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PRINCIPAL VALUES FOR THE SIGNED RIESZ KERNELS OF NON-INTEGER DIMENSION

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ABSTRACT. For positive measures μ in $\mathbf{R}^n$ and $0 < \alpha < 1$, we study the μ-almost everywhere existence of the principal values of the α-Riesz transform of μ,

$$\lim_{\varepsilon\to 0}\int_{|y-x|>\varepsilon} \frac{y-x}{|y-x|^{1+\alpha}}\, d\mu(y).$$

We show that the $L^2(\mu)$-boundedness of the α-Riesz transform implies the existence of the above principal value for μ-almost all $x \in \mathbf{R}^n$. We also prove that if μ has positive and finite upper density μ-almost everywhere, then the set where the principal value does not exist has positive μ-measure.

1. Introduction. For any $0 < \alpha < n$, we shall consider the natural α-dimensional generalization of the Cauchy kernel $1/z$, $z \in \mathbf{C}$, in $\mathbf{R}^n$ defined by

$$K_\alpha(x) = \frac{x}{|x|^{1+\alpha}}, \text{ for } x \in \mathbf{R}^n \setminus \{0\}.$$

For a non-zero Radon measure μ on $\mathbf{R}^n$, the α-Riesz transform of μ is defined, for $x \notin \operatorname{spt}(\mu)$, by

$$R^\alpha\mu(x) = \int K_\alpha(y-x)\, d\mu(y).$$

Here we are using vector-valued integrals, which can be defined in terms of the coordinate functions.

Since the definition above does not make sense in general for $x \in \operatorname{spt}(\mu)$, one considers the truncated α-Riesz transform of μ, defined by

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$$R^\alpha_\varepsilon \mu(x) = \int_{|y-x|>\varepsilon} K_\alpha(y-x)\, d\mu(y),$$

for any $\varepsilon > 0$ and $x \in \mathbf{R}^n$.

Recall also that the α-Riesz transform is said to be bounded in $L^2(\mu)$ if the operators R^α_ε are bounded in $L^2(\mu)$ uniformly in $\varepsilon > 0$.

The maximal α-Riesz transform is defined as

$$R^\alpha_* \mu(x) = \sup_{\varepsilon>0} |R^\alpha_\varepsilon \mu(x)|.$$

Recall that the L^2-boundedness of $R^\alpha_*\mu$ comes from the L^2 boundedness of the α-Riesz transform. This follows from standard Calderón-Zygmund theory if the measure is doubling. In the general case, Nazarov, Treil and Volberg proved in [**12**] that if the α-Riesz transform is bounded in $L^2(\mu)$, then the maximal α-Riesz transform satisfies a Cotlar type inequality (slightly different from the classical one) which yields immediately that R^α_* is bounded in $L^p(\mu)$, when $p \in (1, \infty)$.

In this paper we study the μ-almost everywhere existence of the principal values

$$\lim_{\varepsilon \to 0} R^\alpha_\varepsilon \mu(x). \tag{1}$$

The first result of this paper shows that the $L^2(\mu)$-boundedness of the α-Riesz transform implies the existence of the principal value (1) for μ-almost everywhere $x \in \mathbf{R}^n$. For $\alpha = 1$ and $n = 2$, i.e., the Cauchy integral on $\mathbf{C}$, this is a known result of Tolsa [**16**]. In fact, in [**16**] he proves a more general result that gives a geometric characterization of those positive Radon measures μ on $\mathbf{C}$ such that for any finite complex Radon measure ν on $\mathbf{C}$, the principal value of the Cauchy integral of ν exists for μ-almost everywhere $x \in \mathbf{C}$. Our first result reads as follows

Theorem 1. *Let $0 < \alpha < 1$, and let μ be a positive Radon measure on $\mathbf{R}^n$ with compact support such that the α-Riesz transform $R^\alpha\mu$ is bounded in $L^2(\mu)$. Then the principal value* (1) *exists for μ-almost all $x \in \mathbf{R}^n$.*

Recall that a measure μ is Ahlfors-regular of order α, if there exist positive constants c_1 and c_2 such that for all $x \in \operatorname{spt}\mu$ and all

$0 < r < \operatorname{diam}(\operatorname{spt}\mu)$,

$$c_1 r^\alpha \le \mu B(x,r) \le c_2 r^\alpha.$$

According to Vihtilä (see [**21**]), in $\mathbf{R}^n$, for $0 < \alpha < n$, $\alpha \notin \mathbf{Z}$, there are no non zero Ahlfors-regular measures μ of order α for which $R^\alpha \mu$ is bounded in L^2. Hence Theorem 1 does not say anything about Ahlfors-regular measures.

