^2 = -k_1 n^2 (1 + O(1/n^2))$, $\lambda_{n,2}^2 = -k_2 n^2 (1 + O(1/n^2))$.

We further suppose that $k_1 > k_2$. In order to prove (3.4), we put

$$f(x) = \sum_{n=1}^{\infty} n a_n \sin(nx), \quad g(x) = \sum_{n=1}^{\infty} n b_n \cos(nx) \tag{3.5}$$

with $\sum_{n=1}^{\infty} n^2 (a_n^2 + b_n^2) < \infty$.

By density, it is enough to show that (3.4) holds for $f, g \in C_0^\infty(0,\pi)$. Obviously the solution of (3.2) is given by

$$\begin{aligned} v(x,t) &= \sum_{n=1}^{\infty} a_n \sin\left[\sqrt{k_2}n\left(1+o\left(\frac{1}{n}\right)\right)t\right]\sin(nx), \\ \phi(x,t) &= \sum_{n=1}^{\infty} b_n \sin[\sqrt{k_1}n\left(1+o\left(\frac{1}{n}\right)\right)t\Big]\cos(nx), \end{aligned} \tag{3.6}$$

which implies that

$$\begin{aligned} v_x(\rho,t) &= \sum_{n=1}^{\infty} na_n \sin\left[\sqrt{k_2}n\left(1+o\left(\frac{1}{n}\right)\right)t\right]\cos(n\rho), \\ \phi_x(\rho,t) &= -\sum_{n=1}^{\infty} nb_n \sin\left[\sqrt{k_1}n\left(1+o\left(\frac{1}{n}\right)\right)t\right]\sin(n\rho). \end{aligned} \tag{3.7}$$

If we consider the righthand side of (3.7) as a Fourier series in t (see [**4**, Theorem 4.1] for details), we obtain the existence of a constant C dependent only on T such that

$$\|v_x(\rho,\cdot)\|^2_{L^2(0,T)}+\|\phi_x(\rho,\cdot)\|^2_{L^2(0,T)} \le C\left[\sum_{n=1}^{\infty} n^2(a_n^2+b_n^2)\right] < +\infty, \tag{3.8}$$

which is exactly (3.4). The case $k_1 < k_2$ can be proved by simply adapting the proof of the case $k_1 > k_2$, so we skip the details. The proof is completed. □

Proof of Theorem 3.1. Due to the linearity of (1.1) and the well-known properties of the Timoshenko beam equation (see [**10, 11, 12**]), it is enough to consider the case $w_0 = w_1 = \varphi_0 = \varphi_1 = 0$. Suppose again $f, g \in C_0^\infty(0,\pi)$, and let (v,ϕ) be the solution of (3.2). Now, we use the multipliers v and ϕ to the first equation and the second equation of (1.1). Integrating by parts, we obtain that

$$\begin{aligned} \int_0^\pi w(x,\tau)f(x)\,dx = -k_1 &\int_0^\tau\int_0^\pi [w(x,t)\phi_x(x,t)-\varphi_x(x,t)v(x,t)]\,dx\,dt \\ &- \int_0^\tau u_1(t)[v_x(\xi_1,t)-v_x(\eta_1,t)]\,dt, \end{aligned} \tag{3.9}$$

and

(3.10)
$$\int_0^\pi \varphi(x,\tau)g(x)\,dx = k_1\int_0^\tau\int_0^\pi [w(x,t)\phi_x(x,t)-\varphi_x(x,t)v(x,t)]\,dx\,dt - \int_0^\tau u_2(t)[\phi_x(\xi_2,t)-\phi_x(\eta_2,t)]\,dt.$$

Taking the sum of (3.9) and (3.10), we can obtain

$$\int_0^\pi w(x,\tau)f(x)\,dx + \int_0^\pi \varphi(x,\tau)g(x)\,dx = -\int_0^\tau u_1(t)[v_x(\xi_1,t)-v_x(\eta_1,t)]\,dt - \int_0^\tau u_2(t)[\phi_x(\xi_2,t)-\phi_x(\eta_2,t)]\,dt. \tag{3.11}$$

Lemma 3.2 implies that

$$\left|\int_0^\tau u_1(t)[v_x(\xi_1,t)-v_x(\eta_1,t)]\,dt + \int_0^\tau u_2(t)[\phi_x(\xi_2,t)-\phi_x(\eta_2,t)]\,dt\right|^2 \le C[\|u_1\|^2_{L^2(0,T)}\cdot\|f\|^2_{L^2(0,\pi)} + \|u_2\|^2_{L^2(0,T)}\cdot\|g\|^2_{L^2(0,\pi)}].$$

So, by (3.11), we obtain that $(w(\cdot,\tau),\varphi(\cdot,\tau))\in L^2(0,\pi)\times L^2(0,\pi)$, for all $\tau\in[0,T]$. By replacing τ by $\tau+h$ in (3.11) we easily get that

$$(w,\varphi)\in C([0,T];L^2(0,\pi)\times L^2(0,\pi)), \tag{3.12}$$

which implies that

$$(w_{xx},\varphi_{xx})\in C([0,T];H^{-2}(0,\pi)\times H^{-2}(0,\pi)). \tag{3.13}$$

