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PERIODICITY OF DELAYED REACTION-DIFFUSION HIGH-ORDER COHEN-GROSSBERG NEURAL NETWORKS WITH DIRICHLET BOUNDARY CONDITIONS

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ABSTRACT. In this paper, we study delayed reaction-diffusion high-order Cohen-Grossberg neural networks with Dirichlet boundary conditions. By using some inequality techniques and constructing the Lyapunov functional method, some sufficient conditions in which the diffusion coefficients will affect the periodicity and the global exponential stability of solutions are given to ensure the existence and convergence of the periodic oscillatory solution. Finally, an example is given to verify the theoretical analysis.

1. Introduction. Cohen-Grossberg neural networks (CGNNs) which include the traditional neural networks as special cases were first introduced in 1983 by Cohen and Grossberg in [**6**]. CGNNs have been extensively investigated and successfully applied to parallel computation, associative memory and optimization problems, etc. [**1–5, 8–10, 12, 15, 20–26**]. Because these applications heavily depend on the dynamical behaviors of the networks, analysis of the dynamical behaviors is the necessary step to design of neural networks.

However, both in biological and artificial neural networks, diffusion effect cannot be avoided when electrons are moving in asymmetric electromagnetic field; thus, we must consider that the activations vary in space as well as in time. References [**11, 14, 17–18, 19, 23, 26, 27**] have considered the stability of neural networks with diffusion terms, which are expressed by partial differential equations.

Keywords and phrases. Cohen-Grossberg neural networks, reaction-diffusion terms, Dirichlet boundary condition, Lyapunov functional.

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Studies on the dynamical behaviors of neural networks not only involve a discussion of stability properties but also involve many other dynamic behaviors such as periodic oscillatory behavior, bifurcation and chaos [**22, 25**]. In many applications, the properties of periodic oscillatory solutions to the high-order CGNNs are of great interest; for example, the human brain is in periodic oscillatory chaos, hence it is of prime importance to study periodic oscillatory and chaos phenomena of the high-order neural networks.

Moreover, because of the finite processing speed of information, it is sometimes necessary to take account of time delays in the modeling of the biological or artificial neural networks. Time delays may lead to bifurcation, oscillation, divergence or instability which may be harmful to a system; thus, the study of neural dynamics with consideration of the delayed problem becomes extremely important to manufacture high quality neural networks. References [**2**–**5, 8**–**11, 14, 17**–**22**–**27**] have studied the stability of delayed neural networks.

To the best of our knowledge, few authors have considered the existence and the global exponential stability of periodic oscillatory solution of delayed reaction-diffusion high-order CGNNs with Dirichlet boundary conditions. But it is important in theories and applications, and also is a very challenging problem. In this paper, evoking by the methods used in [**26**] and utilizing the Sobolev inequality [**7**], we investigate the existence and global exponential stability of periodic solutions to delayed reaction diffusion high-order CGNNs with Dirichlet boundary conditions and show that the diffusion terms will improve the existence and global exponential stability of periodic solution.

In this paper, we will consider the following delayed high-order Cohen-Grossberg neural networks with reaction-diffusion terms:
(1)

$$\begin{cases} \frac{\partial u_i(x,t)}{\partial t} = \sum_{k=1}^m \frac{\partial}{\partial x_k}\left(D_{ik}\frac{\partial u_i(x,t)}{\partial x_k}\right) - a_i(u_i(x,t)) \\ [b_i(u_i(x,t)) - \sum_{j=1}^n c_{ij} f_j(u_j(x,t)) - \sum_{j=1}^n \omega_{ij} f_j(u_j(x,t-t_{ij}(t))) \\ - \sum_{j,k=1}^n d_{ijk} f_j(u_j(x,t)) f_k(u_k(x,t)) \\ - \sum_{j,k=1}^n e_{ijk} f_j(u_j(x,t-t_{ij}(t))) f_k(u_k(x,t-t_{ik}(t))) + I_i(t)], \\ u_i(x,t) = \varphi_i(x,t), \quad -\tau \le t \le 0,\ x \in \Omega, \\ u_i(x,t) = 0, \quad x \in \partial\Omega, \end{cases}$$

where $i = 1, \dots, n$. The parameter n is the number of neurons in the networks. $u(x,t) = (u_1(x,t),\dots,u_n(x,t))^T$, $u_i(x,t)$ denotes the state of the ith neural unit at time t and in space $x \in \Omega$. Ω is a bounded open domain in $\mathbf{R}^m$ with smooth boundary $\partial\Omega$ and $mes\Omega > 0$ denotes the measure of Ω. $D_{ik} \geq 0$ corresponds to the transmission diffusion coefficient along the ith unit. $a_i(\cdot)$ and $b_i(\cdot)$ represent an amplification function and an appropriately behaved function, respectively. c_{ij} represents the strength of the jth neuron connecting on the ith neuron at time t and in space x, and ω_{ij} represents the strength of the jth neuron connecting on the ith neuron at time $t - t_{ij}(t)$ and in space x. d_{ijk} and e_{ijk} represent the second-order strength of the neuron interconnections within the network without delays and with delays, respectively. t_{ij} corresponds to the transmission delays along the axon of the jth neuron form the ith neuron and satisfies $0 \leq t_{ij}(t) \leq \tau$ and $\dot{t}_{ij}(t) \leq \rho < 1$, where $\tau > 0$ is a given constant. f_i shows how the ith neuron reacts to the input. $I(t) = (I_1(t),\dots,I_n(t))^T$, I_i is the input from outside the system.

Remark 1. When $D_{ik} \equiv 0$, system (1) becomes the system which expressed by ordinary differential equations. Furthermore, while $d_{ijk} \equiv 0$ and $e_{ijk} \equiv 0$, system (1) becomes the system which is the first-order CGNNs.

This paper is organized as follows. In Section 2, some preliminaries and the main result are given. In Section 3, by employing some inequality techniques and constructing suitable Lyapunov functional, some sufficient conditions are obtained to ensure the existence and the global exponential stability of the periodic oscillatory solution. In Section 4, an example is given to verify the theoretical analysis. Section 5 is the conclusion of the paper.

2. Preliminaries. Let $X = C(\mathbf{R}^m \times [-\tau,0], \mathbf{R}^n)$ be the Banach space of continuous functions which map $\mathbf{R}^m \times [-\tau,0]$ into $\mathbf{R}^n$ with the topology of uniform convergence. $L^2(\Omega)$ is the space of real functions on Ω which are L^2 for the Lebesgue measure. It is a Banach space for the norm:

$$\|u(t)\|_2^2 = \sum_{i=1}^{n} \|u_i(t)\|_2^2 = \sum_{i=1}^{n} \int_\Omega |u_i(x,t)|^2 dx,$$

where $u(t) = u(\cdot,t) = (u_1(\cdot,t),\dots,u_n(\cdot,t))^T$.