Recall that the upper and lower α-densities of μ at $x \in \mathbf{R}^n$ are defined by

$$\theta^{*\alpha}(\mu,x) = \limsup_{r\to 0} \frac{\mu B(x,r)}{r^\alpha} \text{ and } \theta^\alpha_*(\mu,x) = \liminf_{r\to 0} \frac{\mu B(x,r)}{r^\alpha}.$$

Mattila and Preiss [**7**] showed that for $0 < \alpha < n$, if μ is a non-zero Radon measure on $\mathbf{R}^n$ such that for μ-almost all $x \in \mathbf{R}^n$, $0 < \theta^\alpha_*(\mu,x) \le \theta^{*\alpha}(\mu,x) < \infty$ and such that the principal values (1) exist μ-almost everywhere, then α must be an integer and μ must be α-rectifiable, i.e., there exist α-dimensional C^1-submanifolds M_i such that $\mu(\mathbf{R}^n \setminus \cup_i M_i) = 0$.

When $0 < \alpha < 1$, we can generalize this theorem from [**7**], getting rid of the hypothesis of having positive lower density μ-almost everywhere. Namely, we show that for $0 < \alpha < 1$ and μ a measure with positive and finite upper density μ-almost everywhere, it cannot happen that the principal value in (1) exists μ-almost everywhere. Our result reads as follows:

Theorem 2. *Let $0 < \alpha < 1$, and let μ be a positive measure in $\mathbf{R}^n$ such that $0 < \theta^{*\alpha}(\mu,x) < \infty$ for μ-almost all $x \in \mathbf{R}^n$. Then the set where the principal value* (1) *does not exist has positive μ-measure.*

Mattila [**5**] has obtained a result closely related to Theorem 2. He has proved that for a finite, positive measure μ on $\mathbf{C}$ and $0 < \alpha < 2$, if for μ-almost all $x \in \mathbf{C}$,

$$\begin{aligned} \theta^\alpha_*(\mu,x) &> 0, && \text{in the case } \alpha \le 1,\\ 0 < \theta^\alpha_*(\mu,x) \le \theta^{*\alpha}(\mu,x) &< \infty && \text{in the case } \alpha > 1, \end{aligned}$$

and the principal value (1) exists and is finite, then $\alpha = 1$.

The difficulty in extending Theorem 1 and Theorem 2 to values $1 < \alpha < n$ is due to the fact that the Riesz kernels enjoy a special positivity property for $\alpha \leq 1$, which fails for every α in the range $1 < \alpha < n$. This lack of positivity makes the treatment of the case $1 < \alpha < n$ much more difficult (see [**3, 4, 19**] for ways to circumvent this difficulty).

In [**20**], Tolsa generalizes the result from [**7**]. He proves that for $0 < \alpha < n$, $\alpha \in \mathbf{Z}$, if μ is a non-zero Radon measure on $\mathbf{R}^n$ such that for μ-almost all $x \in \mathbf{R}^n$, $0 < \theta^{*\alpha}(\mu, x) < \infty$ and such that the principal value (1) exists μ-almost everywhere, then μ must be α-rectifiable. Getting rid of the hypothesis $0 < \theta_*^\alpha(\mu, x)$ μ-almost everywhere (as in Theorem 2), which was an open problem raised by the authors in [**7**].

The arguments to prove the results from [**7**] and [**5**] use tangent measures and hence, they are very different from the ones used in [**20**] and in this paper. In [**20**], Tolsa obtains precise L^2-estimates of Riesz transforms on Lipschitz graphs, and in this paper the special positivity property that the Riesz kernels enjoy for $0 < \alpha \leq 1$ plays a role analogous to the L^2-estimates (see Section 2).

Mattila and Verdera ([**8**]) have recently shown that L^2-bounded singular integrals in metric spaces with respect to general measures and kernels converge weakly. For measures with zero density they have shown the almost everywhere existence of principal values.

The paper is organized as follows. In Section 2 we review the symmetrization method and its relation with the $L^2(\mu)$-boundedness of the α-Riesz transform. In Section 3 we prove Theorem 1. The fourth and last section is devoted to the proof of Theorem 2 by using an adaptation of a deep result of Nazarov, Treil and Volberg (see [**11, 13, 15**]).

