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EQUAL SUMS OF LIKE POWERS, BOTH POSITIVE AND NEGATIVE

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ABSTRACT. Several mathematicians have studied the problem of finding two sets of integers $x_1, \ldots, x_s$ and $y_1, \ldots, y_s$, such that $\sum_{i=1}^{s} x_i^r = \sum_{i=1}^{s} y_i^r$, $r = k_1, k_2, \ldots, k_n$, where k_i are specified positive integers. The particular case when $r = 1, 2, \ldots, n$ is the well-known Tarry-Escott problem. This paper is concerned with the scarcely investigated problem of finding two or more distinct sets of integers with equal sums of powers for both positive and negative powers, that is to say, integer solutions of diophantine systems $\sum_{i=1}^{s} x_i^r = \sum_{i=1}^{s} y_i^r$, and diophantine chains $\sum_{i=1}^{s} x_{i1}^r = \sum_{i=1}^{s} x_{i2}^r = \cdots = \sum_{i=1}^{s} x_{it}^r$, where, in both cases, the equality holds simultaneously for negative integral exponents $-h_m, \ldots, -h_2, -h_1$ and positive integral exponents $k_1, k_2, \ldots, k_n$. It is proved in the paper that, given any arbitrary set of such exponents, there exists a solution of the aforementioned diophantine chains for a suitable value of s. Parametric or numerical solutions of many diophantine systems and chains are given in the paper, two examples being the system of equations $\sum_{i=1}^{9} x_{i1}^r = \sum_{i=1}^{9} x_{i2}^r$, $r = -2, -1, 1, 3, 5, 7$, and the arbitrarily long chains $\sum_{i=1}^{6} x_{i1}^r = \sum_{i=1}^{6} x_{i2}^r = \cdots = \sum_{i=1}^{6} x_{it}^r$, $r = -1, 1, 2, 3, 4, 5$.

1. Introduction. This paper is concerned with the study of diophantine systems of the type

$$\sum_{i=1}^{s} x_i^r = \sum_{i=1}^{s} y_i^r, \qquad r = -h_m,\ -h_{m-1}, \ldots, -h_2,\ -h_1, \quad k_1,\ k_2, \ldots, k_n, \tag{1}$$

where x_i, y_i, $i = 1, 2, \ldots, s$, are integers, while h_u, $u = 1, 2, \ldots, m$, as well as k_v, $v = 1, 2, \ldots, n$, are positive integers, and the equality holds

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simultaneously for negative exponents $-h_m, -h_{m-1}, \dots, -h_2, -h_1$, and positive exponents $k_1, k_2, \dots, k_n$. More generally, the paper deals with simultaneous diophantine chains of the type

$$\sum_{i=1}^{s} x_{i1}^r = \sum_{i=1}^{s} x_{i2}^r = \cdots = \sum_{i=1}^{s} x_{it}^r, \tag{2}$$

$r = -h_m, -h_{m-1}, \dots, -h_2, -h_1, k_1, k_2, \dots, k_n$, where t is some integer > 2, x_{ij} are integers for $i = 1, 2, \dots, s$, $j = 1, 2, \dots, t$, and, as before, h_u, k_v are all positive integers, so that the equalities in (2) hold simultaneously for the specified exponents which are both positive and negative. The diophantine systems (1) and chains (2) will be called multigrade equations and multigrade chains, respectively, for positive and negative exponents.

We note that various diophantine systems involving equal sums of like powers, with the equality holding only for certain positive exponents, have been investigated by several mathematicians (see, for instance, [**3, 4, 6, 12**]). Specifically, when the positive exponents take the consecutive integer values $1, 2, \dots, n$, we get the well-known Tarry-Escott problem of degree n which has attracted considerable attention [**2, 5, 7, 8, 10**–**12**]. However, the problem of equal sums of like powers, both positive and negative, has scarcely been investigated till now. It seems that apart from the following identities given by Shuwen [**12**],

$$\begin{aligned} &4^r + 10^r + 12^r = 5^r + 6^r + 15^r, && r = -1, 1,\\ &6^r + 14^r + 14^r = 7^r + 9^r + 18^r, && r = -1, 1,\\ &3^r + 40^r = 4^r + 15^r + 24^r = 5^r + 8^r + 30^r, && r = -1, 1, \end{aligned} \tag{3}$$

there are no other published solutions of (1) or (2).

We note that, when h is a positive integer, the degree of the equation

$$\sum_{i=1}^{s} x_i^{-h} = \sum_{i=1}^{s} y_i^{-h}, \tag{4}$$

written as a polynomial equation in the variables x_i, y_i, is $(2s-1)h$, which increases when we increase the number of variables thereby making the problem of equal sums of like powers, both positive and negative, much more difficult than the problem restricted only to positive powers.

In Section 2 of this paper we prove certain general results about the diophantine systems (1) and the diophantine chains (2). Specifically we establish that, given arbitrary positive integers h_u, $u = 1, 2, \dots, m$ and k_v, $v = 1, 2, \dots, n$, there exists a suitable value of the integer s such that there exists a solution in positive integers of the diophantine system (1) and, further, there exists an integer s, possibly different from the previous one, such that there exists a solution in positive integers of the diophantine chains (2). We also obtain certain estimates for the values of s.