For any $\varphi(x,t) \in C(\mathbf{R}^m \times [-\tau,0], \mathbf{R}^n)$, we define $\|\varphi\|_2^2 = \sum_{i=1}^n \|\varphi_i\|_2^2$, where $\varphi(x,t) = (\varphi_1(x,t), \dots, \varphi_n(x,t))^T$,

$$\|\varphi_i\|_2^2 = \int_\Omega |\varphi_i(x)|_\tau^2 \, dx, \qquad |\varphi_i(x)|_\tau = \sup_{-\tau \le s \le 0} |\varphi_i(x,s)|.$$

Definition 1. The solution $\widetilde{u} = \widetilde{u}(x,t;\phi) \in \mathbf{R}^n$ of system (1) with the initial value ϕ is said to be global exponential stability, if there exist positive constants $\gamma > 0$ and $\varepsilon > 0$ such that

$$\|u(t) - \widetilde{u}(t)\| \le \gamma \|\varphi - \phi\| e^{-\varepsilon t}, \quad \text{for all } t \ge 0, \tag{2}$$

where $u = u(x,t;\varphi) \in \mathbf{R}^n$ is any solution to system (1) with the initial value φ.

Definition 2. Let $V : \mathbf{R} \to \mathbf{R}$ be a continuous function, the upper right Dini-derivative D^+V is defined as follows:

$$D^+V = \limsup_{h \to 0^+} \frac{V(t+h) - V(t)}{h}.$$

Throughout the paper, we always assume that system (1) has a smooth solution $u(t,x)$ with the norm $\|u(t)\|_2 = \sqrt{\sum_{i=1}^n \int_\Omega |u_i(x,t)|^2 dx}$, for all $t \in [0, +\infty)$.

In order to deal with the Dirichlet boundary conditions, we introduce the following lemma [**7**]:

Lemma 1. *Let Ω be a bounded open domain in $\mathbf{R}^m$ with smooth boundary $\partial\Omega$. If $u = u(x)$ defined on Ω is a smooth function with $u|_{\partial\Omega} = 0$, then the following inequality holds*:

$$\int_\Omega u^2(x) \, dx \le \left(\frac{|\Omega|}{\omega_m}\right)^{1/m} \int_\Omega \left(\frac{\partial u}{\partial x}\right)^2 dx,$$

where $|\Omega|$ denotes the volume of Ω. ω_m denotes the surface area of unit ball in $\mathbf{R}^m$.

In this paper, we always assume that:

$(H1)$ There exist constants Δ_{ik}, m_i, M_i and such that $0 < m_i \leq a_i(u_i) \leq M_i$, $i = 1, \dots, n$ and $D_{ik} \geq \Delta_{ik} \geq 0$, $(k = 1, \dots, m)$, $\Delta_i^{\min} = \min_k\{\Delta_{ik}\} \geq 0$. Moreover, $a_i(\cdot)$ is differentiable and there exist positive constants Γ_i such that $0 < a_i'(\cdot) \leq \Gamma_i$, $(i = 1, \dots, n)$.

$(H2)$ $I_i(t)$ is an $\omega-$periodic function satisfying $I_i(t+\omega) = I_i(t)$ with the boundary $B_i \geq 0$, $(i = 1, \dots, n)$. $b_i(\cdot) \geq 0$ is differentiable, $b_i(0) = 0$, $i = 1, \dots, n$. Moreover, $\alpha_i \triangleq \inf_{x\in\mathbf{R}}\{\dot{b}_i(x)\} > 0$, where $\dot{b}_i(\cdot)$ denotes the derivative of $b_i(\cdot)$, $i = 1, \dots, n$.

$(H3)$ There exist positive constants Λ_i, β_i such that

$$\|f_i\| = \sup_u |f_i(u)| \leq \Lambda_i, \quad i = 1, \dots, n. \tag{3}$$

$$|f_i(x_i) - f_i(y_i)| \leq \beta_i|x_i - y_i|, \quad \text{for all } x_i, y_i \in \mathbf{R}, \quad i = 1, \dots, n. \tag{4}$$

Remark 2. By $(H3)$, we know that the functions f_i, $(i = 1, \dots, n)$ also satisfy

$$|f_i(x_i)f_j(x_j) - f_i(y_i)f_j(y_j)| \leq \gamma_{ij}|x_i - y_i| + \gamma_{ji}|x_j - y_j|,$$
$$\forall\ x_i, x_j, y_i, y_j \in \mathbf{R}, \quad (i, j = 1, \dots, n),$$

where $\gamma_{ij} = \beta_i\Lambda_j$, and $\beta_i, \Lambda_i (i = 1, \dots, n)$ are defined in $(H3)$.

$(H4)$ There exist constants $\delta > 0$, $p_{ij}, q_{ij}, m_{ijk}, n_{ijk}, r_{ijk}, s_{ijk}, p_{ij}^*$, q_{ij}^*, $m_{ijk}^*, n_{ijk}^*, r_{ijk}^*, s_{ijk}^*$ such that for $i = 1, \dots, n$, the following holds

$$\begin{aligned} -2\alpha_i m_i - 2\frac{\Delta_i^{\min}}{\delta} &+ \sum_{j=1}^n \Gamma_i\big((|c_{ij}| + |\omega_{ij}|)\Lambda_j + B_i\big) \\ &+ \sum_{j=1}^n M_i\Big(|c_{ij}|^{2p_{ij}}\beta_j^{2q_{ij}} + |\omega_{ij}|^{2p_{ij}^*}\beta_j^{2q_{ij}^*}\Big) \\ &+ \sum_{j=1}^n M_j\left(|c_{ji}|^{2-2p_{ji}}\beta_i^{2-2q_{ji}} + \frac{|\omega_{ji}|^{2-2p_{ji}^*}\beta_i^{2-2q_{ji}^*}}{1-\rho}\right) \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{j,k=1}^{n} (M_i|d_{ijk}|^{2m_{ijk}} \gamma_{jk}^{2n_{ijk}} + M_j|d_{jik}|^{2-2m_{jik}} \gamma_{ik}^{2-2n_{jik}}) \\
&+ \sum_{j,k=1}^{n} (M_i|d_{ijk}|^{2r_{ijk}} \gamma_{kj}^{2s_{ijk}} + M_k|d_{kji}|^{2-2r_{kji}} \gamma_{ij}^{2-2s_{kji}}) \\
&+ \sum_{j,k=1}^{n} \left(M_i|e_{ijk}|^{2m^*_{ijk}} \gamma_{jk}^{2n^*_{ijk}} + \frac{M_j|e_{jik}|^{2-2m^*_{jik}} \gamma_{ik}^{2-2n^*_{jik}}}{1-\rho} \right) \\
&+ \sum_{j,k=1}^{n} \left(M_i|e_{ijk}|^{2r^*_{ijk}} \gamma_{kj}^{2s^*_{ijk}} + \frac{M_k|e_{kji}|^{2-2r^*_{kji}} \gamma_{ij}^{2-2s^*_{kji}}}{1-\rho} \right) \\
&< 0.
\end{aligned}
$$

Remark 3. $(H4)$ includes the constants $\Delta_i^{\min}$ defined according to the diffusion coefficients D_{ik}, which implies that the diffusion coefficients will affect the global exponential stability and the existence of the periodic oscillatory solution to system (1).