2. L^2 bounds for the α-Riesz transforms and the symmetrization method. For $0 < \alpha < 1$, we can use the technique of symmetrizing the kernels to study the L^2-boundedness of the Riesz transforms. The symmetrization process for the Cauchy kernel introduced in [**9**] has been successfully applied in these last years to many problems of analytic capacity and L^2 boundedness of the Cauchy integral operator (see [**6, 10, 17, 18**] for example; the survey [**2**] and the book [**14**] contain many other interesting references). Given three distinct points in the plane, z_1, z_2 and z_3, one finds out, by an elementary computation that

$$c(z_1,z_2,z_3)^2=\sum_\sigma \frac{1}{(z_{\sigma(1)}-z_{\sigma(3)})\overline{(z_{\sigma(2)}-z_{\sigma(3)})}} \tag{2}$$

where the sum is taken over the six permutations of the set $\{1,2,3\}$, and $c(z_1,z_2,z_3)$ is the Menger curvature, that is, the inverse of the radius of the circle through z_1, z_2 and z_3. In particular, (2) shows that the sum on the right hand side is a non-negative quantity.

It can be shown that for $0<\alpha<1$ the symmetrization of the Riesz kernel $K_\alpha(x)=x/|x|^{1+\alpha}$ also gives a positive quantity. On the other hand for $1<\alpha<n$ the phenomenon of change of signs appears when symmetrizing the kernel K_α, as one can easily check.

For $0<\alpha<n$ the quantity

$$\sum_\sigma \frac{x_{\sigma(2)}-x_{\sigma(1)}}{|x_{\sigma(2)}-x_{\sigma(1)}|^{1+\alpha}}\cdot\frac{x_{\sigma(3)}-x_{\sigma(1)}}{|x_{\sigma(3)}-x_{\sigma(1)}|^{1+\alpha}}, \tag{3}$$

where the sum is taken over the six permutations of the set $\{1,2,3\}$, is the obvious analogue of the right hand side of (2) for the Riesz kernel K_α. Notice that (3) is exactly

$$2\,p_\alpha(x_1,x_2,x_3),$$

where $p_\alpha(x_1,x_2,x_3)$ is defined as the sum in (3) taken only on the three permutations $(1,2,3)$, $(2,3,1)$, $(3,1,2)$.

The relationship between the quantity $p_\alpha(x,y,z)$ and the L^2 estimates of the operator with kernel K_α is as follows. Take a positive finite Radon measure μ in $\mathbf{R}^n$ which satisfies the growth condition $\mu B(x,r)\le C_0 r^\alpha$, $x\in\mathbf{R}^n$, $r>0$. Then (see in [**10, 14**] the argument for $\alpha=1$, or see ([**15**, page 960]) for the case $0<\alpha<1$)

$$\left|\int |R_\varepsilon^\alpha\mu(x)|^2\,d\mu(x)-\frac{1}{3}p_{\alpha,\varepsilon}(\mu)\right|\le MC_0^2\|\mu\|, \tag{4}$$

where M is some positive constant depending on n and α and

$$p_{\alpha,\varepsilon}(\mu)=\iiint_{S_\varepsilon} p_\alpha(x,y,z)\,d\mu(x)\,d\mu(y)\,d\mu(z),$$

with

$$S_\varepsilon = \{(x,y,z) : |x-y| > \varepsilon,\ |x-z| > \varepsilon \text{ and } |y-z| > \varepsilon\}.$$

Thus,

$$p_\alpha(\mu) \le 3 \sup_\varepsilon \int |R_\varepsilon^\alpha \mu(x)|^2 \, d\mu(x) + C\|\mu\|, \tag{5}$$

where

$$p_\alpha(\mu) = \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} p_\alpha(x,y,z) \, d\mu(x) \, d\mu(y) \, d\mu(z).$$

In [**15**] the following useful lemma was proved:

Lemma 3. *Let $0 < \alpha < 1$, and let μ be a positive Borel measure with $0 < \theta^{*\alpha}(\mu, x) < \infty$ for μ-almost all $x \in \mathbf{R}^n$. Then*

$$\iiint p_\alpha(x_1, x_2, x_3) \, d\mu(x_1) \, d\mu(x_2) \, d\mu(x_3) = +\infty.$$

Remark. Lemma 3 together with (5) gives us that if $0 < \alpha < 1$, and if μ is a positive Borel measure with $0 < \theta^{*\alpha}(\mu, x) < \infty$ for μ-almost all $x \in \mathbf{R}^n$, then the α-Riesz transform is not bounded in $L^2(\mu)$.