As (w,φ) satisfies (1.1), from (3.13) we obtain that

$$(w_{tt},\varphi_{tt})\in L^2([0,T];H^{-2}(0,\pi)\times H^{-2}(0,\pi)). \tag{3.14}$$

From (3.12) and (3.14), by applying the intermediate derivative theorem (see [**8**]) it follows that

$$(w_t,\varphi_t)\in L^2([0,T];H^{-1}(0,\pi)\times H^{-1}(0,\pi)). \tag{3.15}$$

Combining (3.12) with (3.15) and the general lifting result (see [**6**]), we can obtain

$$(w,\varphi)\in C([0,T];L^2(0,\pi)\times L^2(0,\pi))\cap C^1([0,T];H^{-1}(0,\pi)\times H^{-1}(0,\pi)).$$

This completes the proof of Theorem 3.1. □

4. Proof of the main results. In this section we prove our main results of this paper.

Let us put

$$\begin{aligned} v_0(x) &= \sum_{n=1}^{\infty} a_n \sin(nx), \quad v_1(x) = \sum_{n=1}^{\infty} nc_n \sin(nx),\\ \phi_0(x) &= \sum_{n=1}^{\infty} b_n \cos(nx), \quad \phi_1(x) = \sum_{n=1}^{\infty} nd_n \cos(nx), \end{aligned} \tag{4.1}$$

with $\sum_{n=1}^{\infty} n^2(a_n^2+b_n^2+c_n^2+d_n^2)<+\infty$.

A simple calculation shows that the solution of (2.1) is given by

$$\begin{aligned} v(x,t) &= \sum_{n=1}^{\infty}\bigg[a_n\cos\bigg(\sqrt{k_2}n\bigg(1+o\bigg(\frac{1}{n}\bigg)\bigg)t\bigg)\\ &\qquad + c_n\sin\bigg(\sqrt{k_2}n\bigg(1+o\bigg(\frac{1}{n}\bigg)\bigg)t\bigg)\bigg]\sin(nx),\\ \phi(x,t) &= \sum_{n=1}^{\infty}\bigg[b_n\cos\bigg(\sqrt{k_1}n\bigg(1+o\bigg(\frac{1}{n}\bigg)\bigg)t\bigg)\\ &\qquad + d_n\sin\bigg(\sqrt{k_1}n\bigg(1+o\bigg(\frac{1}{n}\bigg)\bigg)t\bigg)\bigg]\cos(nx), \end{aligned} \tag{4.2}$$

which implies that

$$\begin{aligned}(4.3)\quad &\int_0^T \{[v_x(\xi_1,t)-v_x(\eta_1,t)]^2+[\phi_x(\xi_2,t)-\phi_x(\eta_2,t)]^2\}\,dt\\
&= 4\int_0^T \bigg\{\sum_{n=1}^{\infty} n^2\bigg[a_n\cos\bigg(\sqrt{k_2}n\bigg(1+o\bigg(\frac{1}{n}\bigg)\bigg)t\bigg)\\
&\quad + c_n\sin\bigg(\sqrt{k_2}n\bigg(1+o\bigg(\frac{1}{n}\bigg)\bigg)t\bigg)\bigg]^2\\
&\quad\times \sin^2\bigg[\frac{n(\xi_1+\eta_1)}{2}\bigg]\sin^2\bigg[\frac{n(\xi_1-\eta_1)}{2}\bigg]\bigg\}\,dt\\
&\quad + 4\int_0^T \bigg\{\sum_{n=1}^{\infty} n^2\bigg[b_n\cos\bigg(\sqrt{k_1}n\bigg(1+o\bigg(\frac{1}{n}\bigg)\bigg)t\bigg)\\
&\quad + d_n\sin\bigg(\sqrt{k_1}n\bigg(1+o\bigg(\frac{1}{n}\bigg)\bigg)t\bigg)\bigg]^2\\
&\quad\times \cos^2\bigg[\frac{n(\xi_2+\eta_2)}{2}\bigg]\sin^2\bigg[\frac{n(\xi_2-\eta_2)}{2}\bigg]\bigg\}\,dt.\end{aligned}$$

Proof of Theorem 2.1. By Proposition 2.1, the conclusion of Theorem 2.1 is equivalent to the existence of a constant $C>0$ such that

$$\begin{aligned}(4.4)\quad &\int_0^T \{[v_x(\xi_1,t)-v_x(\eta_1,t)]^2+[\phi_x(\xi_2,t)-\phi_x(\eta_2,t)]^2\}\,dt\\
&\geq C[\|v_0\|^2_{H^{-1}(0,\pi)}+\|v_1\|^2_{H^{-2}(0,\pi)}+\|\phi_0\|^2_{H^{-1}(0,\pi)}+\|\phi_1\|^2_{H^{-2}(0,\pi)}],\end{aligned}$$

for all $(v_0,v_1,\phi_0,\phi_1)\in H^1(0,\pi)\times L^2(0,\pi)\times H^1(0,\pi)\times L^2(0,\pi)$.