Remark 4. In $(H4)$, the positive constant δ depends on the volume of the domain Ω and the surface area of the unit ball in $\mathbf{R}^m$.

The main result of this paper is the following theorem:

Theorem 1. *If $(H1)$–$(H4)$ hold, then system (1) has only one $\omega-$periodic solution and all other solutions converge exponentially to it as $t \to +\infty$.*

Remark 5. Let $p_{ij} = q_{ij} = m_{ijk} = n_{ijk} = r_{ijk} = s_{ijk} = p^*_{ij} = q^*_{ij} = m^*_{ijk} = n^*_{ijk} = r^*_{ijk} = s^*_{ijk} = 1$ or $p_{ij} = q_{ij} = m_{ijk} = n_{ijk} = r_{ijk} = s_{ijk} = p^*_{ij} = q^*_{ij} = m^*_{ijk} = n^*_{ijk} = r^*_{ijk} = s^*_{ijk} = 1/2$, then $(H4)$ can be changed into the following:

$(H5)$

$$
\begin{aligned}
&-2\alpha_i m_i - 2\frac{\Delta_i^{\min}}{\delta} + \sum_{j=1}^{n} \Gamma_i\big((|c_{ij}| + |\omega_{ij}|)\Lambda_j + B_i\big) \\
&\qquad + \sum_{j=1}^{n} \left[M_i\Big(|c_{ij}|^2\beta_j^2 + |\omega_{ij}|^2\beta_j^2\Big) + \frac{2-\rho}{1-\rho} M_j \right]
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{j,k=1}^{n} (M_i |d_{ijk}|^2 \gamma_{jk}^2 + M_j) + \sum_{j,k=1}^{n} (M_i |d_{ijk}|^2 \mu_{jk}^2 + M_k) \\
&+ \sum_{j,k=1}^{n} \left(M_i |e_{ijk}|^2 \gamma_{jk}^2 + \frac{1}{1-\rho} M_j \right) \\
&+ \sum_{j,k=1}^{n} \left(M_i |e_{ijk}|^2 \mu_{jk}^2 + \frac{1}{1-\rho} M_k \right) \\
&< 0
\end{aligned}
$$

or

$(H6)$

$$
\begin{aligned}
-2\alpha_i m_i - 2\frac{\Delta_i^{\min}}{\delta} &+ \sum_{j=1}^{n} \Gamma_i \big((|c_{ij}| + |\omega_{ij}|) \Lambda_j + B_i \big) \\
&+ \sum_{j=1}^{n} \left[M_i \Big(|c_{ij}|\beta_j + |\omega_{ij}|\beta_j \Big) + M_j \left(|c_{ji}|\beta_i + \frac{|\omega_{ji}|\beta_i}{1-\rho} \right) \right] \\
&+ \sum_{j,k=1}^{n} (M_i |d_{ijk}| \gamma_{jk} + M_j |d_{jik}| \gamma_{ik}) \\
&+ \sum_{j,k=1}^{n} (M_i |d_{ijk}| \gamma_{kj} + M_k |d_{kji}| \gamma_{ij}) \\
&+ \sum_{j,k=1}^{n} \left(M_i |e_{ijk}| \gamma_{jk} + \frac{M_j |e_{jik}| \gamma_{ik}}{1-\rho} \right) \\
&+ \sum_{j,k=1}^{n} \left(M_i |e_{ijk}| \gamma_{kj} + \frac{M_k |e_{kji}| \gamma_{ij}}{1-\rho} \right) \\
&< 0.
\end{aligned}
$$

Then we have the following corollary:

Corollary 1. *If $(H1)$–$(H3)$ hold, furthermore, $(H5)$ or $(H6)$ holds, then system (1) has only one $\omega-$periodic solution and all other solutions converge exponentially to it as $t \to +\infty$.*

3. The proof of Theorem 1. In this section, we will discuss the periodicity of the solution to system (1).

Proof of Theorem 1. For any smooth vector functions φ and ψ, let $u(x,t;\varphi) = (u_1(x,t;\varphi),\dots,u_n(x,t;\varphi))^T$ and $u(x,t;\psi) = (u_1(x,t;\psi),\dots,u_n(x,t;\psi))^T$ denote the solutions to system (1) which satisfy the assumptions $(H1)$–$(H4)$ through $(\varphi,0)$ and $(\psi,0)$, respectively.

Let $v_i(x,t) = u_i(x,t;\varphi) - u_i(x,t;\psi)$, $i = 1,\dots n$; then it follows that

$$
\begin{aligned}
\frac{\partial v_i(x,t)}{\partial t} = &\sum_{k=1}^{m} \frac{\partial}{\partial x_k}\left(D_{ik}\frac{\partial v_i(x,t)}{\partial x_k}\right) \\
&- a_i(u_i(x,t;\varphi))\Bigg[b_i(u_i(x,t;\varphi)) - \sum_{j=1}^{n} c_{ij} f_j(u_j(x,t;\varphi)) \\
&- \sum_{j=1}^{n} \omega_{ij} f_j(u_j(x,t-t_{ij}(t);\varphi)) \\
&- \sum_{j,k=1}^{n} d_{ijk} f_j(u_j(x,t;\varphi)) f_k(u_k(x,t;\varphi)) \\
&- \sum_{j,k=1}^{n} e_{ijk} f_j(u_j(x,t-t_{ij}(t);\varphi)) f_k(u_k(x,t-t_{ik}(t);\varphi)) + I_i(t)\Bigg] \\
&+ a_i(u_i(x,t;\psi))\Bigg[b_i(u_i(x,t;\psi)) - \sum_{j=1}^{n} c_{ij} f_j(u_j(x,t;\psi)) \\
&- \sum_{j=1}^{n} \omega_{ij} f_j(u_j(x,t-t_{ij}(t);\psi)) \\
&- \sum_{j,k=1}^{n} d_{ijk} f_j(u_j(x,t;\psi)) f_k(u_k(x,t;\psi)) \\
&- \sum_{j,k=1}^{n} e_{ijk} f_j(u_j(x,t-t_{ij}(t);\psi)) f_k(u_k(x,t-t_{ik};\psi)) + I_i(t)\Bigg], \qquad (5)\\
v_i(x,t) = &\ \varphi_i(x,t) - \psi_i(x,t), \qquad -\tau \le t \le 0, \quad x \in \Omega, \\
v_i(x,t) = &\ 0, \qquad x \in \partial\Omega.
\end{aligned}
$$