3. Proof of Theorem 1. It is well known (see [**1**, page 56], for example) that if the α-Riesz transform $R^\alpha \mu$ is bounded in $L^2(\mu)$, then there exists some positive constant C_0 such that

$$\mu B(x,r) \le C_0 r^\alpha, \text{ for } x \in \mathbf{R}^n, \text{ and } r > 0.$$

Therefore, for all $x \in \mathbf{R}^n$ we have $\theta^{*\alpha}(\mu, x) \le C_0$. The remark after Lemma 3 tells us that, in this situation, L^2-boundedness of the α-Riesz transform can only happen if the upper density is zero. Hence, since we are assuming that the α-Riesz transform $R^\alpha \mu$ is bounded in $L^2(\mu)$, Theorem 1 will be proved if we show that the principal value $\lim_{\varepsilon \to 0} R_\varepsilon^\alpha \mu(x)$ exists for μ-almost all x in the set

$$Z = \{x \in \operatorname{spt}(\mu) : \theta^{*\alpha}(x, \mu) = 0\}.$$

The proof of this fact follows ideas of [**16**].

It suffices to verify the following statement. For every $\delta > 0$, there is an n_0 such that there exists a $K_\delta \subset Z$ for which $\mu(Z \setminus K_\delta) \leq \delta$ and

$$|R^\alpha_{2^{-n}}\mu(x) - R^\alpha_{2^{-m}}\mu(x)| \leq \delta, \text{ for } x \in K_\delta, \quad m, n > n_0. \tag{6}$$

Then setting $B = \cap_{j=1}^\infty \cup_{i=j}^\infty K_{2^{-i}}$, we find that $\mu(Z \setminus B) = 0$ and $\lim_{\varepsilon\to 0} R^\alpha_\varepsilon \mu(x)$ exists for $x \in B$.

For future reference, if $F, G \subset \mathbf{R}^n$ are μ-measurable, set

$$p_\mu(x, F, G) = \int_F \int_G p_\alpha(x, x_2, x_3)\, d\mu(x_2)\, d\mu(x_3)$$

and

$$p_\mu(F) = \int_F \int_F \int_F p_\alpha(x, x_2, x_3)\, d\mu(x_1)\, d\mu(x_2)\, d\mu(x_3).$$

Let $\gamma > 0$ be fixed later. For each m set

$$G_{\gamma,m} = \left\{ x \in Z : \mu B(x,r) \leq \gamma r^\alpha \text{ and } p_\mu(x, B(x,r), \mathbf{R}^n) \leq \gamma^2 \right. \\ \left. \text{for } 0 < r \leq \frac{1}{m} \right\}.$$

Then $G_{\gamma,m} \subset G_{\gamma,m+1}$. The L^2-boundedness of the α-Riesz transform together with (5) gives us $p_\mu(x, \mathbf{R}^n, \mathbf{R}^n) < +\infty$ for μ-almost all $x \in \mathbf{R}^n$ and hence $\lim_{r\to 0} p_\mu(x, B(x,r), \mathbf{R}^n) = 0$. Therefore,

$$\mu\left(Z \setminus \bigcup_m G_{\gamma,m} \right) = 0. \tag{7}$$

Fix $\gamma \leq (100\sqrt{3M+1})^{-1}\delta$ (where M is the constant in (4)). By (7) there is an m_1 such that $\mu(Z \setminus G_{\gamma,m_1}) \leq \delta/4$. Let $\rho > 0$ be fixed later. There are an open neighborhood U of G_{γ,m_1} and a compact subset K of G_{γ,m_1}, such that $\mu(U \setminus G_{\gamma,m_1}) \leq \rho$ and $\mu(G_{\gamma,m_1} \setminus K) \leq \rho$.