By applying the Ball-Slemrod generalization of Imgham's inequality (see [**2**]), from (4.3) we obtain that a constant $C>0$ exists such that

$$\begin{aligned}(4.5)\quad &\int_0^T \{[v_x(\xi_1,t)-v_x(\eta_1,t)]^2+[\phi_x(\xi_2,t)-\phi_x(\eta_2,t)]^2\}\,dt\\
&\geq C\bigg\{\sum_{n=1}^{\infty} n^2(a_n^2+c_n^2)\sin^2\bigg[\frac{n(\xi_1+\eta_1)}{2}\bigg]\cdot\sin^2\bigg[\frac{n(\xi_1-\eta_1)}{2}\bigg]\\
&\quad+\sum_{n=1}^{\infty} n^2(b_n^2+d_n^2)\cos^2\bigg[\frac{n(\xi_2+\eta_2)}{2}\bigg]\cdot\sin^2\bigg[\frac{n(\xi_2-\eta_2)}{2}\bigg]\bigg\}.\end{aligned}$$

As $(\xi_1+\eta_1)/2$, $(\xi_1-\eta_1)/2$ and $(\xi_2-\eta_2)/2$ are in $\mathbf{S}$, $(\xi_2+\eta_2)/2$ is in $\mathbf{B}$, from (2.3) and (2.4) we can obtain that there exists a constant $C>0$ such that for n large enough we have

$$\text{(4.6)}\quad \int_0^T \{[v_x(\xi_1,t)-v_x(\eta_1,t)]^2+[\phi_x(\xi_2,t)-\phi_x(\eta_2,t)]^2\}\,dt \geq C\left[\sum_{n=1}^{\infty} n^{-2}(a_n^2+c_n^2+b_n^2+d_n^2)\right],$$

which is exactly (2.2) for $\alpha=1$. The proof has been completed. □

Proof of Theorem 2.2. As $(\xi_1+\eta_1)/2$, $(\xi_1-\eta_1)/2$ and $(\xi_2-\eta_2)/2$ are in $\mathbf{B}_\varepsilon$, $(\xi_2+\eta_2)/2$ is in $\mathbf{B}$, from (2.4) and (2.5), it follows that

$$\text{(4.7)}\quad \begin{aligned} &\left|\sin\left[\frac{n(\xi_1+\eta_1)}{2}\right]\right| \geq \frac{C}{n^{1+\varepsilon}}, \quad \left|\sin\left[\frac{n(\xi_1-\eta_1)}{2}\right]\right| \geq \frac{C}{n^{1+\varepsilon}},\\ &\left|\sin\left[\frac{n(\xi_2-\eta_2)}{2}\right]\right| \geq \frac{C}{n^{1+\varepsilon}}, \quad \left|\cos\left[\frac{n(\xi_2+\eta_2)}{2}\right]\right| \geq C, \quad \forall n\geq 1.\end{aligned}$$

Consider again the solution (v,ϕ) of (2.1) with the initial data given by (4.1). By applying (4.3) and (4.7) we obtain

$$\text{(4.8)}\quad \begin{aligned}\int_0^T \{[v_x(\xi_1,t)-v_x(\eta_1,t)]^2+[\phi_x(\xi_2,t)-\phi_x(\eta_2,t)]^2\}\,dt &\geq C[\|v_0\|^2_{H^{-1-\varepsilon}(0,\pi)}+\|v_1\|^2_{H^{-2-\varepsilon}(0,\pi)}\\ &\quad+\|\phi_0\|^2_{H^{-1-\varepsilon}(0,\pi)}+\|\phi_1\|^2_{H^{-2-\varepsilon}(0,\pi)}].\end{aligned}$$

For $\alpha=1+\varepsilon$, $\varepsilon>0$, by Proposition 2.1 it follows that all initial data in $H^{1+\varepsilon}(0,\pi)\times H^{\varepsilon}(0,\pi)\times H^{1+\varepsilon}(0,\pi)\times H^{\varepsilon}(0,\pi)$ are exactly L^2-controllable in (ξ_1,η_1), (ξ_2,η_2) at time T, for any $T>0$. This completes the proof of Theorem 2.2. □

One can observe that Theorems 2.1 and 2.2 give no information on the controllability of initial data in $H^\alpha(0,\pi)\times H^{\alpha-1}(0,\pi)\times H^\alpha(0,\pi)\times H^{\alpha-1}(0,\pi)$, with $\alpha<1$. Because the characterization of the space of L^2-controllable initial data depends strongly on the location of two piezoelectric actuators that are attached to a Timoshenko beam. A natural question concerning the exact controllability of initial data in $H^\alpha(0,\pi) \times H^{\alpha-1}(0,\pi) \times H^\alpha(0,\pi) \times H^{\alpha-1}(0,\pi)$, with $0 \le \alpha < 1$, is about the appropriate location of the actuators that are attached to a Timoshenko beam. What conditions on the positions of the actuators must be imposed in order to achieve controllability? This problem appears to be open.

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