And, by simple calculation, we see that $v_i(x,t)$ also satisfies the following equation:

$$\begin{aligned}
\frac{\partial v_i(x,t)}{\partial t} &= \sum_{k=1}^{m} \frac{\partial}{\partial x_k}(D_{ik}\frac{\partial v_i(x,t)}{\partial x_k}) \\
&- \Big(a_i(u_i(x,t;\varphi)) - a_i(u_i(x,t;\psi))\Big) \\
&\times \Big[b_i(u_i(x,t;\psi)) - \sum_{j=1}^{n} c_{ij} f_j(u_j(x,t;\psi)) \\
&- \sum_{j=1}^{n} \omega_{ij} f_j(u_j(x,t-t_{ij}(t);\psi)) \\
&- \sum_{j,k=1}^{n} d_{ijk} f_j(u_j(x,t;\psi)) f_k(u_k(x,t;\psi)) \\
&- \sum_{j,k=1}^{n} e_{ijk} f_j(u_j(x,t-t_{ij}(t);\psi)) f_k(u_k(x,t-t_{ik}(t);\psi)) + I_i(t)\Big] \\
&- a_i(u_i(x,t;\varphi)) \Big[\Big(b_i(u_i(x,t;\varphi)) - b_i(u_i(x,t;\psi))\Big) \\
&- \sum_{j=1}^{n} c_{ij}\Big(f_j(u_j(x,t;\varphi)) - f_j(u_j(x,t;\psi))\Big) \\
&- \sum_{j=1}^{n} \omega_{ij}\Big(f_j(u_j(x,t-t_{ij}(t);\varphi)) - f_j(u_j(x,t-t_{ij}(t);\psi))\Big) \\
&- \sum_{j,k=1}^{n} d_{ijk}\Big(f_j(u_j(x,t;\varphi)) f_k(u_k(x,t;\varphi)) \\
&- f_j(u_j(x,t;\psi)) f_k(u_k(x,t;\psi))\Big) \\
&- \sum_{j,k=1}^{n} e_{ijk}\Big(f_j(u_j(x,t-t_{ij}(t);\varphi)) f_k(u_k(x,t-t_{ik}(t);\varphi)) \\
&- f_j(u_j(x,t-t_{ij}(t);\psi)) f_k(u_k(x,t-t_{ik}(t);\psi))\Big)\Big].
\end{aligned}$$

By $(H4)$, there exists a small positive constant λ which satisfies

$$0 < \lambda < \min\left\{\frac{1}{2}, \min_i\left\{m_i\alpha_i + \frac{\Delta_i^{\min}}{\delta}\right\}\right\}$$

such that
(6)

$$
\begin{aligned}
W_i =&\lambda - \alpha_i m_i - \frac{\Delta_i^{\min}}{\delta} \\
&+ \frac{1}{2}\Bigg[\sum_{j=1}^{n} \Gamma_i\big(\big(|c_{ij}| + |\omega_{ij}|\big)\Lambda_j + B_i\big) \\
&+ \sum_{j=1}^{n} (M_i |c_{ij}|^{2p_{ij}} \beta_j^{2q_{ij}} + M_j |c_{ji}|^{2-2p_{ji}} \beta_i^{2-2q_{ji}}) \\
&+ \sum_{j=1}^{n} \left(M_i |\omega_{ij}|^{2p_{ij}^*} \beta_j^{2q_{ij}^*} + \frac{M_j |\omega_{ji}|^{2-2p_{ji}^*} \beta_i^{2-2q_{ji}^*} e^{2\lambda\tau}}{1-\rho} \right) \\
&+ \sum_{j,k=1}^{n} (M_i |d_{ijk}|^{2m_{ijk}} \gamma_{jk}^{2n_{ijk}} + M_j |d_{jik}|^{2-2m_{jik}} \gamma_{ik}^{2-2n_{jik}}) \\
&+ \sum_{j,k=1}^{n} (M_i |d_{ijk}|^{2r_{ijk}} \gamma_{kj}^{2s_{ijk}} + M_k |d_{kji}|^{2-2r_{kji}} \gamma_{ij}^{2-2s_{kji}}) \\
&+ \sum_{j,k=1}^{n} \left(M_i |e_{ijk}|^{2m_{ijk}^*} \gamma_{jk}^{2n_{ijk}^*} + \frac{M_j |e_{jik}|^{2-2m_{jik}^*} \gamma_{ik}^{2-2n_{jik}^*} e^{2\lambda\tau}}{1-\rho} \right) \\
&+ \sum_{j,k=1}^{n} \left(M_i |e_{ijk}|^{2r_{ijk}^*} \gamma_{kj}^{2s_{ijk}^*} + \frac{M_k |e_{kji}|^{2-2r_{kji}^*} \gamma_{ij}^{2-2s_{kji}^*} e^{2\lambda\tau}}{1-\rho} \right) \Bigg] \\
\leq& \; 0.
\end{aligned}
$$

Taking the Lyapunov functional as follows:

$$
\begin{aligned}
V(t) = \sum_{i=1}^{n} \int_{\Omega} \Bigg[&|v_i(x,t)|^2 e^{2\lambda t} + \sum_{j=1}^{n} M_i |\omega_{ij}|^{2-2p_{ij}^*} \beta_j^{2-2q_{ij}^*} \\
&\cdot \int_{t-t_{ij}}^{t} |v_j(x,s)|^2 e^{2\lambda(s+t_{ij})}\, ds \\
&+ \sum_{j,k=1}^{n} M_i |e_{ijk}|^{2-2m_{ijk}^*} \gamma_{jk}^{2-2n_{ijk}^*} \int_{t-t_{ij}}^{t} |v_j(x,s)|^2 e^{2\lambda(s+t_{ij})} ds \\
&+ \sum_{j,k=1}^{n} M_i |e_{ijk}|^{2-2r_{ijk}^*} \gamma_{kj}^{2-2s_{ijk}^*}
\end{aligned}
$$