In Section 3 we prove two theorems concerning the diophantine system

$$\sum_{i=1}^{s} x_i^r = \sum_{i=1}^{s} y_i^r, \quad r = -n,\ -n+1, \dots,\ -2,\ -1, \tag{5}$$

and the simultaneous diophantine chains

$$\begin{gathered}\sum_{i=1}^{s} x_{i1}^r = \sum_{i=1}^{s} x_{i2}^r = \cdots = \sum_{i=1}^{s} x_{it}^r,\\ r = -n,\ -n+1, \dots,\ -2,\ -1.\end{gathered} \tag{6}$$

Since the diophantine system (5) may equivalently be written as

$$\sum_{i=1}^{s} \left(\frac{1}{x_i}\right)^r = \sum_{i=1}^{s} \left(\frac{1}{y_i}\right)^r, \quad r = 1, 2, \dots, n, \tag{7}$$

we will refer to the system (5) or (7) as the nth degree Tarry-Escott problem for reciprocals. The two theorems of Section 3 provide a method of generating new solutions of higher degree Tarry-Escott problems for reciprocals, or the diophantine chains (6) starting from known solutions of (5) or (6), respectively.

In Sections 4 and 5 we obtain parametric or numerical solutions of several diophantine systems with equal sums of like powers, both positive and negative, some examples being the following multigrades:

$$x_1^r + x_2^r + x_3^r = y_1^r + y_2^r + y_3^r,\ r = -2,\ 2,$$
$$x_1^r + x_2^r + x_3^r + x_4^r = y_1^r + y_2^r + y_3^r + y_4^r,\ r = -2,\ -1,\ 1,\ 2,$$
$$\sum_{i=1}^{9} x_{i1}^r = \sum_{i=1}^{9} x_{i2}^r,\ r = -2,\ -1,\ 1,\ 3,\ 5,\ 7,$$

and the arbitrarily long chains,

$$\sum_{i=1}^{6} x_{i1}^r = \sum_{i=1}^{6} x_{i2}^r = \cdots = \sum_{i=1}^{6} x_{it}^r, \quad r = -1,\ 1,\ 2,\ 3,\ 4,\ 5,$$
$$\sum_{i=1}^{7} x_{i1}^r = \sum_{i=1}^{7} x_{i2}^r = \cdots = \sum_{i=1}^{7} x_{it}^r, \quad r = -2,\ -1,\ 1,\ 3,\ 5,$$

where t is any integer > 2.

2. Some general results. In this section we will prove certain general results about the diophantine systems (1) and the diophantine chains (2) and finally show that, given arbitrary positive integers $h_u,\ k_v$, there exist solutions in positive integers of (1) and (2) for suitably chosen values of s. We first note that all equations in (1) and (2) are homogeneous; hence, any solution in rational numbers yields, on appropriate scaling, a solution in integers. It therefore suffices to obtain rational numerical or parametric solutions of (1) or (2).

A solution of (1) or (2) will be considered as trivial if all equations are trivially satisfied by the given solution. Henceforth, unless otherwise stated, when we refer to a solution of (1) or (2) , we will actually mean a nontrivial solution of (1) or (2) .

A solution in integers of the multigrade equations (1) or the chains (2) will be called primitive if $\gcd(x_1,\ x_2, \dots, x_s,\ y_1,\ y_2, \dots, y_s) = 1$ in case of a solution of (1), and if $\gcd(x_{11},\ x_{12}, \dots, x_{ij}, \dots, x_{st}) = 1$ in case of a solution of (2). Given a solution in integers of (1) or (2), we may by removing a suitable common factor, obtain a primitive solution of (1) or (2). We will denote by $N_t(-h_m,\ -h_{m-1}, \dots, -h_2,\ -h_1,\ k_1,\ k_2, \dots, k_n)$ the least value of s such that there exists a solution of the chain equations (2). In particular, $N_2(-h_m, -h_{m-1}, \dots, -h_2, -h_1, k_1, k_2, \dots, k_n)$

will denote the least value of s such that there exists a solution of the diophantine system (1). It is well-known that[**10**, page 616, Theorem 4]

$$N_2(1,\, 2, \dots, n) \geq n+1. \tag{8}$$

Lemma 1. *Given arbitrary positive integers $k_1,\, k_2, \dots, k_n$, there exists a primitive solution in positive integers of the simultaneous diophantine chains*

$$\sum_{i=1}^{s} x_{i1}^r = \sum_{i=1}^{s} x_{i2}^r = \cdots = \sum_{i=1}^{s} x_{it}^r, \qquad r = k_1,\; k_2, \dots, k_n, \tag{9}$$

if we take $s \geq 1+k_1+k_2+\cdots+k_n$. It follows that

$$N_t(k_1,\, k_2, \dots, k_n) \leq 1+k_1+k_2+\cdots+k_n, \tag{10}$$

for all positive integer values of $t \geq 2$.

