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SELECTION THEOREMS AND MINIMAL MAPPINGS IN A CLUSTER SETTING

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ABSTRACT. The paper introduces a generalized concept of the cluster points of multi-function as a unified approach to the study of selection theorems, closed graph theorems and minimal multi-functions. The crucial notion is that of generalized lower and upper quasi continuity with respect to a given cluster system.

1. Introduction. The basic notion of this paper is a cluster process with respect to a given nonempty system $\mathcal{E}$ of the nonempty subsets of a topological space X, and the investigation is focused on the mutual connection between the original multifunction F and its resultant cluster multi-function $\mathcal{E}_F$. This concept enables us to describe the techniques for finding selections and to study the properties of multi-functions from many points of view. The main results of the paper concern the connection between a semi-$\mathcal{E}$-continuous multi-function and its $\mathcal{E}$-cluster multi-function. Further, the theorems concerning closed graph, minimality and the existence of quasi continuous selection for u-$\mathcal{E}$-continuous multi-functions are given.

2. Basic symbols and terminology. In the sequel X, Y are topological spaces, the symbols $\overline{A}$, A°, $\mathbf{N}$ and $\mathbf{R}$, respectively, denote the closure of a set A, the interior of A, the natural numbers $\{0, 1, 2, \dots\}$, and the reals with their usual topology. A compact/locally compact space is understood as a T_2 space, hence any locally compact space is regular (even completely regular).

A set S is quasi-open (terminology by [**20**] where a connection between quasi-continuity and the quasi-open sets (also called semi-open) was accented), if for any open set H intersecting S there is a

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nonempty open set $H_0 \subset H$ such that $H_0 \subset S$, or equivalently, if S is of the form $G \cup A$, where G is an open and A is a nowhere dense subset of $\overline{G}$.

By a multi-function F we understand a nonempty subset of the Cartesian product $X \times Y$ with the values $\{y \in Y : [x,y] \in F\} =: F(x)$ (it can be empty valued at some points). By $\mathrm{Dom}\,(F)$, we denote domain of F, i.e., the set of all arguments x in which $F(x)$ is nonempty. For a multi-function with domain A we use notation $F : A \to Y$ or we say that F is defined on A. If $\mathrm{Dom}\,(F)$ is a dense set, F is said to be densely defined. By $\overline{F}$, we denote a multifunction given by the closure of F in $X \times Y$.

A function $f : A \to Y$ is understood as a strictly single valued multi-function with values $\{f(x)\}$, $x \in A$.

A multi-function S is a submulti-function of F (S can be empty valued at some points of $\mathrm{Dom}\,(F)$), if $S \subset F$, i.e., $S(x) \subset F(x)$ for all $x \in X$. That means $\mathrm{Dom}\,(S) \subset \mathrm{Dom}\,(F)$. To stress that $S(x) \neq \varnothing$ for all $x \in \mathrm{Dom}\,(F)$ we will say that S is a nonempty valued submulti-function of F, hence $\mathrm{Dom}\,(S) = \mathrm{Dom}\,(F)$. A function $f : A \to Y$ is a selection of a multifunction F on a set $A \subset X$, if $f(x) \in F(x)$ for all $x \in A$ ($\mathrm{Dom}\,(f) = A \subset \mathrm{Dom}\,(F)$). If $\mathrm{Dom}\,(f) = \mathrm{Dom}\,(F)$, we say briefly f is a selection of F.

For any set $W \subset Y$ the upper and lower inverse images are defined as $F^+(W) = \{x \in X : F(x) \subset W\}$, $F^-(W) = \{x \in X : F(x) \cap W \neq \varnothing\}$. Let us note that $X \backslash F^-(W) = F^+(Y \backslash W)$ and $X \backslash F^+(W) = F^-(Y \backslash W)$.

The basic types of continuity for a multi-function are lower and upper semi continuity, briefly lsc and usc.

Definition 1. A multi-function F is lsc (usc) at x, if for any open set V for which $F(x) \cap V \neq \varnothing$ ($F(x) \subset V$) there is an open set $U \ni x$ such that $F(u) \cap V \neq \varnothing$ ($F(u) \subset V$) for any $u \in U$. The multi-function F is lsc (usc), if it is so at any point $x \in \mathrm{Dom}\,(F)$. That means $F^-(V)$ ($F^+(V)$) is open for any open set $V \subset Y$. Finally, F is *usco* at x, if F is nonempty compact valued and usc at x.

3. Cluster systems. Any nonempty system $\mathcal{E} \subset 2^X \setminus \{\varnothing\}$ will be called a cluster system. For some special cluster systems we will use special notation. For example, $\mathcal{O}$, $\mathcal{B}r$, $\mathcal{A}$, $\mathcal{D}$ is a cluster system containing all nonempty open sets, all sets which are of second category with the Baire property, all sets which are not nowhere dense, all sets which are of second category, respectively (all mentioned topological properties are meant with respect to topology on X). Further, $\mathcal{B}$ denotes a cluster system such that $\mathcal{B} \subset \mathcal{O} \cup \mathcal{B}r$ (any $E \in \mathcal{B}$ is the nonempty open set or a set of second category with the Baire property) and $\mathcal{E}^\circ = 2^X \setminus \{\varnothing\}$. Let us stress that $\mathcal{E}$ is always a nonempty system and any set $E \in \mathcal{E}$ is nonempty. Apart from these, we can consider a cluster system containing only subsets of the domain of F or a cluster system derived from some property of F (for example, $\mathcal{C}^u/\mathcal{C}^l = \{A : \varnothing \neq A \subset C^u/C^l\}$, where C^u/C^l is the set of all points at which F is *usc*/*lsc*. For a function, the resultant cluster multi-function is called a densely continuous form [**7**]).

The next two definitions introduce the notion of an $\mathcal{E}$-cluster point and lower and upper $\mathcal{E}$-continuity as a basic tool for investigation of the properties of multi-functions. In this form it was firstly studied in [**12**], next in [**9, 14, 15**], for functions see [**5**].