$$\cdot \int_{t-t_{ik}}^{t} |v_k(x,s)|^2 e^{2\lambda(s+t_{ik})}\, ds \Big] dx.$$

Calculating $D^+V(t)$ along system (5), we have

$$\begin{aligned} D^+V(t) = &\sum_{i=1}^{n} \int_{\Omega} \Big[2\lambda |v_i(x,t)|^2 e^{2\lambda t} + 2v_i(x,t) e^{2\lambda t} \frac{\partial v_i(x,t)}{\partial t} \\ &+ \sum_{j=1}^{n} M_i |\omega_{ij}|^{2-2p^*_{ij}} \beta_j^{2-2q^*_{ij}} (|v_j(x,t)|^2 e^{2\lambda(t+t_{ij})} \\ &- |v_j(x,t-t_{ij})|^2 e^{2\lambda t}) \\ &+ \sum_{j,k=1}^{n} M_i (|e_{ijk}|^{2-2m^*_{ijk}} \gamma_{jk}^{2-2n^*_{ijk}} (|v_j(x,t)|^2 e^{2\lambda(t+t_{ij})} \\ &- |v_j(x,t-t_{ij})|^2 e^{2\lambda t}) \\ &+ \sum_{j,k=1}^{n} M_i |e_{ijk}|^{2-2r^*_{ijk}} \gamma_{kj}^{2-2s^*_{ijk}}) (|v_k(x,t)|^2 e^{2\lambda(t+t_{ik})} \\ &- |v_k(x,t-t_{ik})|^2 e^{2\lambda t}) \Big] dx \\ = &\sum_{i=1}^{n} \int_{\Omega} \Big\{ 2\lambda |v_i(x,t)|^2 e^{2\lambda t} + 2v_i(x,t) e^{2\lambda t} \\ &\cdot \Big\{ \sum_{k=1}^{m} \frac{\partial}{\partial x_k} \Big(D_{ik} \frac{\partial v_i(x,t)}{\partial x_k} \Big) \\ &- \Big(a_i(u_i(x,t;\varphi)) - a_i(u_i(x,t;\psi)) \Big) \Big[b_i(u_i(x,t;\psi)) \\ &- \sum_{j=1}^{n} c_{ij} f_j(u_j(x,t;\psi)) \\ &- \sum_{j=1}^{n} \omega_{ij} f_j(u_j(x,t-t_{ij};\psi)) \\ &- \sum_{j,k=1}^{n} d_{ijk} f_j(u_j(x,t;\psi)) f_k(u_k(x,t;\psi)) \\ &- \sum_{j,k=1}^{n} e_{ijk} f_j(u_j(x,t-t_{ij};\psi)) f_k(u_k(x,t-t_{ik};\psi)) + I_i(t) \Big] \end{aligned}$$

$$
\begin{aligned}
& - a_i(u_i(x,t;\varphi))\Big[\Big(b_i(u_i(x,t;\varphi)) - b_i(u_i(x,t;\psi))\Big)\\
& - \sum_{j=1}^{n} c_{ij}\Big(f_j(u_j(x,t;\varphi)) - f_j(u_j(x,t;\psi))\Big)\\
& - \sum_{j=1}^{n} \omega_{ij}\Big(f_j(u_j(x,t-t_{ij};\varphi)) - f_j(u_j(x,t-t_{ij};\psi))\Big)\\
& - \sum_{j,k=1}^{n} d_{ijk}\Big(f_j(u_j(x,t;\varphi))f_k(u_k(x,t;\varphi))\\
& - f_j(u_j(x,t;\psi))f_k(u_k(x,t;\psi))\Big) \qquad (7)\\
& - \sum_{j,k=1}^{n} e_{ijk}\Big(f_j(u_j(x,t-t_{ij};\varphi))f_k(u_k(x,t-t_{ik};\varphi))\\
& - f_j(u_j(x,t-t_{ij};\psi))f_k(u_k(x,t-t_{ik};\psi))\Big)\Big]\Big\}\\
& + \sum_{j=1}^{n} M_i|\omega_{ij}|^{2-2p_{ij}^*}\beta_j^{2-2q_{ij}^*}(|v_j(x,t)|^2 e^{2\lambda(t+t_{ij})}\\
& - |v_j(x,t-t_{ij})|^2 e^{2\lambda t})\\
& + \sum_{j,k=1}^{n} M_i|e_{ijk}|^{2-2m_{ijk}^*}\gamma_{jk}^{2-2n_{ijk}^*}(|v_j(x,t)|^2 e^{2\lambda(t+t_{ij})}\\
& - |v_j(x,t-t_{ij})|^2 e^{2\lambda t})\\
& + \sum_{j,k=1}^{n} M_i|e_{ijk}|^{2-2r_{ijk}^*}\gamma_{kj}^{2-2s_{ijk}^*}(|v_k(x,t)|^2 e^{2\lambda(t+t_{ik})}\\
& - |v_k(x,t-t_{ik})|^2 e^{2\lambda t})\Big\}\,dx.
\end{aligned}
$$

Noting $(H1)$, by the Green formula, the boundary condition and Lemma 1, it follows that

$$
\int_\Omega \sum_{k=1}^{m} \frac{\partial}{\partial x_k}\Big(D_{ik}\frac{\partial v_i(x,t)}{\partial x_k}\Big) v_i(x,t)dx = \int_\Omega \sum_{k=1}^{m} \frac{\partial}{\partial x_k}\Big(D_{ik}\frac{\partial v_i(x,t)}{\partial x_k} v_i(x,t)\Big)dx
$$

$$
\begin{aligned}
&-\int_\Omega \sum_{k=1}^m \frac{\partial v_i(x,t)}{\partial x_k} D_{ik} \frac{\partial v_i(x,t)}{\partial x_k}\,dx\\
&= -\sum_{k=1}^m \int_\Omega D_{ik}\left(\frac{\partial v_i(x,t)}{\partial x_k}\right)^2 dx\\
&\le -\Delta_i^{\min}\left(\frac{\omega_m}{|\Omega|}\right)^{1/m}\int_\Omega |v_i(x,t)|^2 dx\\
&= -\frac{\Delta_i^{\min}}{\delta}\int_\Omega |v_i(x,t)|^2 dx,
\end{aligned}
$$

where $\delta = (|\Omega|/\omega_m)^{1/m} > 0$.