Proof. The proof is a straightforward extension of the proof given by Wright in [**13**] to establish the existence of simultaneous diophantine chains of the type (9) with $k_1,\, k_2, \dots, k_n$ being the consecutive integers $1,\, 2, \dots, n$, and is accordingly omitted. In fact, the proof shows that there exist arbitrarily many distinct solutions of (9) when $s \geq 1+k_1+k_2+\cdots+k_n$. □

Lemma 2. *There is a one-to-one correspondence between the primitive solutions in positive integers of the simultaneous diophantine chains*

$$\sum_{i=1}^{s} x_{i1}^r = \sum_{i=1}^{s} x_{i2}^r = \cdots = \sum_{i=1}^{s} x_{it}^r, \qquad r = -h_m,\, -h_{m-1}, \dots, -h_2,\, -h_1,\; k_1,\, k_2, \dots, k_n, \tag{11}$$

and

$$\sum_{i=1}^{s} X_{i1}^r = \sum_{i=1}^{s} X_{i2}^r = \cdots = \sum_{i=1}^{s} X_{it}^r, \qquad r = h_m,\, h_{m-1}, \dots, h_2,\, h_1,\; -k_1,\, -k_2, \dots, -k_n, \tag{12}$$

where h_u, $u = 1, 2, \dots, m$ and k_v, $v = 1, 2, \dots, n$, are arbitrary positive integers and $t \geq 2$.

Proof. Any primitive positive integer solution x_{ij}, $i = 1, 2, \dots, s$, $j = 1, 2, \dots, t$, of (11) immediately yields a positive rational solution $X_{ij} = x_{ij}^{-1}$, $i = 1, 2, \dots, s$, $j = 1, 2, \dots, t$, of (12), which on appropriate scaling leads to a unique primitive positive integer solution of (12). The inverse correspondence is similarly established. It is easily seen that this is a one-to-one correspondence and the proof is complete. □

Lemma 3. *Given any arbitrary positive integers $h_1, h_2, \dots, h_m$, there exists a primitive solution in positive integers of the simultaneous diophantine chains*

$$\sum_{i=1}^{s} x_{i1}^r = \sum_{i=1}^{s} x_{i2}^r = \cdots = \sum_{i=1}^{s} x_{it}^r, \quad r = -h_m, -h_{m-1}, \dots, -h_2, -h_1, \tag{13}$$

if we take $s \geq 1 + h_1 + h_2 + \cdots + h_m$. It follows that

$$N_t(-h_1, -h_2, \dots, -h_m) \leq 1 + h_1 + h_2 + \cdots + h_m, \tag{14}$$

for all positive integer values of $t \geq 2$.

Proof. If we take $s \geq 1 + h_1 + h_2 + \cdots + h_m$, it follows from Lemma 1 that there exists a primitive solution in positive integers of the diophantine chains

$$\sum_{i=1}^{s} x_{i1}^r = \sum_{i=1}^{s} x_{i2}^r = \cdots = \sum_{i=1}^{s} x_{it}^r, \quad r = h_m, h_{m-1}, \dots, h_2, h_1, \tag{15}$$

and now following Lemma 2, we get a primitive positive integer solution of the simultaneous diophantine chains (13). This proves the lemma. □

As in the case of the simultaneous diophantine equations (9), there exist arbitrarily many distinct solutions of (13) when $s \geq 1 + h_1 + h_2 + \cdots + h_m$.

Theorem 1. *Given arbitrary positive integers $h_1, h_2, \dots, h_m$ and $k_1, k_2, \dots, k_n$, there exists a solution in positive integers of the simultaneous diophantine equations* (1) *if we take $s \geq 2(1 + \sum_{u=1}^{m} h_u)(1 +$

$\sum_{v=1}^{n} k_v$), *and further, there exists a solution in positive integers of the simultaneous diophantine chains* (2) *if we take* $s \geq 2(t-1)(1+\sum_{u=1}^{m} h_u)(1+\sum_{v=1}^{n} k_v)$. *It follows that when* $t \geq 2$,

$$(16)\quad N_t(-h_m, \ldots, -h_1,\ k_1, \ldots, k_n) \leq 2(t-1)\left(1+\sum_{u=1}^{m} h_u\right)\left(1+\sum_{v=1}^{n} k_v\right).$$

Proof. In view of Lemmas 1 and 3, for appropriately chosen integers p and q,

$$(17)\qquad p \leq \left(1+\sum_{u=1}^{m} h_u\right), \quad q \leq \left(1+\sum_{v=1}^{n} k_v\right),$$

there exist positive integers a_i, b_i, $i = 1, 2, \ldots, p$, and positive integers c_j, d_j, $j = 1, 2, \ldots, q$, such that

$$(18)\qquad \sum_{i=1}^{p} a_i^r = \sum_{i=1}^{p} b_i^r, \quad r = -h_m, -h_{m-1}, \ldots, -h_2, -h_1,$$

and

$$(19)\qquad \sum_{i=1}^{q} c_i^r = \sum_{i=1}^{q} d_i^r, \quad r = k_1, k_2, \ldots, k_n.$$

It follows that

$$(20)\qquad \left(\sum_{i=1}^{p} a_i^r - \sum_{i=1}^{p} b_i^r\right)\left(\sum_{i=1}^{q} c_j^r - \sum_{j=1}^{q} d_j^r\right) = 0,$$

for $r = -h_m, -h_{m-1}, \ldots, -h_2, -h_1,\ k_1, k_2, \ldots, k_n$, and hence we get

$$(21)\qquad \begin{gathered}\sum_{i=1}^{p}\sum_{j=1}^{q}\{(a_ic_j)^r + (b_id_j)^r\} = \sum_{i=1}^{p}\sum_{j=1}^{q}\{(a_id_j)^r + (b_ic_j)^r\}, \\ r = -h_m, -h_{m-1}, \ldots, -h_2, -h_1,\ k_1, k_2, \ldots, k_n.\end{gathered}$$

This gives the desired solution and it immediately follows that

$$(22)\quad N_2(-h_m, \ldots, -h_1,\ k_1, \ldots, k_n) \leq 2pq \leq 2\left(1+\sum_{u=1}^{m} h_u\right)\left(1+\sum_{v=1}^{n} k_v\right).$$

This proves the relation (16) when $t = 2$.