Definition 2. A point $y \in Y$ is an $\mathcal{E}$-cluster point of F at a point x, if for any open sets $V \ni y$ and $U \ni x$, there is a set $E \in \mathcal{E}$, $E \subset U$ such that $F(e) \cap V \neq \varnothing$ for any $e \in E$, i.e., $E \subset U \cap F^-(V)$. The set of all $\mathcal{E}$-cluster points of F at x is denoted by $\mathcal{E}_F(x)$. A multi-function $\mathcal{E}_F$ with the values $\mathcal{E}_F(x)$ is called an $\mathcal{E}$-cluster multi-function of F, provided that $\mathcal{E}_F(x)$ is nonempty at least at one point $x \in X$.

Remark 1. Note that $\mathcal{E}_F$ is empty-valued outside of the closure of $\mathrm{Dom}\,(F)$. At the points from $\overline{\mathrm{Dom}\,(F)}$ it can also be empty valued. For example, if $F \subset \mathbf{R} \times \mathbf{R}$ is defined $F(x) = \{1\}$ for $x > 0$, F is identical with the Dirichlet function for $x < -1$ and it is empty valued otherwise, then $\mathcal{O}_F(x) = \{1\}$ for $x \geq 0$ and $\mathcal{O}_F$ is empty valued otherwise.

Definition 3 (Semi $\mathcal{E}$-continuity). A multi-function F is l-$\mathcal{E}$-continuous (u-$\mathcal{E}$-continuous) at $x \in \mathrm{Dom}\,(F)$, if for any open sets V, U such that $V \cap F(x) \neq \varnothing$ ($F(x) \subset V$) and $x \in U$ there is a set $E \in \mathcal{E}$, $E \subset U \cap \mathrm{Dom}\,(F)$ such that $F(e) \cap V \neq \varnothing$ ($F(e) \subset V$) for any $e \in E$. Correspondingly the global definition is given by the local one at each point of $\mathrm{Dom}\,(F)$. If F is l-$\mathcal{E}$-continuous and u-$\mathcal{E}$-continuous, then we say F is $\mathcal{E}$-continuous.

For the system $\mathcal{B}r$, the l-$\mathcal{B}r$-continuity (u-$\mathcal{B}r$-continuity) will be called lower (upper) Baire continuity, respectively ([**1, 12**]) and for $\mathcal{O}$ it is the well-known notion of lower (upper) quasi continuity ([**20**]).

In [**15**] we can find the next example of functions/multi-functions and the cluster systems which are helpful to understand the definition of semi $\mathcal{E}$-continuity and the cluster multifunction.

Example 1. Let $X = Y = \mathbf{R}$ and $\mathcal{E} \in \{\mathcal{O}, \mathcal{A}, \mathcal{B}r\}$. Consider the next functions/multi-functions ($\mathbf{Q}$ is the set of all rational numbers).

(1) $f(x) = 1$, if $x \geq 0$ and $f(x) = 0$ otherwise,

(2) $F(x) = \langle 0, 1\rangle$, if $x \in \mathbf{Q}$ and $F(x) = \{0\}$ otherwise,

(3) g is a characteristic function of the Cantor set,

(4) $G(x) = \{x/|x|\}$ if $x \neq 0$ and $G(0) = S$, $\varnothing \neq S \subset \mathbf{R}$.

(1) The function f is $\mathcal{E}$-continuous and $\mathcal{E}_f(x) = \{1\}$ for $x > 0$, $\mathcal{E}_f(0) = \{0, 1\}$ and $\mathcal{E}_f(x) = \{0\}$ for $x < 0$.

(2) The multi-function F is $\mathcal{A}$-continuous on $\mathbf{R}$, u-$\mathcal{B}r$-continuous on $\mathbf{R}$ and l-$\mathcal{B}r$-continuous on $\mathbf{R} \setminus \mathbf{Q}$, u-$\mathcal{O}$-continuous on $\mathbf{Q}$ and l-$\mathcal{O}$-continuous on $\mathbf{R} \setminus \mathbf{Q}$. Moreover, for any $x \in \mathbf{R}$ we have $\mathcal{A}_F(x) = \langle 0, 1\rangle$ and $\mathcal{O}_F(x) = \mathcal{B}r_F(x) = \{0\}$.

(3) The function g is not $\mathcal{E}$-continuous on the Cantor set and it is $\mathcal{E}$-continuous otherwise. $\mathcal{E}_g(x) = \{0\}$ for any $x \in \mathbf{R}$.

(4) G is $\mathcal{E}$-continuous if and only if $S \subset \{-1, 1\}$ and u-$\mathcal{E}$-continuous if and only if $S \cap \{-1, 1\} \neq \varnothing$. $\mathcal{E}_G(x) = \{x/|x|\}$ for $x \neq 0$ and $\mathcal{E}_G(0) = \{-1, 1\}$.

Using the cluster concept, l-$\mathcal{E}$-continuity at x can by characterized by the inclusion $\mathcal{E}_F(x) \supset F(x) \neq \varnothing$ and global l-$\mathcal{E}$-continuity by the inclusion $\mathcal{E}_F \supset F$. The natural question arises if u-$\mathcal{E}$-continuity can be characterized by the inclusion $\mathcal{E}_F \subset F$. The answer is negative. The function f from Example 1 is u-$\mathcal{O}$-continuous, but $\mathcal{O}_f \not\subset f$. On the other hand, for the multi-function F from Example 1 we have $\mathcal{O}_F \subset F$, but F is not u-$\mathcal{O}$-continuous.

Lemma 1. *For any net $\{x_t\}$ converging to x and $y_t \in \mathcal{E}_F(x_t)$, $\mathcal{E}_F(x)$ contains all accumulation points of the net $\{y_t\}$.*

Proof. Let y be an accumulation point of $\{y_t\}$. Then for any open sets $V \ni y$ and $U \ni x$ there are frequently given indexes t' such that

$x_{t'} \in U$ and $y_{t'} \in V \cap \mathcal{E}_F(x_{t'})$. Hence there is an $E \in \mathcal{E}$, $E \subset U$ such that $F(e) \cap V \neq \varnothing$ for any $e \in E$. That means $y \in \mathcal{E}_F(x)$. $\square$

Remark 2. (a) From Lemma 1 it follows that multi-function $\mathcal{E}_F$ has a closed graph, hence also closed values. That means $\mathcal{E}_F^-(K)$ is closed for any nonempty compact set K or equivalently, $\mathcal{E}_F^+(G) = X \setminus \mathcal{E}_F^-(Y \setminus G)$ is open for any open G with compact complement. Hence $\mathcal{E}_F$ is *usc* with respect to topology given by the original open sets with compact complement in Y (compact complement topology). This continuity is also called c-upper semi continuity [**8, 19**]. Consequently, if $\mathcal{E}_F^-(K)$ is dense in an open set G, then $G \subset \mathcal{E}_F^-(K)$.