Hence, it follows from (7) that we get

$$
\begin{aligned}
D^+V(t) \le e^{2\lambda t}\sum_{i=1}^n \int_\Omega &[2\lambda |v_i(x,t)|^2 - 2\left(m_i\alpha_i + \frac{\Delta_i^{\min}}{\delta}\right)|v_i(x,t)|^2\\
&+ \sum_{j=1}^n \Gamma_i\big(\big(|c_{ij}| + |\omega_{ij}|\big)\Lambda_j + B_i\big)|v_i(x,t)|^2\\
&+ \sum_{j=1}^n M_i(|c_{ij}|\beta_j|v_i(x,t)||v_j(x,t)|\\
&+ |\omega_{ij}|\beta_j|v_i(x,t)||v_j(x,t-t_{ij})|)\\
&+ \sum_{j,k=1}^n M_i|d_{ijk}|(\gamma_{jk}|v_i(x,t)||v_j(x,t)|\\
&+ \gamma_{kj}|v_i(x,t)||v_k(x,t)|)\\
&+ \sum_{j,k=1}^n M_i|e_{ijk}|(\gamma_{jk}|v_i(x,t)||v_j(x,t-t_{ij})|\\
&+ \gamma_{kj}|v_i(x,t)||v_k(x,t-t_{ik})|)\\
&+ \sum_{j=1}^n M_i|\omega_{ij}|^{2-2p_{ij}^*}\beta_j^{2-2q_{ij}^*}(|v_j(x,t)|^2 e^{2\lambda t_{ij}}\\
&- |v_j(x,t-t_{ij})|^2)\\
&+ \sum_{j,k=1}^n M_i|e_{ijk}|^{2-2m_{ijk}^*}\gamma_{jk}^{2-2n_{ijk}^*}(|v_j(x,t)|^2 e^{2\lambda t_{ij}}
\end{aligned}
$$

$$
\begin{aligned}
&- |v_j(x,t-t_{ij})|^2) \\
&+ \sum_{j,k=1}^{n} M_i |e_{ijk}|^{2-2r^*_{ijk}} \gamma_{kj}^{2-2s^*_{ijk}} (|v_k(x,t)|^2 e^{2\lambda t_{ik}} \\
&- |v_k(x,t-t_{ik})|^2)]\, dx \\
\le{}& e^{2\lambda t} \sum_{i=1}^{n} \int_\Omega \Bigg\{ |v_i(x,t)|^2 \Bigg(2\lambda - 2m_i\alpha_i - 2\frac{\Delta_i^{\min}}{\delta} \\
&+ \sum_{j=1}^{n} \Gamma_i \big((|c_{ij}| + |\omega_{ij}|) \Lambda_j + B_i \big) \\
&+ \sum_{j=1}^{n} (M_i |c_{ij}|^{2p_{ij}} \beta_j^{2q_{ij}} + M_j |c_{ji}|^{2-2p_{ji}} \beta_i^{2-2q_{ji}} \\
&+ M_i |\omega_{ij}|^{2p^*_{ij}} \beta_j^{2q^*_{ij}}) \\
&+ \sum_{j,k=1}^{n} (M_i |d_{ijk}|^{2m_{ijk}} \gamma_{jk}^{2n_{ijk}} + M_j |d_{jik}|^{2-2m_{jik}} \gamma_{ik}^{2-2n_{jik}} \\
&+ M_i |d_{ijk}|^{2r_{ijk}} \gamma_{kj}^{2s_{ijk}} + M_k |d_{kji}|^{2-2r_{kji}} \gamma_{ij}^{2-2s_{kji}}) \Bigg) \\
&+ \sum_{j,k=1}^{n} (M_i |e_{ijk}|^{2m^*_{ijk}} \gamma_{jk}^{2n^*_{ijk}} + M_i |e_{ijk}|^{2r^*_{ijk}} \gamma_{kj}^{2s^*_{ijk}}) |v_i(x,t)|^2 \\
&+ \sum_{j,k=1}^{n} (M_i |e_{ijk}|^{2-2m^*_{ijk}} \gamma_{jk}^{2-2n^*_{ijk}} |v_j(x,t-t_{ij})|^2 \\
&+ M_i |e_{ijk}|^{2-2r^*_{ijk}} \gamma_{kj}^{2-2s^*_{ijk}} |v_k(x,t-t_k)|^2) \\
&+ \sum_{j=1}^{n} M_i |\omega_{ij}|^{2-2p^*_{ij}} \beta_j^{2-2q^*_{ij}} |v_j(x,t-t_{ij})|^2 \\
&+ \sum_{j=1}^{n} M_i |\omega_{ij}|^{2-2p^*_{ij}} \beta_j^{2-2q^*_{ij}} (|v_j(x,t)|^2 e^{2\lambda t_{ij}} \\
&- |v_j(x,t-t_{ij})|^2) \\
&+ \sum_{j,k=1}^{n} M_i |e_{ijk}|^{2-2m^*_{ijk}} \gamma_{jk}^{2-2n^*_{ijk}} (|v_j(x,t)|^2 e^{2\lambda t_{ij}}
\end{aligned}
$$

$$
\begin{aligned}
&- |v_j(x, t - t_{ij})|^2) \\
&+ \sum_{j,k=1}^{n} M_i |e_{ijk}|^{2-2r^*_{ijk}} \gamma_{kj}^{2-2s^*_{ijk}} (|v_k(x,t)|^2 e^{2\lambda t_{ik}} \\
&- |y_k(x, t - t_k)|^2) \Big\} \, dx.
\end{aligned}
$$

That is,

$$
\begin{aligned}
D^+V(t) \le{}& e^{2\lambda t} \sum_{i=1}^{n} \int_\Omega \Big[|v_i(x,t)|^2 \Big(2\lambda - 2m_i\alpha_i - 2\frac{\Delta_i^{\min}}{\delta} \\
&+ \sum_{j=1}^{n} \Gamma_i \big(\big(|c_{ij}| + |\omega_{ij}|\big) \Lambda_j + B_i \big) \\
&+ \sum_{j=1}^{n} (M_i |c_{ij}|^{2p_{ij}} \beta_j^{2q_{ij}} + M_j |c_{ji}|^{2-2p_{ji}} \beta_i^{2-2q_{ji}} \\
&+ M_i |\omega_{ij}|^{2p^*_{ij}} \beta_j^{2q^*_{ij}}) \\
&+ \sum_{j,k=1}^{n} (M_i |d_{ijk}|^{2m_{ijk}} \gamma_{jk}^{2n_{ijk}} + M_j |d_{jik}|^{2-2m_{jik}} \gamma_{ik}^{2-2n_{jik}} \\
&+ M_i |d_{ijk}|^{2r_{ijk}} \gamma_{kj}^{2s_{ijk}} + M_k |d_{kji}|^{2-2r_{kji}} \gamma_{ij}^{2-2s_{kji}}) \Big) \\
&+ \sum_{j,k=1}^{n} (M_i |e_{ijk}|^{2m^*_{ijk}} \gamma_{jk}^{2n^*_{ijk}} + M_i |e_{ijk}|^{2r^*_{ijk}} \gamma_{kj}^{2s^*_{ijk}}) |v_i(x,t)|^2 \\
&+ \sum_{j=1}^{n} M_i |\omega_{ij}|^{2-2p^*_{ij}} \beta_j^{2-2q^*_{ij}} e^{2\lambda\tau} |v_j(x,t)|^2 \\
&+ \sum_{j,k=1}^{n} M_i |e_{ijk}|^{2-2m^*_{ijk}} \gamma_{jk}^{2-2n^*_{ijk}} e^{2\lambda\tau} |v_j(x,t)|^2 \\
&+ \sum_{j,k=1}^{n} M_i |e_{ijk}|^{2-2r^*_{ijk}} \gamma_{kj}^{2-2s^*_{ijk}} e^{2\lambda\tau} |v_k(x,t)|^2 \Big] \, dx \\
={}& 2e^{2\lambda t} \sum_{i=1}^{n} W_i \int_\Omega |v_i(x,t)|^2 dx \le 0,
\end{aligned}
$$