Since there exist arbitrarily many solutions of (18) and (19) when p and q satisfy the inequalities (17), it follows that there exist arbitrarily many solutions of (1) when $s \geq 2(1 + \sum_{u=1}^{m} h_u)(1 + \sum_{v=1}^{n} k_v)$. Thus if we take $s_1 = 2(1 + \sum_{u=1}^{m} h_u)(1 + \sum_{v=1}^{n} k_v)$, for any given arbitrary integer $t > 2$, there exist $t - 1$ distinct positive integer solutions of (1) with $s = s_1$, that is, there exist positive integers a_{ij}, b_{ij}, $i = 1, 2, \dots, s_1$, $j = 1, 2, \dots, t-1$, such that for each j, $j = 1, 2, \dots, t-1$,

$$\sum_{i=1}^{s_1} a_{ij}^r = \sum_{i=1}^{s_1} b_{ij}^r, \qquad (23)$$
$$r = -h_m,\ -h_{m-1}, \dots, -h_2,\ -h_1,\ k_1,\ k_2, \dots, k_n.$$

On adding $\sum_{\substack{w=1,\\ w\neq j}}^{t-1} b_{iw}^r$ to both sides, for each j, $j = 1, 2, \dots, t-1$, we get

$$\sum_{i=1}^{s_1} a_{ij}^r + \sum_{i=1}^{s_1} \sum_{\substack{w=1,\\ w\neq j}}^{t-1} b_{iw}^r = \sum_{i=1}^{s_1} \sum_{w=1}^{t} b_{iw}^r, r = -h_m, \dots, -h_1,\ k_1, \dots, k_n. \qquad (24)$$

Since the righthand side of (24) is independent of j, we get for each r, $r = -h_m,\ -h_{m-1}, \dots, -h_2,\ -h_1,\ k_1,\ k_2, \dots, k_n$,

$$\begin{aligned} \sum_{i=1}^{s_1} a_{i1}^r + \sum_{i=1}^{s_1} \sum_{w=2}^{t-1} b_{iw}^r &= \sum_{i=1}^{s_1} a_{i2}^r + \sum_{i=1}^{s_1} \sum_{\substack{w=1,\\ w\neq 2}}^{t-1} b_{iw}^r = \cdots \\ &\vdots \\ &= \sum_{i=1}^{s_1} a_{ij}^r + \sum_{i=1}^{s_1} \sum_{\substack{w=1,\\ w\neq j}}^{t-1} b_{iw}^r = \cdots \\ &\vdots \\ &= \sum_{i=1}^{s_1} a_{i,(t-1)}^r + \sum_{i=1}^{s_1} \sum_{w=1}^{t-2} b_{iw}^r \\ &= \sum_{i=1}^{s_1} \sum_{w=1}^{t-1} b_{iw}^r. \end{aligned} \qquad (25)$$

This gives a solution in positive integers of the simultaneous chains (2) with

$$s = s_1(t-1) = 2(t-1)(1+\sum_{u=1}^{m} h_u)(1+\sum_{v=1}^{n} k_v).$$

This proves the relation (16) when $t > 2$ and the proof is complete. □

3. Equal sums of powers of reciprocals. In this section we prove two theorems concerning equal sums of powers of reciprocals of integers in the special case when we consider only the consecutive exponents $1, 2, \ldots, n$ of reciprocals of integers. Starting from a given solution of equal sums of powers of reciprocals for the consecutive exponents $1, 2, \ldots, n$, these theorems provide new solutions of this type for the consecutive exponents $1, 2, \ldots, n$ as well as for the consecutive exponents $1, 2, \ldots, n, n+1$. By continuing the latter process, we may obtain a solution of equal sums of powers of reciprocals for the consecutive exponents $1, 2, \ldots, N$, where N is any positive integer greater than n.

Theorem 2. *If x_{ij}, $i = 1, 2, \ldots, s$, $j = 1, 2, \ldots, t$, is a solution in integers of the simultaneous diophantine chains*

$$\sum_{i=1}^{s}\left(\frac{1}{x_{i1}}\right)^r = \sum_{i=1}^{s}\left(\frac{1}{x_{i2}}\right)^r = \cdots = \sum_{i=1}^{s}\left(\frac{1}{x_{it}}\right)^r, \quad r = 1, 2, \ldots, n, \tag{26}$$

then another solution in integers of these simultaneous diophantine chains is given by

$$X_{ij} = \frac{Mx_{ij}}{d_1x_{ij}+d_2}, \quad i = 1, 2, \ldots, s, \; j = 1, 2, \ldots, t, \tag{27}$$

where d_1, d_2 and M are arbitrary integers such that X_{ij} is a nonzero integer for $i = 1, 2, \ldots, s$, $j = 1, 2, \ldots, t$.