For a set V, let $\mu_{|V}$ denote the restriction of μ to V. For $x \in K$ and n, m such that $2^{-n}, 2^{-m} < d := \operatorname{dist}(K, U^c)$, we have

$$\begin{aligned} \left| R^\alpha_{2^{-n}}\mu(x) - R^\alpha_{2^{-m}}\mu(x) \right| &= \left| R^\alpha_{2^{-n}}\mu_{|U}(x) - R^\alpha_{2^{-m}}\mu_{|U}(x) \right| \\ &\leq \left| R^\alpha_{2^{-n}}\mu_{U\setminus K}(x) - R^\alpha_{2^{-m}}\mu_{U\setminus K}(x) \right| \\ &\quad + \left| R^\alpha_{2^{-n}}\mu_{|K}(x) - R^\alpha_{2^{-m}}\mu_{|K}(x) \right| \\ &= I(x) + II(x). \end{aligned}$$

Notice that $I(x) \leq 2R_*^\alpha \mu_{|U\setminus K}(x)$, and set

$$K_1 = \{x \in K : R_*^\alpha \mu_{|U\setminus K}(x) \leq \delta/4\}.$$

We claim that (choosing $\rho > 0$ appropriately) we have $\mu(K \setminus K_1) \leq \delta/4$. To see this, notice that the $L^2(\mu)$-boundedness of R_*^α gives us

$$\begin{aligned}\mu(K \setminus K_1) &= \mu\Big\{x \in K : R_*^\alpha \mu_{|U\setminus K}(x) > \delta/4\Big\}\\ &\leq \frac{16}{\delta^2}\|R_*^\alpha\|_{L^2(\mu)}^2 \mu(U \setminus K)\\ &\leq \|R_*^\alpha\|_{L^2(\mu)}^2 \frac{32\rho}{\delta^2};\end{aligned}$$

hence, choosing

$$\rho \leq \frac{\delta^3}{128\|R_*^\alpha\|_{L^2(\mu)}^2},$$

we have $\mu(K \setminus K_1) \leq \delta/4$ and

$$I(x) \leq \frac{\delta}{2} \text{ for } x \in K_1.$$

Now we deal with the term $II(x)$. Since we are assuming that the α-Riesz transform is bounded in $L^2(\mu)$, we can consider a weak * limit operator $\widetilde{R}^\alpha$ of the truncated operators $\{R_{\varepsilon_n}^\alpha\}_n$ in the Banach space of the operators bounded in $L^2(\mu)$, for some sequence $\{\varepsilon_n\}_n \to 0$. Then, for a compactly supported $f \in L^2(\mu)$ we have $\widetilde{R}^\alpha f(y) = R^\alpha f(y)$ for μ-almost all $y \notin \operatorname{spt}(f)$ (this is a consequence of the fact that if $y \in \operatorname{spt}(\mu)\setminus\operatorname{spt}(f)$ is a Lebesgue point of $\widetilde{R}^\alpha f$; then $\widetilde{R}^\alpha f(y) = R^\alpha f(y)$).

The Lebesgue points of $\widetilde{R}^\alpha \mu_{|K}$ with respect to $\mu_{|K}$ satisfy

$$\lim_{r\to 0} \frac{1}{\mu_{|K}B(x,r)} \int_{B(x,r)} \widetilde{R}^\alpha \mu_{|K}(y)\, d\mu_{|K}(y) = \widetilde{R}^\alpha \mu_{|K}(x).$$

Hence there exists some $\beta > 0$ such that the set

$$K_\delta = \Big\{x \in K_1 : \Big|\frac{1}{\mu_{|K}B(x,r)} \int_{B(x,r)} \widetilde{R}^\alpha \mu_{|K}\, d\mu_{|K} - \widetilde{R}^\alpha \mu_{|K}(x)\Big| \leq \frac{\delta}{100}, \quad \text{for } 0 < r < \beta, \Big\}$$

satisfies $\mu(K_1 \setminus K_\delta) \le \delta/4$. Choose $\beta > 0$ such that

$$0 < \beta \le \min\left(d, \frac{1}{2m_1}, \frac{\delta}{1600 C_1 C_0 m_1}\right),$$

where $C_1 > 0$ will be determined later.

We claim that, for all $x \in K_\delta$,

$$II(x) \le \frac{\delta}{2}, \text{ if } 2^{-n}, 2^{-m} < \beta, \tag{8}$$

so that if we take n_0 such that $2^{-n_0} \le \beta$ and $\rho \le \delta/4$, then (6) holds and

$$\mu(Z \setminus K_\delta) \le \mu(Z \setminus G_{\gamma,m_1}) + \mu(G_{\gamma,m_1} \setminus K) + \mu(K \setminus K_1) + \mu(K_1 \setminus K_\delta) \le \delta.$$

So we are only left with the task of proving claim (8). Assume $m < n$. We distinguish between two cases:

1. For $k = 2, \dots, n-m$, we have $2^{1+\alpha}\mu_{|K}B(x, 2^{k-1-n}) < \mu_{|K}B(x, 2^{k-n})$. Then

$$\begin{aligned} II(x) &= \left| R^\alpha_{2^{-n}}\mu_{|K}(x) - R^\alpha_{2^{-m}}\mu_{|K}(x) \right| \\ &\le \sum_{k=1}^{n-m} \int_{2^{k-1-n} \le |x-y| \le 2^{k-n}} \frac{d\mu_{|K}(y)}{|y-x|^\alpha} \\ &\le \sum_{k=1}^{n-m} \frac{2^{(1+\alpha)(k-n+m)}\mu_{|K}B(x, 2^{-m})}{2^{(k-1-n)\alpha}} \\ &= 2^\alpha \frac{\mu_{|K}B(x, 2^{-m})}{2^{-m\alpha}} 2^{m-n} \sum_{k=1}^{n-m} 2^k \\ &\le 2^{1+\alpha} \frac{\mu_{|K}B(x, 2^{-m})}{2^{-m\alpha}} \le 2^{1+\alpha}\gamma \le \frac{\delta}{2}. \end{aligned} \tag{9}$$

2. For some $2 \le k \le n-m$, we have $2^{1+\alpha}\mu_{|K}B(x, 2^{k-1-n}) \ge \mu_{|K}B(x, 2^{k-n})$. Let $2 \le m^* \le M^* \le n-m$ be the least and largest integers with this property. Then arguing as in (9) we obtain

$$\left| R^\alpha_{2^{-n}}\mu_{|K}(x) - R^\alpha_{2^{m^*-n-1}}\mu_{|K}(x) \right| \le 2^{1+\alpha} \frac{\mu_{|K}B(x, 2^{m^*-n-1})}{2^{(m^*-n-1)\alpha}} \le 2^{1+\alpha}\gamma$$

and

$$\left|R^\alpha_{2^{M^*-n}}\mu_{|K}(x) - R^\alpha_{2^{-m}}\mu_{|K}(x)\right| \le 2^{1+\alpha}\frac{\mu_{|K}B(x,2^{-m})}{2^{-m\alpha}} \le 2^{1+\alpha}\gamma.$$

Moreover,

$$\left|R^\alpha_{2^{m^*-n-1}}\mu_{|K}(x) - R^\alpha_{2^{m^*-n}}\mu_{|K}(x)\right| \le \frac{\mu B(x,2^{m^*-n})}{2^{(m^*-n-1)\alpha}} \le 2^\alpha\gamma.$$

Set $F = K \cap B(x,(2m_1)^{-1})$. Putting all of the above estimates together and using that $2^{m^*-n}, 2^{M^*-n} \le \beta \le (2m_1)^{-1}$, we obtain

$$\begin{aligned} II(x) &= \left|R^\alpha_{2^{-n}}\mu_{|K}(x) - R^\alpha_{2^{-m}}\mu_{|K}(x)\right| \\ &\le 2^{2+\alpha}\gamma + 2^\alpha\gamma + \left|R^\alpha_{2^{m^*-n}}\mu_{|K}(x) - R^\alpha_{2^{M^*-n}}\mu_{|K}(x)\right| \\ &\le 10\gamma + \left|R^\alpha_{2^{m^*-n}}\mu_{|K}(x) - R^\alpha_{2^{M^*-n}}\mu_{|K}(x)\right| \\ &\le \frac{\delta}{10} + \left|R^\alpha_{2^{m^*-n}}\mu_{|F}(x) - R^\alpha_{2^{M^*-n}}\mu_{|F}(x)\right|. \end{aligned}$$

If we show that for $x \in K_\delta$,

$$\left|R^\alpha_{2^{m^*-n}}\mu_{|F}(x) - \left[\widetilde{R}^\alpha\mu_{|K}(x) - R^\alpha\mu_{|K\setminus F}(x)\right]\right| \le \frac{\delta}{5} \tag{10}$$

and

$$\left|R^\alpha_{2^{M^*-n}}\mu_{|F}(x) - \left[\widetilde{R}^\alpha\mu_{|K}(x) - R^\alpha\mu_{|K\setminus F}(x)\right]\right| \le \frac{\delta}{5} \tag{11}$$

hold, then claim (8) will be proved.