where W_i is defined in (6). It means that

$$\begin{aligned}V(t) \le V(0) = \sum_{i=1}^{n} \int_{\Omega} \Big[&|\varphi_i(x,0) - \psi_i(x,0)|^2 \\ &+ \sum_{j=1}^{n} M_i |\omega_{ij}|^{2-2p^*_{ij}} \beta_j^{2-2q^*_{ij}} \\ &\cdot \int_0^{t_{ij}} |\varphi_j(x,s-t_{ij}) - \psi_j(x,s-t_{ij})|^2 e^{2\lambda s} ds \\ &+ \sum_{j,k=1}^{n} M_i |e_{ijk}|^{2-2m^*_{ijk}} \gamma_{jk}^{2-2n^*_{ijk}} \\ &\cdot \int_0^{t_{ji}} |\varphi_j(x,s-t_{ji}) - \psi_j(x,s-t_{ji})|^2 e^{2\lambda s} ds \\ &+ \sum_{j,k=1}^{n} M_i |e_{ikj}|^{2-2r^*_{ikj}} \gamma_{jk}^{2-2s^*_{ikj}} \\ &\cdot \int_0^{t_{ik}} |\varphi_j(x,s-t_{ij}) - \psi_j(x,s-t_{ij})|^2 e^{2\lambda s} ds \Big] dx.\end{aligned}$$

Let

$$\begin{aligned}\Upsilon = \max_i \Big\{ 1 + (e^{2\lambda\tau} - 1) \sum_{j,k=1}^{n} \frac{M_i}{2\lambda} \Big(&|\omega_{ij}|^{2-2p^*_{ij}} \beta_j^{2-2q^*_{ij}} \\ &+ |e_{ijk}|^{2-2m^*_{ijk}} \gamma_{jk}^{2-2n^*_{ijk}} \\ &+ |e_{ikj}|^{2-2r^*_{ikj}} \gamma_{jk}^{2-2s^*_{ikj}} \Big) \Big\} > 1.\end{aligned}$$

For $V(t) \ge \|v(t)\|_2^2 e^{2\lambda t}$, it holds:

$$\|v(t)\|_2^2 \le \Upsilon \|\varphi - \psi\|_2^2 e^{-2\lambda t},$$

that is,

$$\|u(\cdot,t;\varphi) - u(\cdot,t;\psi)\|_2 \le \sqrt{\Upsilon} \|\varphi - \psi\|_2 e^{-\lambda t}. \tag{8}$$

For $\tau > 0$, we can choose a positive integer N and a positive constant κ such that

$$\sqrt{\Upsilon} e^{-\lambda(N\omega - \tau)} \le \kappa < 1. \tag{9}$$

If we define a Poincaré mapping $P : C \to C$ (here C denotes the continuous function space) by $P(\Phi) = u(x, \omega + \theta; \Phi)$, $(\theta \in [-\tau, 0])$, then $P^N(\Phi) = u(x, N\omega + \theta; \Phi)$, $\theta \in [-\tau, 0]$. Let $t = N\omega$; by (8)–(9), we get

$$\begin{aligned}\|P^N(\varphi) - P^N(\psi)\|_2 &= \|u(x, N\omega + \theta; \varphi) - u(x, N\omega + \theta; \psi)\|_2 \\ &\le \sqrt{\Upsilon}\|\varphi - \psi\|_2 e^{-\lambda(N\omega+\theta)} \\ &\le \sqrt{\Upsilon}\|\varphi - \psi\|_2 e^{-\lambda(N\omega-\tau)} \\ &\le \kappa\|\varphi - \psi\|_2.\end{aligned}$$

This implies that P^N is a contraction mapping because of $0 < \kappa < 1$. According to the Banach fixed point theorem, there exists a unique point $\varphi^* \in C$ such that $P^N(\varphi^*) = \varphi^*$. Since $P^N(P(\varphi^*)) = P(P^N(\varphi^*)) = P(\varphi^*)$, then $P(\varphi^*)$ is also a fixed point of P^N. By the uniqueness of the fixed point of P^N, we have

$$P(\varphi^*) = \varphi^* \text{ that is } u(x, \omega + \theta; \varphi^*) = \varphi^*.$$

Let $u(x, \omega + \theta; \varphi^*)$ be the solution of CGNNs (1) through $(\varphi^*, 0)$; then $u(x, t + \omega + \theta; \varphi^*)$ is also the solution of CGNNs (1) through $(u(x, \omega + \theta; \varphi^*), 0)$. Obviously, for all $t \ge 0$, it follows that

$$u(x, t+\omega+\theta; \varphi^*) = u(x, t+\theta; u(x, \omega+\theta; \varphi^*)) = u(x, t+\theta; \varphi^*),\ \forall \theta \in [-\tau, 0].$$

Hence, for all $t \ge 0$, it follows that

$$u(x, t + \omega; \varphi^*) = u(x, t; \varphi^*).$$

This shows that $u(x, t; \varphi^*)$ is exactly one ω-periodic solution of CGNNs (1) and, according to (8), it is easy to see that all other solutions of CGNNs (1) converge exponentially to it as $t \to +\infty$. We have finished the proof of Theorem 1. □

4. Example. Consider high-order Cohen-Grossberg neural networks with delays and reaction-diffusion terms

$$\frac{\partial u_1(x,t)}{\partial t} = \frac{\partial}{\partial x}(D_1 \frac{\partial u_1(x,t)}{\partial x}) - a_1(u_1(x,t))\Big[b_1(u_1(x,t))$$

$$
\begin{aligned}
&- \sum_{j=1}^{2} c_{1j} f_j(u_j(x,t)) - \sum_{j=1}^{2} \omega_{1j} f_j(u_j(x,t-t_j)) \\
&- \sum_{j,k=1}^{2} d_{1jk} f_j(u_j(x,t)) f_k(u_k(x,t)) \\
&- \sum_{j,k=1}^{2} e_{1jk} f_j(u_j(x,t-t_j)) f_k(u_k(x,t-t_k) + I_1(t))\Big],
\end{aligned}
$$