Proof. It follows from (26) that there exist constants S_r, $r = 1, 2, \ldots, n$, such that for each j, $j = 1, 2, \ldots, t$,

$$\sum_{i=1}^{s}\left(\frac{1}{x_{ij}}\right)^r = S_r. \tag{28}$$

Therefore, for each j, $j = 1, 2, \ldots, t$, and each r, $r = 1, 2, \ldots, n$,

$$
\begin{aligned}
\sum_{i=1}^{s}\left(\frac{1}{X_{ij}}\right)^r &= M^{-r}\sum_{i=1}^{s}\left(d_1+\frac{d_2}{x_{ij}}\right)^r \\
&= M^{-r}\sum_{i=1}^{s}\left(\sum_{h=0}^{r}{}^rC_h d_1^{r-h}\left(\frac{d_2}{x_{ij}}\right)^h\right) \\
&= M^{-r}(d_1^r + {}^rC_1 d_1^{r-1} d_2 S_1 + \cdots + d_2^r S_r).
\end{aligned} \tag{29}
$$

Thus $\sum_{i=1}^{s}(1/X_{ij})^r$ is independent of j for each value of r and the theorem follows. □

It follows from Theorem 2 that if we have a solution of (26) in positive and negative integers, we can, by an appropriate choice of d_1, d_2 and M, obtain a solution of (26) in positive integers only.

Theorem 3. *If x_i, y_i, $i = 1, 2, \ldots, s$, is a solution in integers of the diophantine system*

$$
\sum_{i=1}^{s}\left(\frac{1}{x_i}\right)^r = \sum_{i=1}^{s}\left(\frac{1}{y_i}\right)^r, \quad r = 1, 2, \ldots, n, \tag{30}
$$

then a solution in integers of the diophantine system

$$
\sum_{i=1}^{2s}\left(\frac{1}{X_i}\right)^r = \sum_{i=1}^{2s}\left(\frac{1}{Y_i}\right)^r, \quad r = 1, 2, \ldots, n, n+1, \tag{31}
$$

is given by

$$
\begin{aligned}
X_i &= Mx_i, \quad X_{s+i} = Md_2y_i/(d_1y_i + d_2), \\
Y_i &= My_i, \quad Y_{s+i} = Md_2x_i/(d_1x_i + d_2),
\end{aligned} \tag{32}
$$

where $i = 1, 2, \ldots, s$ in each case while d_1, d_2 and M are arbitrary nonzero integers such that X_i, Y_i are integers for $i = 1, 2, \ldots, 2s$.

Proof. Combining $\sum_{i=1}^{s}(1/X_i)^r = \sum_{i=1}^{s}(1/Y_i)^r$, $r = 1, 2, \ldots, n$, which is an immediate consequence of (30), with $\sum_{i=s+1}^{2s}(1/X_i)^r =$

$\sum_{i=s+1}^{2s}(1/Y_i)^r$, $r = 1, 2, \dots, n$, which is a consequence of Theorem 2, proves the relations (31) for the exponents $1, 2, \dots, n$. To prove (31) when $r = n+1$, we observe that

$$\begin{aligned}
&\sum_{i=1}^{2s}\left\{\left(\frac{1}{X_i}\right)^{n+1} - \left(\frac{1}{Y_i}\right)^{n+1}\right\} \\
&= M^{-n-1}\sum_{i=1}^{s}\left[\left(\frac{1}{x_i}\right)^{n+1} - \left(\frac{1}{y_i}\right)^{n+1} + \left(\frac{d_1 y_i + d_2}{d_2 y_i}\right)^{n+1}\right. \\
&\qquad\qquad\qquad\qquad\qquad \left. - \left(\frac{d_1 x_i + d_2}{d_2 x_i}\right)^{n+1}\right] \\
&= M^{-n-1}\sum_{i=1}^{s}\left[\left(\frac{1}{x_i}\right)^{n+1} - \left(\frac{1}{y_i}\right)^{n+1}\right. \\
&\qquad\qquad \left. - d_2^{-n-1}\left\{\left(d_1 + \frac{d_2}{x_i}\right)^{n+1} - \left(d_1 + \frac{d_2}{y_i}\right)^{n+1}\right\}\right] \\
&= M^{-n-1}\sum_{i=1}^{s}\left[-d_2^{-n-1}\sum_{h=0}^{n}{}^{n+1}C_h d_1^{n+1-h} d_2^h\left\{\left(\frac{1}{x_i}\right)^h - \left(\frac{1}{y_i}\right)^h\right\}\right] \\
&= -(Md_2)^{-n-1}\left[\sum_{h=0}^{n}{}^{n+1}C_h d_1^{n+1-h} d_2^h \sum_{i=1}^{s}\left\{\left(\frac{1}{x_i}\right)^h - \left(\frac{1}{y_i}\right)^h\right\}\right] \\
&= 0.
\end{aligned}$$

This proves the theorem. □

4. Multigrade equations with both positive and negative exponents. In this section we obtain parametric or numerical solutions of several diophantine systems of type (1). Solutions of several other diophantine systems of this type follow from the more general results on multigrade chains obtained in Section 5.

4.1. Multigrade equations for exponents $r = -1, 1$. In this subsection we obtain a solution of the simultaneous diophantine equations

$$\sum_{i=1}^{s} x_i^r = \sum_{i=1}^{s} y_i^r, \quad r = -1, 1, \tag{33}$$

where s is any integer ≥ 2.