(b) If $\mathcal{E}^1 \subset \mathcal{E}^2$, then $\mathcal{E}_F^1 \subset \mathcal{E}_F^2$ and if $F_1 \subset F_2$, then $\mathcal{E}_{F_1} \subset \mathcal{E}_{F_2}$.

(c) $\mathcal{E}_F \subset \overline{F} = \mathcal{E}_F^\circ$. Hence any multi-function F is l-$\mathcal{E}^\circ$-continuous.

4. Lower $\mathcal{E}$-continuity and closed graph theorems. In this paragraph the notion of $\mathcal{E}$-closedness of the graph will be introduced, and we present a quasi variant of the results from [**17**] in the cluster setting of multi-functions. We will also deal with lower quasi continuity and with the structure of the domain of $\mathcal{E}_F$. The existence of a lower quasi continuous submulti-function is given, provided there is an l-$\mathcal{E}$-continuous submulti-function.

Lemma 2. *If F is l-$\mathcal{E}$-continuous, then $\mathcal{E}_F$ is l-$\mathcal{E}$-continuous and $\mathcal{E}_F = \mathcal{E}_{\mathcal{E}_F}$.*

Proof. From the inclusion $F \subset \mathcal{E}_F$ it follows that $\mathcal{E}_F \subset \mathcal{E}_{\mathcal{E}_F}$ (see Remark 2). Hence $\mathcal{E}_F$ is l-$\mathcal{E}$-continuous. Let $y \in \mathcal{E}_{\mathcal{E}_F}(x)$, $U \ni x$, $V \ni y$, U, V are open. Then there is an $E \in \mathcal{E}$ and $E \subset U \cap \mathcal{E}_F^-(V)$. Let $y_0 \in \mathcal{E}_F(e) \cap V$, $e \in E$. Hence, there is an $E_0 \in \mathcal{E}$ such that $E_0 \subset U \cap F^-(V)$. That means $y \in \mathcal{E}_F(x)$ and $\mathcal{E}_F = \mathcal{E}_{\mathcal{E}_F}$. $\square$

Lemma 3. *If F is l-$\mathcal{E}^1$-continuous and l-$\mathcal{E}^2$-continuous, then $\mathcal{E}_F^1 = \mathcal{E}_F^2$.*

Proof. We will prove the inclusion $\mathcal{E}_F^1 \subset \mathcal{E}_F^2$ (the inverse is dual). Let $y \in \mathcal{E}_F^1(x)$, $U \ni x$, $V \ni y$, U, V are open. Then there is an $E^1 \in \mathcal{E}^1$ such that $E^1 \subset U \cap F^-(V)$. The existence of a set $E^2 \in \mathcal{E}^2$ such that $E^2 \subset U \cap F^-(V)$ follows from l-$\mathcal{E}^2$-continuity at chosen $e^1 \in E^1$. Hence, $y \in \mathcal{E}_F^2(x)$ and $\mathcal{E}_F^1 = \mathcal{E}_F^2$. $\square$

Since any multifunction F is l-$\mathcal{E}_F^\circ$-continuous (Remark 2), we have the next global characterization of l-$\mathcal{E}$-continuity.

Corollary 1. *A multi-function F is l-$\mathcal{E}$-continuous, if and only if $\mathcal{E}_F = \overline{F}$.*

As we have mentioned, if $F(x) \neq \varnothing$, the local l-$\mathcal{E}$-continuity is characterized by the inclusion $F(x) \subset \mathcal{E}_F(x)$. Since $\mathcal{E}_F(x)$ is closed, F is l-$\mathcal{E}$-continuous at x if and only if the inclusion $\overline{F(x)} \subset \mathcal{E}_F(x)$ holds.

Lemma 4. *Let Y be locally compact, and let F be l-$\mathcal{E}$-continuous, $\mathcal{E} \subset \mathcal{A}$. Then $\mathcal{E}_F(x) \neq \varnothing$ for any $x \in \overline{\operatorname{Dom}(F)} \setminus S$, where S is nowhere dense and the domain of $\mathcal{E}_F$ is a nonempty quasi open set.*

Proof. Let $x \in \overline{\operatorname{Dom}(F)}$. Then for any open set G containing x there is an $a \in G \cap \operatorname{Dom}(F)$. Since $\varnothing \neq F(a) \subset \mathcal{E}_F(a)$, we can take $y \in \mathcal{E}_F(a)$. Then for an open set V containing y and $\overline{V}$ compact, there is an $E \in \mathcal{E}$, $E \subset G$, such that $F(e) \cap \overline{V} \neq \varnothing$ for any $e \in E \subset \operatorname{Dom}(F)$. Since E is not nowhere dense, there is an open set $H \subset G$ such that E is dense in H. Then, for any $z \in H \cap E$, we have $\varnothing \neq F(z) \cap \overline{V} \subset \mathcal{E}_F(z) \cap \overline{V}$. Then $\mathcal{E}_F(z) \neq \varnothing$ for any $z \in H \subset G$, by Remark 2. That means $\mathcal{E}_F$ is nonempty valued at any point of $\overline{\operatorname{Dom}(F)}$ except for a nowhere dense set S. Now we will sketch that $\operatorname{Dom}(\mathcal{E}_F)$ is quasi open. Let G be open and $G \cap \operatorname{Dom}(\mathcal{E}_F) \neq \varnothing$. Then there is an $a \in G \cap \operatorname{Dom}(F)$ and the proof can continue as above. $\square$

Corollary 2. *Let Y be locally compact and F l-$\mathcal{E}$-continuous, $\mathcal{E} \subset \mathcal{A}$. If $\operatorname{Dom}(F)$ is dense, then $\mathcal{E}_F$ is nonempty valued except for a nowhere dense set.*

Theorem 1. *Let Y be a locally compact topological space, and let F be l-$\mathcal{E}$-continuous, $\mathcal{E} \subset \mathcal{A}$. Then $\mathcal{E}_F$ is lower quasi continuous with a quasi open domain.*