(10)

$$
\begin{aligned}
\frac{\partial u_2(x,t)}{\partial t} &= \frac{\partial}{\partial x}(D_2 \frac{\partial u_2(x,t)}{\partial x}) - a_2(u_2(x,t)) \\
&\cdot \Big[b_2(u_2(x,t)) - \sum_{j=1}^{2} c_{2j} f_j(u_j(x,t)) \\
&- \sum_{j=1}^{2} \omega_{2j} f_j(u_j(x,t-t_j)) \\
&- \sum_{j,k=1}^{2} d_{2jk} f_j(u_j(x,t)) f_k(u_k(x,t)) \\
&- \sum_{j,k=1}^{2} e_{2jk} f_j(u_j(x,t-t_j)) f_k(u_k(x,t-t_k) + I_2(t))\Big],
\end{aligned}
$$

$$
u_i(x,t) = \varphi_i(x,t), \quad -\tau \le t \le 0,\ x \in \Omega,\ i = 1,2
$$

$$
\frac{\partial u_i(x,t)}{\partial n} = 0, \quad x \in \partial\Omega,\ i = 1,2.
$$

Let $\Delta_1^{\min} = \Delta_{11} = D_1 > 0$, $\Delta_2^{\min} = \Delta_{21} = D_2 > 0$. Ω is a bounded open domain with $|\Omega| \le 1$, and we can have $\delta = 1/\pi$. Let $a_1 = 6 + \sin u_1(x,t)$ and $a_2 = 3 - \sin u_2(x,t)$ which satisfy $(H1)$ with $m_1 = 5$, $M_1 = 7$, $m_2 = 2$, $M_2 = 4$, $\Gamma_1 = 1$, $\Gamma_2 = 1$. $b_1 = 5u_1(x,t) + (1/2)\sin u_1(x,t)$ and $b_2 = 6u_2(x,t) + (1/2)\sin u_2(x,t)$ satisfy $(H2)$ with $\alpha_1 = 9/2$, $\alpha_2 = 11/2$. $f_1 = \tanh u_1(x,t)$ and $f_2 = \tanh u_2(x,t)$ satisfy $(H3)$ with $\beta_1 = 1$, $\beta_2 = 1$, $\gamma_{ij} = 1$, $i,j = 1,2$ and $\Lambda_1 = 1$, $\Lambda_2 = 1$. Because $c_{ij} = d_{ijk} = 0$, $i,j,k = 1,2$ and $\omega_{11} = 1/20$, $\omega_{12} = 1/60$, $\omega_{21} = 1/50$, $\omega_{22} = 1/10$. $e_{111} = 1/20$, $e_{112} = e_{121} = -1/64$, $e_{122} = 1/12$, $e_{211} = 1/40$, $e_{212} = e_{221} = -1/32$, $e_{222} = -1/30$. $I_1 = 2 + \sin t$, $I_2 = 1 - 3\cos t$ with $B_1 = 3$ and $B_2 = 4$.

By simple calculation, we can choose

$$p_{ij} = q_{ij} = m_{ijk} = n_{ijk} = r_{ijk} = s_{ijk} = p^*_{ij}$$
$$= q^*_{ij} = m^*_{ijk} = n^*_{ijk} = r^*_{ijk} = s^*_{ijk} = \frac{1}{2},$$
$$(i, j = 1, 2)$$

such that

$$\begin{aligned}
-2\alpha_1 m_1 - 2\frac{\Delta_1^{\min}}{\delta} &+ \sum_{j=1}^{2} \Gamma_1\big((|c_{1j}| + |\omega_{1j}|)\Lambda_j + B_1\big) \\
&+ \sum_{j=1}^{2}(M_1|\omega_{1j}|\beta_j + M_j|\omega_{j1}|\beta_1) \\
&+ \sum_{j,k=1}^{2}(M_1|e_{1jk}|\gamma_{jk} + M_j|e_{j1k}|\gamma_{1k}) \\
&+ \sum_{j,k=1}^{2}(M_1|e_{1jk}|\mu_{jk} + M_k|e_{kj1}|\mu_{j1}) \\
&< -2 \cdot \frac{9}{2} \cdot 5 + 1 \cdot \left[\left(\frac{1}{20} + \frac{1}{60}\right) \cdot 1 + 3\right] \\
&+ \left[7 \cdot \left(\frac{1}{20} + \frac{1}{60}\right) + 7 \cdot \frac{1}{20} + 4 \cdot \frac{1}{50}\right] \\
&+ 2 \cdot \left[7 \cdot \left(\frac{1}{20} + \frac{1}{64} + \frac{1}{64} + \frac{1}{12}\right) + 7\right. \\
&\left. \cdot \left(\frac{1}{20} + \frac{1}{64}\right) + 4 \cdot \left(\frac{1}{40} + \frac{1}{32}\right)\right] \\
&= -45 + 7\frac{509}{800} < 0.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
-2\alpha_2 m_2 - 2\frac{\Delta_2^{\min}}{\delta} &+ \sum_{j=1}^{2} \Gamma_2\big((|c_{2j}| + |\omega_{2j}|)\Lambda_j + B_2\big) \\
&+ \sum_{j=1}^{2}(M_2|\omega_{2j}|\beta_j + M_j|\omega_{j2}|\beta_2)
\end{aligned}$$

$$+ \sum_{j,k=1}^{2} (M_2|e_{2jk}|\gamma_{jk} + M_j|e_{j2k}|\gamma_{2k})$$
$$+ \sum_{j,k=1}^{2} (M_2|e_{2jk}|\mu_{jk} + M_k|e_{kj2}|\mu_{j2}) < 0.$$

That means $(H6)$ holds. It follows from Corollary 1 that system (10) has only one $\omega-$periodic solution and all other solutions converge exponentially to it as $t \to +\infty$.

5. Conclusions. In this paper, the periodic behaviors of delayed reaction diffusion high-order Cohen-Grossberg neural network models with Dirichlet boundary conditions are studied. By employing inequality techniques and constructing the Lyapunov functional, some sufficient conditions have been given to ensure the existence of the periodic solution. These conditions are useful in the design and applications of delayed reaction diffusion high-order Cohen-Grossberg neural networks model, and the method in this paper may be extended for more complex networks.

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