If x_i, $i = 1, 2, \dots, s$, are positive integers or rational numbers, and we take $y_i = m/x_i$, $i = 1, 2, \dots, s$, where m is an arbitrary parameter, both equations of the diophantine system (33) reduce to the same condition that immediately yields $m = \sum_{i=1}^{s} x_i / \sum_{i=1}^{s} x_i^{-1}$, and we thus obtain a solution in positive rational numbers of the diophantine system (33) for any arbitrary integer $s \geq 2$. A solution in positive integers is obtained on appropriate scaling.

When $s = 3$ we can find several parametric solutions of (33). We give one such rational solution in which x_i, $i = 1, 2, 3$ are arbitrary. We write

$$y_1 = (x_2 - x_1)t + x_1,\ y_2 = (x_3 - x_2)t + x_2,\ y_3 = (x_1 - x_3)t + x_3, \tag{34}$$

when (33) holds for all values of x_i and t with $r = 1$, while the condition for $r = -1$ leads to a cubic equation in t which can be readily solved to yield the following solution of (33) with $s = 3$:

$$\begin{aligned} y_1 &= \frac{(x_1^2 - x_2x_3)(x_2^2 - x_3x_1)x_3}{(x_1 - x_3)(x_2 - x_3)(x_1x_2 + x_2x_3 + x_3x_1)}, \\ y_2 &= \frac{(x_2^2 - x_3x_1)(x_3^2 - x_1x_2)x_1}{(x_1 - x_2)(x_1 - x_3)(x_1x_2 + x_2x_3 + x_3x_1)}, \\ y_3 &= \frac{(x_1^2 - x_2x_3)(x_1x_2 - x_3^2)x_2}{(x_1 - x_2)(x_2 - x_3)(x_1x_2 + x_2x_3 + x_3x_1)}. \end{aligned} \tag{35}$$

4.2. Multigrade equations for exponents $r = -2, 2$. In this subsection we obtain a solution of the simultaneous diophantine equations

$$\sum_{i=1}^{3} x_i^r = \sum_{i=1}^{3} y_i^r, \quad r = -2,\ 2. \tag{36}$$

We note that when we substitute $y_1 = m/x_1$, $y_2 = m/x_2$, $x_3 = m/y_3$, where m is an arbitrary parameter, both the equations of diophantine system (33) reduce to the same condition which is as follows:

$$x_1^2x_2^2y_3^2(x_1^2 + x_2^2 - y_3^2) + (x_1^2x_2^2 - x_1^2y_3^2 - x_2^2y_3^2)m^2 = 0. \tag{37}$$

When we make the further substitutions,

$$x_1 = az + b, \quad x_2 = a - bz, \quad y_3 = az - b, \tag{38}$$

where a, b, z are arbitrary parameters, equation (37) reduces to

$$\begin{aligned}(a^2z^2 - b^2)^2(a^2 - b^2z^2)^2 \\ - (a^4z^4 - 4ab^3z^3 + 6a^2b^2z^2 - 4a^3bz + b^4)m^2 = 0.\end{aligned} \tag{39}$$

Equation (37) will have a rational solution for m if

$$\phi(z) = a^4z^4 - 4ab^3z^3 + 6a^2b^2z^2 - 4a^3bz + b^4, \tag{40}$$

is a perfect square. Now $\phi(z)$ is a quartic in z and, following the usual procedure described by Dickson [**9**, page 639], we can find values of z that would make $\phi(z)$ a perfect square. Two such values of z are given by

$$\begin{aligned} z_1 &= (a^3 + a^2b + b^3)/(a^3 + ab^2 - b^3), \\ z_2 &= -(a^3 + ab^2 - b^3)/(a^3 - a^2b - ab^2), \end{aligned} \tag{41}$$

and these lead to two rational solutions of the diophantine system (36) which, on clearing denominators, may be written as

$$\begin{aligned} x_1 &= a^{10} + 2a^9b + 3a^8b^2 + 10a^7b^3 - 10a^3b^7 \\ &\quad + 3a^2b^8 - 2ab^9 + b^{10}, \\ x_2 &= a^{10} - a^9b + 3a^8b^2 - 2a^7b^3 - 6a^6b^4 - 6a^4b^6 + 2a^3b^7 \\ &\quad + 3a^2b^8 + ab^9 + b^{10}, \\ x_3 &= -a^{10} - 2a^9b + 2a^7b^3 + 9a^6b^4 + 9a^4b^6 \\ &\quad - 2a^3b^7 + 2ab^9 - b^{10}, \\ y_1 &= -a^{10} + 4a^7b^3 - a^6b^4 + a^4b^6 + 4a^3b^7 + b^{10}, \\ y_2 &= -a^{10} - 3a^9b - 3a^8b^2 - 2a^7b^3 + 2a^6b^4 - 6a^5b^5 - 2a^4b^6 \\ &\quad - 2a^3b^7 + 3a^2b^8 - 3ab^9 + b^{10}, \\ y_3 &= a^{10} + 3a^8b^2 + 2a^7b^3 - 2a^6b^4 + 2a^4b^6 \\ &\quad + 2a^3b^7 - 3a^2b^8 - b^{10}, \end{aligned} \tag{42}$$