To prove (10) split it as follows:

$$\left|R^\alpha_{2^{m^*-n}}\mu_{|F}(x) - \left[\widetilde{R}^\alpha\mu_{|K}(x) - R^\alpha\mu_{|K\setminus F}(x)\right]\right| \le A + B,$$

where

$$A = \left|R^\alpha_{2^{m^*-n}}\mu_{|F}(x) - \frac{1}{\mu_{|K}B(x,2^{m^*-n-1})}\int_{B(x,2^{m^*-n-1})}\widetilde{R}^\alpha\mu_{|F}(y)\,d\mu_{|K}(y)\right|$$

and

$$B = \left| \frac{1}{\mu_{|K} B(x, 2^{m^*-n-1})} \int_{B(x,2^{m^*-n-1})} \widetilde{R}^\alpha \mu_{|F} d\mu_{|K} - \left[\widetilde{R}^\alpha \mu_{|K}(x) - R^\alpha \mu_{|K \setminus F}(x) \right] \right|.$$

We first deal with term A.

(12)

$$A \le \left| R^\alpha_{2^{m^*-n}} \mu_{|F}(x) - \frac{1}{\mu_{|K} B(x, 2^{m^*-n-1})} \int_{B(x,2^{m^*-n-1})} \widetilde{R}^\alpha \mu_{|F \setminus B(x,2^{m^*-n})}(y)\, d\mu_{|K}(y) \right| + \left| \frac{1}{\mu_{|K} B(x, 2^{m^*-n-1})} \int_{B(x,2^{m^*-n-1})} \widetilde{R}^\alpha \mu_{|F \cap B(x,2^{m^*-n})}(y)\, d\mu_{|K}(y) \right|.$$

Notice that, since $x \in Z$ and diam $(F) \le 1/m_1$, then $\mu(F \cap B(x,r)) \le \gamma r^\alpha$, for all $r > 0$. Hence, integrating on annulus, we get for μ-almost all $y \in B(x, 2^{m^*-n-1})$ and some positive constant $C_1 \le 3$,

$$\begin{aligned} \left| R^\alpha_{2^{m^*-n}} \mu_{|F}(x) - \widetilde{R}^\alpha \mu_{|F \setminus B(x,2^{m^*-n})}(y) \right| &= \left| R^\alpha \mu_{|F \setminus B(x,2^{m^*-n})}(x) - R^\alpha \mu_{|F \setminus B(x,2^{m^*-n})}(y) \right| \\ &\le C_1 |x-y| \sum_{k=0}^\infty \frac{\mu B(x, 2^{k+1+m^*-n})}{(2^{k+m^*-n})^{1+\alpha}} \\ &\le C_1 2^\alpha \gamma \le 6\gamma. \end{aligned}$$

Now integration on $F \cap B(x, 2^{m^*-n-1})$ gives

$$\left| R^\alpha_{2^{m^*-n}} \mu_{|F}(x) - \frac{1}{\mu_{|K} B(x, 2^{m^*-n-1})} \times \int_{B(x,2^{m^*-n-1})} \widetilde{R}^\alpha \mu_{|F \setminus B(x,2^{m^*-n})}\, d\mu_{|K} \right| \le 6\gamma.$$

For the second term in (12), we will use that the $L^2(\mu_{|F})$ norm of the α-Riesz transform is small. By Cauchy-Schwartz inequality and (4),

$$\begin{aligned}
(13)\quad & \left|\frac{1}{\mu_{|K}B(x,2^{m^*-n-1})}\int_{B(x,2^{m^*-n-1})}\widetilde{R}^\alpha\mu_{F\cap B(x,2^{m^*-n})}(y)\,d\mu_{|K}(y)\right| \\
&\le \frac{\|\widetilde{R}^\alpha\mu_{|F\cap B(x,2^{m^*-n})}\|_{L^2(\mu|F\cap B(x,2^{m^*-n}))}}{\mu_{|K}B(x,2^{m^*-n-1})^{1/2}} \\
&\le \frac{\sup_{\varepsilon>0}\|R^\alpha_\varepsilon\mu_{|F\cap B(x,2^{m^*-n})}\|_{L^2(\mu|F\cap B(x,2^{m^*-n}))}}{\mu_{|K}B(x,2^{m^*-n-1})^{1/2}} \\
&\le \frac{\left[p_\mu(F\cap B(x,2^{m^*-n}))+3A\gamma^2\mu(F\cap B(x,2^{m^*-n}))\right]^{1/2}}{\mu_{|K}B(x,2^{m^*-n-1})^{1/2}} \\
&\le \gamma\sqrt{3M+1}\left(\frac{\mu_{|K}B(x,2^{m^*-n})}{\mu_{|K}B(x,2^{m^*-n-1})}\right)^{1/2} \\
&\le 2\gamma\sqrt{3M+1},
\end{aligned}$$

where inequality (13) comes from

$$p_\mu(y,F,\mathbf{R}^n)\le p_\mu\left(y,B\left(x,\frac{1}{m_1}\right),\mathbf{R}^n\right)\le\gamma^2,\ \text{ for } y\in F.$$