Proof. Let U, V be open, $a \in U, V \cap \mathcal{E}_F(a) \neq \varnothing$. Then there is an open set V_0 with compact closure, $\overline{V_0} \subset V$ and $V_0 \cap \mathcal{E}_F(a) \neq \varnothing$. That means there is an $E \in \mathcal{E}$, $E \subset U \cap F^-(\overline{V_0})$. A multi-function F is l-$\mathcal{E}$-continuous, i.e., $\varnothing \neq F(x) \subset \mathcal{E}_F(x)$ for all $x \in \operatorname{Dom}(F)$, hence $E \subset F^-(\overline{V_0}) \subset \mathcal{E}_F^-(\overline{V_0})$. Since E is not nowhere dense, there is an open set $G \subset U$ such that E is dense in G. Since $\overline{V_0}$ is compact, $G \subset \mathcal{E}_F^-(\overline{V_0}) \subset \mathcal{E}_F^-(V)$ (by Remark 2), that means $\mathcal{E}_F$ is lower quasi continuous at a. Finally, $\mathcal{E}_F$ has a quasi open domain, by Lemma 4. $\square$

Definition 4. A multi-function F has an $\mathcal{E}$-closed graph at x, if $\mathcal{E}_F(x) \subset F(x)$ and F has an $\mathcal{E}$-closed graph, if it has an $\mathcal{E}$-closed graph at any point from X. If F has an $\mathcal{E}^\circ$-closed graph/$\mathcal{E}^\circ$-closed graph at x, i.e., $\overline{F} = \mathcal{E}_F^\circ \subset F/\mathcal{E}_F^\circ(x) \subset F(x)$, then we use standard terminology that F has a closed graph/closed graph at x.

Remark 3. The notion of $\mathcal{E}$-closedness of the graph is more general than closedness of the graph, because if F has a closed graph, then $\mathcal{E}_F \subset \overline{F} = F$. On the other hand, multi-function F from Example 1 has a $\mathcal{B}r$-closed graph ($\mathcal{B}r_F(x) = \{0\}$ for all x), but its graph is not closed.

The next theorem deals with the existence of a lower quasi continuous submulti-function with closed graph, provided it has an l-$\mathcal{E}$-continuous submulti-function (see item (1)). Moreover, its item (2) can be considered as a quasi variant of closed graph theorems. Nearly continuous functions with the closed graph were investigated in [**17**].

Theorem 2. *Let Y be locally compact, and let F have an $\mathcal{E}$-closed graph, $\mathcal{E} \subset \mathcal{A}$. If F has an l-$\mathcal{E}$-continuous* (*nonempty valued*) *submulti-function, then*

(1) *F has a lower quasi continuous* (*nonempty valued*) *submulti-function with closed graph, which is defined on a quasi open set.*

(2) *Moreover, if F is l-$\mathcal{E}$-continuous* (*densely defined l-$\mathcal{E}$-continuous*), *then F is lower quasi continuous with a quasi open domain* (*lower quasi continuous with a quasi open domain, which complement is a nowhere dense*) *and F has a closed graph.*

Proof. (1) Let G be an l-$\mathcal{E}$-continuous (nonempty valued) submulti-function of F. By Theorem 1, $\mathcal{E}_G$ is lower quasi continuous on a quasi open domain. Since $G \subset \mathcal{E}_G \subset \mathcal{E}_F \subset F$, $\mathcal{E}_G$ is a desirable lower quasi continuous (nonempty valued) submulti-function of F.

(2) Moreover, if F is l-$\mathcal{E}$-continuous, then $F \subset \mathcal{E}_F$ (from l-$\mathcal{E}$-continuity) and $F \supset \mathcal{E}_F$ (from $\mathcal{E}$-closed graph). Hence $F = \mathcal{E}_F$ and from Theorem 1, F is lower quasi continuous on a quasi open domain. If F is densely defined l-$\mathcal{E}$-continuous, then the complement of its domain is nowhere dense by Corollary 2. □

Lemma 5. *If Y is locally compact, then $\mathcal{E}_F$ is usco on a relatively* (*in the domain of $\mathcal{E}_F$*) *open set.*

Proof. If $\mathcal{E}_F$ is *usco* at x, then there is an open set $W \supset \mathcal{E}_F(x)$ with compact closure $\overline{W} =: C$ ($\mathcal{E}_F(x)$ is compact and Y is regular locally compact) and an open set G containing x such that $\mathcal{E}_F(g) \subset W \subset C$ for any $g \in G$. So $\mathcal{E}_F$ is bounded by a compact set C on G and from c-upper semi continuity, it follows that $\mathcal{E}_F$ is *usco* on $G \cap \mathrm{Dom}\,(\mathcal{E}_F)$. It is clear from the following. Let $x_0 \in G \cap \mathrm{Dom}\,(\mathcal{E}_F)$, $\mathcal{E}_F(x_0) \subset V$ and V be open. The set $(Y \setminus V) \cap C$ is compact and its complement $V \cup (Y \setminus C)$ is open containing $\mathcal{E}_F(x_0)$; hence, there is an open set $H \subset G$ containing x_0 such that $\mathcal{E}_F(h) \subset V \cup (Y \setminus C)$ for any $h \in H$. Since $\mathcal{E}_F(h) \subset C$, $\mathcal{E}_F(h) \subset V$ for any $h \in H$. That means $\mathcal{E}_F$ is *usc* at x_0. Hence, $\mathcal{E}_F(x_0)$ is *usco* at any point of $G \cap \mathrm{Dom}\,(\mathcal{E}_F)$. $\square$

Corollary 3. *Let X be Baire, Y a locally compact metric space, F a densely defined compact valued multi-function with $\mathcal{E}$-closed graph and $\mathcal{E} \subset \mathcal{A}$. If F has an l-$\mathcal{E}$-continuous nonempty valued submulti-function, then F has a lower quasi continuous nonempty compact valued submulti-function F_0 such that F_0 has closed graph,and it is usco on an open set for which a complement in X is nowhere dense and lsc except for a set of first category. Moreover, if F is l-$\mathcal{E}$-continuous itself, then F $(= \mathcal{E}_F)$ has all properties as F_0.*

Proof. From Theorem 2 item (1), there is a nonempty compact valued and lower quasi continuous submulti-function F_0 of F, F_0 has closed graph and $\mathrm{Dom}\,(F_0) = \mathrm{Dom}\,(F)$ is quasi open and dense (F is densely defined), so $\mathrm{Dom}\,(F_0) = G \cup A$, where G is open dense and A is nowhere dense. By [**13**, Theorem 2], F_0 is *usco* on a residual set in G. Since $F_0 = \mathcal{O}_{F_0}$, by Lemma 5, F_0 is *usco* on an open residual set in G, hence, on an open set with a nowhere dense complement. By [**13**, Theorem 1], F_0 is *lsc* except for a set of first category. $\square$

5. Selections and $\mathcal{E}$-minimality. This section is devoted to the existence of a quasi continuous selection, provided an $\mathcal{E}$-continuous one exists and we will introduce $\mathcal{E}$-minimality as a basic tool in searching for $\mathcal{E}$-continuous selection.