and

$$
\begin{aligned}
x_1 &= -a^{10} - 2a^9b + a^8b^2 - 8a^7b^3 + 2a^6b^4 - 4a^5b^5 - 2a^4b^6 \\
&\quad + 2a^3b^7 - 4a^2b^8, \\
x_2 &= a^{10} + 3a^9b - a^8b^2 + 2a^6b^4 - 5a^5b^5 + 4a^4b^6 - 2a^3b^7 \\
&\quad - 2a^2b^8 + 2ab^9 - 2b^{10}, \\
x_3 &= a^{10} - 2a^9b - 7a^6b^4 - 5a^4b^6 + 2a^3b^7 - 3a^2b^8 + 2ab^9, \\
y_1 &= a^{10} - 4a^8b^2 - 6a^7b^3 - a^6b^4 + 4a^5b^5 + a^4b^6 + 4a^3b^7 \\
&\quad + a^2b^8 + 2ab^9 - 2b^{10}, \\
y_2 &= -a^{10} + a^9b + a^8b^2 + 4a^7b^3 + 2a^6b^4 + 11a^5b^5 + 4a^4b^6 \\
&\quad + 2a^3b^7 - 4a^2b^8, \\
y_3 &= -a^{10} - 4a^9b - 3a^8b^2 + 4a^6b^4 + 2a^4b^6 - 2a^3b^7 + 4ab^9.
\end{aligned} \tag{43}
$$

4.3. Multigrade equations for exponents $r = -2, -1, 1, 2$. In this subsection we obtain a solution of the simultaneous diophantine equations

$$\sum_{i=1}^{4} x_i^r = \sum_{i=1}^{4} y_i^r, \tag{44}$$

where $r = -2, -1, 1, 2$.

A solution of the diophantine equation (44) valid simultaneously for the exponents $r = -1, 1$, only, obtained as in subsection 4.1, is given by $y_i = m/x_i$, $1 = 1, 2, 3, 4$, where

$$m = \frac{x_1x_2x_3x_4(x_1 + x_2 + x_3 + x_4)}{(x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_1 + x_4x_1x_2)}. \tag{45}$$

Direct computation now shows that with these values of y_i, equation (44) will also be satisfied for both the exponents 2 and -2 if x_i, $i = 1, 2, 3, 4$ are chosen such that the following condition holds:

$$x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 = 0. \tag{46}$$

Thus when x_1, x_2, x_3 are arbitrary nonzero rational numbers such that $\sum_{i=1}^{3} x_i \neq 0$, and we choose $x_4 = -(x_1x_2+x_2x_3+x_3x_1)/(x_1+x_2+x_3)$ so that condition (46) is satisfied, then it is easily seen that $\sum_{i=1}^{4} x_i \neq 0$, $m \neq 0$, and we obtain a rational solution, and hence a solution in integers of the simultaneous diophantine equations (44) valid for the exponents $r = -2, -1, 1, 2$.

As a numerical example, starting with $x_1 = -4$, $x_2 = 2$, $x_3 = 1$, we get the solution

$$(-230)^r + (-92)^r + 23^r + 46^r = (-220)^r + (-110)^r + 22^r + 55^r,\\ r = -2, -1, 1, 2.$$

4.4. Multigrade equations for exponents $r = -6, -2, -1, 1, 2, 6$**.** In this subsection we obtain a numerical solution of the simultaneous diophantine equations

$$\sum_{i=1}^{18} x_i^r = \sum_{i=1}^{18} y_i^r, \quad r = -6, -2, -1, 1, 2, 6. \tag{47}$$

We will use the following identities which have been established earlier in [**6**, page 846]

$$83^r + 211^r + (-300)^r = (-124)^r + (-185)^r + 303^r,\\ r = 1, 2, 6. \tag{48}$$

$$43^r + 371^r + (-372)^r = 140^r + 307^r + (-405)^r,\\ r = 1, 2, 6. \tag{49}$$

It follows from (48) that when M is any nonzero integer,

$$(M/83)^r + (M/211)^r + (-M/300)^r\\ = (-M/124)^r + (-M/185)^r + (M/303)^r, \tag{50}$$

for $r = -1, -2, -6$. We take $M = 2^2.3.5^2.31.37.83.101.211$ and, combining identities (49) and (50) as described in Theorem 1, we get the following two sets, of 18 integers each, which provide a solution of the multigrade equations (47):

$$\begin{array}{rrrr}
\{-2727918817200, & -1506896015025, & -1073067591600, & -1010027599260, \\
-813540145500, & -752695377581, & -687183850500, & -460598905200, \\
-87239625973, & 124037382900, & 281223754000, & 315323949300, \\
616683517700, & 754724206092, & 1070183001300, & 1332446832900, \\
1987924710375, & 2720585702100\} & & \\
\text{and} & & & \\
\{-2969911615500, & -1821037203825, & -1220587098780, & -1168259071500, \\
-747251689200, & -622850352877, & -284035991540, & -211063611225, \\
-141469663740, & 86375867300, & 403842642000, & 745242948100, \\
821675546955, & 885569222100, & 1026636114000, & 1223877090960, \\
1825945659900, & 2251266335700\} & &
\end{array}$$

As shown in [**6**], we can obtain infinitely many solutions of the multigrade equations

$$x_1^r + x_2^r + x_3^r = y_1^r + y_2^r + y_3^r, \quad r = 1,\ 2,\ 6, \tag{51}$$

and hence, we can obtain infinitely many solutions of the multigrade equations (47).