Hence,

$$A\le(6+2\sqrt{3M+1})\gamma\le\frac{2\delta}{25}.$$

To estimate B, we will use that x is a Lebesgue point of $\widetilde{R}^\alpha\mu_{|K}$ with respect to $\mu_{|K}$. Write

$$\begin{aligned}
B\le&\left|\frac{1}{\mu_{|K}B(x,2^{m^*-n-1})}\int_{B(x,2^{m^*-n-1})}\widetilde{R}^\alpha\mu_{|K}(y)\,d\mu_{|K}(y)-\widetilde{R}^\alpha\mu_{|K}(x)\right| \\
&+\left|R^\alpha\mu_{|K\setminus B(x,(2m_1)^{-1})}(x)-\frac{1}{\mu_{|K}B(x,2^{m^*-n-1})}\right. \\
&\qquad\qquad\left.\int_{B(x,2^{m^*-n-1})}\widetilde{R}^\alpha\mu_{|K\setminus B(x,(2m_1)^{-1})}\,d\mu_{|K}\right| \\
&=: B_1+B_2.
\end{aligned}$$

The definition of K_δ gives us $B_1\le\delta/100$. For B_2, notice that if $|x-y|\le 2^{m^*-n-1}\le\beta$, then integrating on annulus and using the

α-growth of the measure (with constant C_0) we get,

$$\begin{aligned}\left|R^\alpha \mu_{|K\setminus B(x,(2m_1)^{-1})}(x) - R^\alpha \mu_{|K\setminus B(x,(2m_1)^{-1})}(y)\right| \\ &\leq C_1\beta \sum_{k=0}^{\infty} \frac{\mu_{|K} B(x, 2^{k+1}(2m_1)^{-1})}{(2^k(2m_1)^{-1})^{1+\alpha}} \\ &\leq 2^{\alpha+2} C_1 m_1 \beta C_0.\end{aligned}$$

Hence, since for μ-almost all $y \in B(x, 2^{m^*-n-1})$,

$$\widetilde{R}^\alpha \mu_{|K\setminus B(x,(2m_1)^{-1})}(y) = R^\alpha \mu_{|K\setminus B(x,(2m_1)^{-1})}(y),$$

if we integrate on $B(x, 2^{m^*-n-1})$ we obtain

$$B_2 \leq 8C_1 m_1 \beta C_0 \leq \frac{\delta}{200},$$

which implies

$$B \leq \frac{3\delta}{200}.$$

Therefore, for all $x \in K_\delta$,

$$A + B \leq \frac{2\delta}{25} + \frac{3\delta}{200} \leq \frac{\delta}{5},$$

which is (10). Inequality (11) can be obtained in a similar way. □

4. Proof of Theorem 2. For the proof Theorem 2 a version of a deep result of Nazarov, Treil and Volberg [**11**] will be needed, (see [**15**]). The result reads as follows:

Theorem 4. *Let $0 < \alpha < n$ and μ be a positive measure on $\mathbf{R}^n$ such that $\theta^{*\alpha}(\mu, x) < +\infty$ for μ-almost all x. Assume that $R^\alpha_* \mu(x) < +\infty$ for μ-almost all x. Then there is a set F with $\mu(F) > 0$ such that the α-Riesz potential R^α is bounded in $L^2(\mu_{|F})$.*

Proof of Theorem 2. It is obvious that if the principal value

$$\lim_{\varepsilon \to 0} R^\alpha_\varepsilon \mu(x)$$

exists for μ-almost all $x \in \mathbf{R}^n$, we have $R_*^\alpha \mu(x) < \infty$ for μ-almost all $x \in \mathbf{R}^n$. Since all the hypotheses of Theorem 4 hold, there exists a compact set $F \subset \mathbf{R}^n$ with $\mu(F) > 0$ such that the α-Riesz transform is bounded in $L^2(\mu_{|F})$. Therefore, $p_\alpha(\mu_{|F}) < \infty$ by the standard argument reproduced in (5). This contradicts Lemma 3. Therefore, the set where the principal value (1) does not exist has positive μ-measure. □

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