The next example shows that a lower quasi continuous multifunction (even *lsc*, see F from Example 2) need not have a quasi continuous selection. On the other hand, we will show in the next paragraph, a multi-function which is u-$\mathcal{E}$-continuous can have quasi continuous selection, despite the fact that the multi-function is neither upper nor lower quasi continuous (more general continuity of the multi-function implies stronger type of continuity of selection (see F from Example 1 which is neither u-$\mathcal{O}$-continuous nor l-$\mathcal{O}$-continuous (it is u-$\mathcal{B}r$-continuous) but it has a quasi continuous selection).

Example 2. Let $X = \mathbf{N}$ be a topological space with co-finite topology (closed sets are all finite ones and $\mathbf{N}$). Then the quasi continuous, as well as continuous functions from X to $\mathbf{R}$ are constant ones. On the other hand, a multi-function F defined as $F(0) = \{0\}$, $F(1) = \{1\}$, $F(x) = \{0, 1\}$ for $x = 2, 3, \dots$ is *lsc* at any x and *usc* except for 0 and 1 (a nowhere dense set), but there is no quasi continuous selection of F.

For further investigation we will use the notion of $\mathcal{E}$-minimality. There are two variants of this notion in the literature, the former with respect to set inclusion and the latter is "pell-mell" modification of lower and upper quasi continuity having the next cluster variant.

Definition 5. A multi-function F is $\mathcal{E}$-minimal at a point x, if $F(x)$ is nonempty and for any open sets U, V such that $U \ni x$ and $V \cap F(x) \neq \varnothing$ there is a set $E \subset U \cap \operatorname{Dom}(F)$, $E \in \mathcal{E}$ such that $F(e) \subset V$ for all $e \in E$. Global definition is given by the local one at each point from $\operatorname{Dom}(F)$.

It is clear any selection of $\mathcal{E}$-minimal multi-function is $\mathcal{E}$-continuous. For system $\mathcal{O}$ we have the well-known notion of minimality [**4, 7, 15, 18**].

Theorem 3. *If Y is a regular topological space, then $\mathcal{E}_f$ is $\mathcal{O}$-minimal on the interior of $\operatorname{Dom}(\mathcal{E}_f)$, provided $\mathcal{E} \subset \mathcal{B}$, i.e., any set from $\mathcal{E}$ is open or of second category with the Baire property.*

Proof. If not in $x \in (\operatorname{Dom}(\mathcal{E}_f))^\circ$, there are the open sets $U \ni x$, V and a set $A \subset U \subset (\operatorname{Dom}(\mathcal{E}_f))^\circ$ dense in U such that $\mathcal{E}_f(x) \cap V \neq \varnothing$ and $\mathcal{E}_f(a) \cap (Y \setminus \overline{V}) \neq \varnothing$ for any $a \in A$. Let $y \in \mathcal{E}_f(x) \cap V$. Then there is a set $E \in \mathcal{E}$, $E \subset U$ such that $f(e) \in V$ for any $e \in E$. The set E is a nonempty open or $E = (G \setminus S) \cup T$ where G is open and of second category and S, T are of first category.

Suppose the first case when E is open. Since A is dense in U there is a point $a \in A \cap E$ such that $\mathcal{E}_f(a) \cap (Y \setminus \overline{V}) \neq \varnothing$. Pick up $z \in \mathcal{E}_f(a) \cap (Y \setminus \overline{V})$. Then there is a set $E_0 \in \mathcal{E}$, $E_0 \subset E$ such that

$f(e_0) \in (Y \setminus \overline{V})$ for any $e_0 \in E_0$ which is a contradiction with the fact that $f(e) \in V$ for any $e \in E$.

In the second case when $E = (G \setminus S) \cup T \subset U$, where G is of second category open and S, T are of first category, intersection $G \cap U$ is of second category and without lost of generality we can suppose that any nonempty open subset of $G \cap U$ is of second category because the interior of the set of all points in which $G \cap U$ is of second category is nonempty. Then again there is a point $a \in A \cap G \cap U$ such that $\mathcal{E}_f(a) \cap (Y \setminus \overline{V}) \neq \varnothing$. Pick up $z \in \mathcal{E}_f(a) \cap (Y \setminus \overline{V})$. Then there is a set $E_0 \in \mathcal{E}$, $E_0 \subset G \cap U$ such that $f(e_0) \in (Y \setminus \overline{V})$ for any $e_0 \in E_0$. In this case E_0 is of second category, then $\varnothing \neq E_0 \setminus S \subset E$. But $f(e) \in V$ for any $e \in E$, which is contradiction. □

The first application of minimality can be seen from the next corollary.

Corollary 4. *Let Y be locally compact and F have $\mathcal{B}$-closed graph. If there is a $\mathcal{B}$-continuous function $f : A \to Y$ which is a selection of F on a set A, then there is a quasi open set $Q \supset A$ and a function $f_0 : Q \to Y$ which is selection of F on Q, and f_0 is quasi continuous on the interior of Q. Hence, if F with $\mathcal{B}$-closed graph is defined on X and it has a $\mathcal{B}$-continuous selection on X, then F has a quasi continuous selection on X.*

Proof. From $\mathcal{B}$-continuity of f, the inclusion $f \subset F$ and from $\mathcal{B}$-closedness of the graph of F, we have $f \subset \mathcal{B}_f \subset \mathcal{B}_F \subset F$. By Lemma 4 (applied on f), $Q := \mathrm{Dom}\,(\mathcal{B}_f)$ is quasi open and Q° is nonempty. By Theorem 3, $\mathcal{B}_f$ is $\mathcal{O}$-minimal on Q°, hence any selection f_0 of $\mathcal{B}_f$ is a desirable selection of F on Q and f_0 is quasi continuous on Q°. □

For the multi-function F with an $\mathcal{E}$-closed graph, $\mathcal{E} \subset \mathcal{A}$, ($\mathcal{B}$-closed graph), Theorem 2 (Corollary 4) can be considered as an equivalence between the existence of a lower quasi continuous submulti-function of F (quasi continuous selection of F) and l-$\mathcal{E}$-continuous submulti-function of F ($\mathcal{B}$-continuous selection of F). Namely, if Y is locally compact, F have an $\mathcal{E}$-closed graph, $\mathcal{E} \subset \mathcal{A}$ ($\mathcal{B}$-closed graph) and any open set contains a set from $\mathcal{E}$, then F defined on X has an l-$\mathcal{E}$-continuous nonempty valued submulti-function ($\mathcal{B}$-continuous selection) if and only if F has a lower quasi continuous nonempty valued submulti-function (quasi continuous selection).

Definition 6. A multi-function F has the Baire property if for any open set V, $F^-(V)$ has the Baire property, i.e., it is of the form $(G \setminus A) \cup B$, where G is open in X and A, B are of first category in X.

Lemma 6. *Let Y be second countable, and let X be a Baire space. If F defined on a residual set T has the Baire property, then F is lower Baire continuous except for a set of first category.*

Proof. Let $\{B_n\}_{n\in\mathbf{N}}$ be a base of Y. Then F is not l-$\mathcal{D}$-continuous on a set

$$B := \bigcup_{n\in\mathbf{N}} \big[F^-(B_n) \setminus \mathrm{D}\,(F^-(B_n))\big],$$

which is a set of first category, so that $A := T \setminus B$ is residual ($\mathrm{D}(S)$ is the set of all points at which S is of second category). Then restriction $F|_A$ is l-$\mathcal{D}$-continuous and, since F has the Baire property, $F|_A$ is l-$\mathcal{B}r$-continuous; hence, F is lower Baire continuous except for a set of first category. $\square$

Corollary 5. *Let X be Baire, Y a locally compact topological space with a countable base, and let F be a $\mathcal{B}$-closed graph. If there is a function $f : S \to Y$ with the Baire property which is selection of F on a residual set S, then there is a quasi continuous function $f_0 : G \to Y$ defined on an open set G with a nowhere dense complement, which is selection of F on G.*

Proof. By the lemma above, f is $\mathcal{B}r$-continuous on a residual set, say A_0 and, by Corollary 4, there is a quasi open set $Q \supset A := A_0 \cap S$ and there is an $f_0 : Q \to Y$ which is quasi continuous on $G := Q^\circ$ and f_0 is a selection of F on G. Hence, Q is residual and quasi open, so the complement of G is nowhere dense. $\square$

6. Upper $\mathcal{E}$-continuity and selection theorems. In two previous sections we have investigated the existence of a lower quasi continuous submulti-function/quasi continuous selection, provided there is an l-$\mathcal{E}$-continuous submulti-function/$\mathcal{B}$-continuous selection (selection with the Baire property). Hence the solution of the existence of an l-$\mathcal{E}$-continuous submulti-function/$\mathcal{E}$-continuous selection is a crucial point for further investigation. This section is devoted to this very problem.

There are many concepts of searching selection with certain property. We can mention three frequently used methods. Two constructive methods (the ε-selection technique and the projection one [**3, 16**]) and the third method is based on Zorn's lemma.

Using an absolutely different method, the existence of a quasi continuous selection for compact valued u-$\mathcal{B}r$-continuous, i.e., upper Baire continuous, multi-function was firstly proved in [**12**]. The idea was based on the existence of a *usco* minimal submulti-function F_0 of $\mathcal{B}r_F$, where F is compact valued u-$\mathcal{B}r$-continuous multi-function defined on a Baire space with values in a metric compact space. Next u-$\mathcal{B}r$-continuity of F guarantees that the intersection $F \cap F_0$ is nonempty, so any selection of $F \cap F_0$ is a quasi continuous selection of F.

Consider a certain family $\mathcal{F}$ of submulti-functions of F with a given property. If any linearly ordered subfamily of $\mathcal{F}$ has a lower bound, then by Zorn's lemma, there is a minimal (with respect to set inclusion) element of $\mathcal{F}$. This method will also be used in our article (Theorem 5). For example, using Zorn's lemma, it is possible to prove the well-known fact that, for any *usco* multi-function F, there is a *usco* multi-function $F_0 \subset F$, which is minimal with respect to set inclusion, i.e., if G is *usco* and $G \subset F_0$, then $G = F_0$, see [**2**]. In a similar way, Cao and Moors in [**1**] recently proved the next theorem concerning the existence of a quasi continuous selection for compact valued u-$\mathcal{B}r$-continuous multi-function, which is an elegant generalization of the result from [**12**].

Theorem 4 (see [**1**]). *Let Y be a T_1 regular space. If $F : X \to Y$ is a compact valued u-$\mathcal{B}r$-continuous multi-function, then F has a quasi continuous selection.*

Using the same method as in [**1**], we can prove a similar theorem in the cluster setting. The result is more general (moreover, X is not implicitly supposed to be Baire) and the proof is shorter (it is sufficient to prove minimality).

Theorem 5. *Let Y be Hausdorff and F be a compact valued u-$\mathcal{E}$-continuous. Then F has an $\mathcal{E}$-continuous selection. Consequently, for $\mathcal{E} = \mathcal{O}$, F has a quasi continuous selection.*

Proof. Let $\mathcal{M}$ be family of all u-$\mathcal{E}$-continuous nonempty compact valued submulti-functions of F which is partially ordered by inclusion. It is a nonempty family since $F \in \mathcal{M}$. For any linearly ordered subfamily $\mathcal{M}_0$, a multi-function $M_0(x) := \cap\{M(x) : M \in \mathcal{M}_0\}$ is a nonempty compact valued submulti-function of F and, for any open sets $V \supset M_0(x)$ and U containing x, there is an $M \in \mathcal{M}_0$ such that $M(x) \subset V$. From u-$\mathcal{E}$-continuity of M, there is a set $E \in \mathcal{E}$, $E \subset \mathrm{Dom}\,(M) \cap U \cap M^+(V)$; hence, for any $e \in E$ we have $M_0(e) \subset M(e) \subset V$. That means M_0 is u-$\mathcal{E}$-continuous and $\mathcal{M}$ has a minimal element M_m with respect to inclusion. Now we will prove that M_m is $\mathcal{E}$-minimal. If not at $a \in \mathrm{Dom}\,(M_m)$, there is a pair of two open sets V intersecting $M_m(a)$ and U containing a such that for any $E \subset U \cap \mathrm{Dom}\,(M_m)$ from $\mathcal{E}$ there is a point $e \in E$ such that $M_m(e) \not\subset V$. Since M_m is u-$\mathcal{E}$-continuous, for all $u \in U \cap \mathrm{Dom}\,(M_m)$ we have $M_m(u) \not\subset V$. Define a multi-function N as $N(x) := M_m(x)$ if $x \in \mathrm{Dom}\,(M_m) \setminus U$ and $N(x) := M_m(x) \cap (Y \setminus V)$ if $x \in U \cap \mathrm{Dom}\,(M_m)$. Then N is a nonempty compact valued submulti-function of F. We will show that N is u-$\mathcal{E}$-continuous. If $x \in \mathrm{Dom}\,(M_m) \setminus U$ there is nothing to prove. Let $x \in U \cap \mathrm{Dom}\,(M_m)$, $N(x) \subset W$, $x \in H \subset U$, H, W open. Then $M_m(x) \subset V \cup W$ and from u-$\mathcal{E}$-continuous of M_m, there is a set $E \in \mathcal{E}$, $E \subset H \cap \mathrm{Dom}\,(M_m)$ such that $M_m(e) \subset V \cup W$ for any $e \in E$. That means $N(e) \subset W$ for any $e \in E$. Hence $N \in \mathcal{M}$ and $N(a) \subsetneqq M_m(a)$, contradiction with minimality of M_m. Finally, any selection of M_m is an $\mathcal{E}$-continuous one of F. □

Global $\mathcal{E}$-continuity on an open set has a very interesting feature. For some cluster systems, global $\mathcal{E}$-continuity of the functions is equivalent with quasi continuity. It is a case when Y is regular and $\mathcal{E} \subset \mathcal{B}$ (any $E \in \mathcal{B}$ is a nonempty open or second category with the Baire property, see [**14**, Theorem 3]) or when $\mathcal{E}_{\mathbf{I}} \subset \{(G \setminus I_1) \cup I_2\}$ where G is a nonempty open and I_1, I_2 are from given an ideal $\mathbf{I}$. We suppose that $G \setminus I$ is not from $\mathbf{I}$ for any open $G \neq \varnothing$ and $I \in \mathbf{I}$; hence, any $E \in \mathcal{E}_{\mathbf{I}}$ is not from $\mathbf{I}$.

Lemma 7. *Let Y be a regular space. If a function $f : H \to Y$ is $\mathcal{E}_{\mathbf{I}}$-continuous ($\mathcal{B}$-continuous) on an open set H, then f is quasi continuous on H.*

Proof. (The case $\mathcal{B}$-continuity follows from [**14**, Theorem 3].) As a reminder, any $E \in \mathcal{E}_{\mathbf{I}}$ is not from $\mathbf{I}$. If not at a point a, then there are open sets V containing $f(a)$ and $U \subset H$ containing a and a set S dense in U such that $f(S) \subset Y \setminus \overline{V}$. Since f is $\mathcal{E}_{\mathbf{I}}$-continuity at a, there is a set $E \subset U$ of the form $(G \setminus I_1) \cup I_2$ such that $f(E) \subset V$. The intersection $U \cap G$ is not from $\mathbf{I}$ (otherwise $E = U \cap ((G \setminus I_1) \cup I_2) \subset (U \cap G) \cup (U \cap I_2) \in \mathbf{I}$, a contradiction); hence, there is a point $s \in U \cap G \cap S$ such that $f(s) \in Y \setminus \overline{V}$. Again, from $\mathcal{E}_{\mathbf{I}}$-continuity at s, there is a set $E' \subset U \cap G$ of the form $(G' \setminus I_1') \cup I_2'$ such that $f(E') \subset Y \setminus \overline{V}$. The intersection $U \cap G \cap G'$ is not from $\mathbf{I}$ (otherwise $E' = U \cap G \cap ((G' \setminus I_1') \cup I_2') \subset (U \cap G \cap G') \cup (U \cap G \cap I_2') \in \mathbf{I}$, a contradiction); hence, there is a point $s' \in (U \cap G \cap G') \setminus (I_1 \cup I_1') \subset (G \setminus I_1) \cup I_2 = E$ such that $f(s') \in Y \setminus \overline{V}$, a contradiction with $f(E) \subset V$. □

As a consequence of Theorem 5 and Lemma 7 we have

Theorem 6 (for upper quasi continuity and a T_1 regular range, see [**1**].) *Let Y be Hausdorff and F compact valued u-$\mathcal{E}$-continuous defined on an open set G. If within functions defined on G, $\mathcal{E}$-continuity on G implies quasi continuity on G, then F has quasi continuous selection on G. Especially if Y is T_1 regular and F is nonempty compact valued u-$\mathcal{E}_{\mathbf{I}}$-continuous* (*u-$\mathcal{B}$-continuous*) *on a open set G, then F has a quasi continuous selection on G.*

The l-$\mathcal{E}$-continuity and $\mathcal{E}$-closedness of the graph imply lower quasi continuity (see Theorem 2), while u-$\mathcal{E}$-continuity and $\mathcal{E}$-closedness of the graph imply only the existence of a lower quasi continuous submulti-function generated by an $\mathcal{E}$-continuous selection ("small" version) or maximal l-$\mathcal{E}$-continuous submult-function ("big" version) as we can see from the next theorem.

Theorem 7. *Let Y be locally compact and $\mathcal{E} \subset \mathcal{A}$. If u-$\mathcal{E}$-continuous compact valued multi-function F has an $\mathcal{E}$-closed graph, then F has a nonempty compact valued lower quasi continuous submulti-function F_0 with a closed graph,* $\mathrm{Dom}\,(F_0) = \mathrm{Dom}\,(F)$ *is a quasi open set and F_0 is generated by $\mathcal{E}$-continuous selection f of F, i.e., $F_0 = \mathcal{E}_f$, or by maximal* (*with respect to inclusion*) *l-$\mathcal{E}$-continuous submulti-function F_m of F, i.e., $F_0 = \mathcal{E}_{F_m}$.*

Proof. By Theorem 5, F has an $\mathcal{E}$-continuous selection f and by Theorem 1, $\mathcal{E}_f := F_0$ is a lower quasi continuous multi-function with a quasi open domain. Since F has an $\mathcal{E}$-closed graph, $F_0 = \mathcal{E}_f \subset \mathcal{E}_F \subset F$ and $\mathrm{Dom}\,(F) = \mathrm{Dom}\,(F_0)$. If F_m is the union of all l-$\mathcal{E}$-continuous nonempty valued submulti-functions of F, then F_m is a maximal l-$\mathcal{E}$-continuous nonempty valued submulti-function of F. Then $\mathcal{E}_{M_m}$ has desirable properties. □

7. Conclusion. Significance of some cluster systems is based on minimizing of the values of a cluster multifunction, as well as on $\mathcal{E}$-closedness of graph which guarantees inclusion $\mathcal{E}_F \subset F$. So, a selection of the former is a selection of the letter.

Especially for an $\mathcal{E}$-continuous selection f of u-$\mathcal{E}$-continuous multifunction F, "largeness" of a multi-function $\mathcal{E}_f$ depends on the cluster system. The multi-function F from Example 1 is u-$\mathcal{A}$-continuous and u-$\mathcal{B}r$-continuous. In the first case, there are many $\mathcal{A}$-continuous selections which can generate different lower quasi continuous multifunctions. For example, any function kd is an $\mathcal{A}$-continuous selection of F, where d is the Dirichlet function and k is a constant from $\langle 0, 1\rangle$. $\mathcal{A}_{kd} = \{0, k\}$ and kd does not have an $\mathcal{A}$-closed graph. On the other hand, considering $\mathcal{B}r$, there is only one $\mathcal{B}r$-continuous selection of F, namely $f(x) = 0$, so $\mathcal{B}r_f(x) = \{0\}$.

Let us consider the next example. Let $K \subset \mathbf{R}$ be closed. There is a disjoint decomposition $A_1, A_2, A_3, \dots$ of $\mathbf{R}$, where A_i are dense in $\mathbf{R}$. Let $\{k_1, k_2, k_3, \dots\}$ be a subset of K and dense in K. Put $g := \sum_{i\in\mathbf{N}},\ k_i\chi_{A_i}$, where χ_{A_i} is characteristic function of A_i. Then g is $\mathcal{A}$-continuous, for which $\mathcal{A}_g$ is identical to K.

It seems that the mutual connection between the original multifunction F and the resultant one $\mathcal{E}_F$ plays an important role for further investigation of the properties of multi-function and searching of selection. From the definitions it implies that a multi-function F is l-$\mathcal{E}$-continuous with an $\mathcal{E}$-closed graph if and only if $F = \mathcal{E}_F$. Hence, under the assumptions of Theorem 2, an invariant multi-function, i.e., for which $F = \mathcal{E}_F$, is lower quasi continuous with a closed graph defined on a quasi open domain. Moreover, under the conditions of Corollary 3, if a compact valued multi-function F is invariant and densely defined, then F is lower quasi continuous and *usco* except for a nowhere dense set and *lsc* except for a set of the first category.

In [**15**] we can find the following characterizations of $\mathcal{E}$-minimality. A multi-function $F : X \to Y$ is $\mathcal{E}$-minimal if and only if $F \subset \mathcal{E}_F \subset \mathcal{E}_G$, whenever G is a nonempty valued submulti-function of F. Further (also in [**15**]), if Y is a regular topological space, then $\bar{f}$ is $\mathcal{O}$-minimal if and only if f is quasi continuous, and $F : X \to Y$ is $\mathcal{O}$-minimal with closed graph if and only if for any selection g of F the equality $F = \mathcal{O}_g$ holds.

Relation $\mathcal{E}_f(x)\cap F(x)\neq\varnothing$ for all $x\in \mathrm{Dom}\,(F)$ is also worth studying, provided $\mathcal{E}_f$ is $\mathcal{O}$-minimal for some function f (not a necessary selection of F). Then any selection of $\mathcal{E}_f\cap F$ is a quasi continuous selection of F. The structure of the set of all points x for which $F(x)=\mathcal{E}_F(x)$, i.e., F is l-$\mathcal{E}$-continuous at x and F has an $\mathcal{E}$-closed graph at x, briefly F is invariant at x, would be interesting for further investigation.

The existence of a continuous selection is not guaranteed, even if F is *usc* and *lsc* is a closed valued multi-function $F: X\to \mathbf{R}$ (hence, F has a closed graph). An example was given in [**11**]. Kupka [**10**] proved that for such an F there is a quasi continuous selection $f: X\to\mathbf{R}$ (X is an arbitrary topological space) such that f is continuous except for a nowhere dense set, and it is upper (lower) continuous, i.e., $f^{-1}(-\infty,a)$ ($f^{-1}(a,\infty)$) is open, for any $a\in\mathbf{R}$. The question is whether there is a more general form of this result. The next theorem gives a partial solution to this question (under more general conditions as for continuity of F, but F is supposed to be compact valued and X is Baire and a role of a selection is substituted by an $\mathcal{O}$-minimal submulti-function).

Theorem 8. *Let X be Baire and Y a locally compact metric. If $F: X\to Y$ is a compact valued upper Baire continuous multi-function with a $\mathcal{B}r$-closed graph, then there is an $\mathcal{O}$-minimal submulti-function $F_0: X\to Y$ of F with closed graph and F_0 is single valued* (*so lsc*) *on a residual set and usco on an open set H for which $X\setminus H$ is nowhere dense.*

Proof. The existence of a $\mathcal{B}r$-continuous selection f of F follows from Theorem 5. By Theorem 3, $F_0 := \mathcal{B}r_f$ is $\mathcal{O}$-minimal, and it is clear that $\mathcal{B}r_f$ is compact valued on a dense open set. Using [**13**, Theorem 2], $\mathcal{B}r_f$ is *usco* except for a set of first category and, by Lemma 5, it is *usco* on an open set H such that $X\setminus H$ is nowhere dense. By [**13**, Theorem 1], $\mathcal{B}r_f$ is *lsc*, so single valued except for a set of first category. It is clear that $f\subset\mathcal{B}r_f\subset\mathcal{B}r_F\subset F$. $\square$

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