5. Multigrade chains with both positive and negative exponents.

In this section we obtain parametric solutions of several multigrade chains with both positive and negative exponents.

5.1. Multigrade chains for exponents $r = -1, 1, 2$.

In this subsection we obtain a parametric solution in integers of the following multigrade chain of arbitrary length:

$$\sum_{i=1}^{3} x_{i1}^r = \sum_{i=1}^{3} x_{i2}^r = \cdots = \sum_{i=1}^{3} x_{it}^r, \quad r = -1,\ 1,\ 2, \tag{52}$$

where t is any positive integer > 2.

For each j, $j = 1,\ 2, \ldots, t$, we define

$$\begin{aligned}
x_{1j} &= Ma_j(a_j + b_j)/D_j, \\
x_{2j} &= Mb_j(a_j + b_j)/D_j, \\
x_{3j} &= -Ma_jb_j/D_j,
\end{aligned} \tag{53}$$

where $D_j = a_j^2 + a_j b_j + b_j^2$, while a_j, b_j and M are arbitrary integers such that x_{ij}, $i = 1, 2, 3$, $j = 1, 2, \ldots, t$, are all nonzero integers. It is easily verified that

$$\sum_{i=1}^{3} x_{ij}^{-1} = 0, \quad \sum_{i=1}^{3} x_{ij} = M, \quad \sum_{i=1}^{3} x_{ij}^2 = M^2. \tag{54}$$

Since for each of the exponents $r = -1, 1, 2$, the sum $\sum_{i=1}^{3} x_{ij}^r$ is the same for each j, it follows that the integers x_{1j}, x_{2j}, x_{3j}, $j = 1, 2, \ldots, t$, provide a solution of the multigrade chain (52). Hence, using (8) we obtain, for arbitrary $t \geq 2$,

$$N_t(-1, 1, 2) = 3. \tag{55}$$

5.2. Multigrade chains for exponents $r = -1, 1, 2, 3$. In this subsection we obtain a parametric solution in integers of the following multigrade chain of arbitrary length:

$$\sum_{i=1}^{4} x_{i1}^r = \sum_{i=1}^{4} x_{i2}^r = \cdots = \sum_{i=1}^{4} x_{it}^r, \quad r = -1, 1, 2, 3, \tag{56}$$

where t is any positive integer > 2.

For each j, $j = 1, 2, \ldots, t$, we define

$$\begin{aligned} x_{1j} &= Ma_j(a_j + b_j)/D_j, \quad & x_{2j} &= Mb_j(a_j + b_j)/D_j, \\ x_{3j} &= Ma_j(a_j - b_j)/D_j, \quad & x_{4j} &= Mb_j(b_j - a_j)/D_j, \end{aligned} \tag{57}$$

where $D_j = a_j^2 + b_j^2$, while a_j, b_j and M are arbitrary integers such that x_{ij}, $i = 1, 2, 3, 4$, $j = 1, 2, \ldots, t$, are all nonzero integers. It is easily verified that

$$\begin{aligned} &\sum_{i=1}^{4} x_{ij}^{-1} = 0, \qquad &\sum_{i=1}^{4} x_{ij} = 2M, \\ &\sum_{i=1}^{4} x_{ij}^2 = 2M^2, &\sum_{i=1}^{4} x_{ij}^2 = 2M^3. \end{aligned} \tag{58}$$

Since for each of the exponents $r = -1, 1, 2, 3$, the sum $\sum_{i=1}^{4} x_{ij}^r$ is the same for each j, it follows that the integers x_{1j}, x_{2j}, x_{3j}, x_{4j}, $j = 1, 2, \dots, t$, provide a solution of the multigrade chain (56). Hence, using (8) we obtain, for arbitrary $t \geq 2$,

$$N_t(-1, 1, 2, 3) = 4. \tag{59}$$

5.3. Multigrade chains for exponents $r = -1, 1, 2, 3, 4, 5$. In this subsection we obtain a parametric solution in integers of the following multigrade chain of arbitrary length:

$$\sum_{i=1}^{6} x_{i1}^r = \sum_{i=1}^{6} x_{i2}^r = \cdots = \sum_{i=1}^{6} x_{it}^r, \quad r = -1, 1, 2, 3, 4, 5, \tag{60}$$

where t is any positive integer > 2.

For each j, $j = 1, 2, \dots, t$, we define

$$\begin{aligned} x_{1j} &= \frac{Ma_j(a_j - b_j)}{D_j}, & x_{2j} &= \frac{Mb_j(b_j - a_j)}{D_j}, \\ x_{3j} &= \frac{Ma_j(2a_j + b_j)}{D_j}, & x_{4j} &= \frac{Mb_j(a_j + 2b_j)}{D_j}, \\ x_{5j} &= \frac{M(a_j + b_j)(a_j + 2b_j)}{D_j}, & x_{6j} &= \frac{M(a_j + b_j)(2a_j + b_j)}{D_j}, \end{aligned} \tag{61}$$

where $D_j = a_j^2 + a_j b_j + b_j^2$, while a_j, b_j and M are arbitrary integers such that x_{ij}, $i = 1, 2, \dots, 6$, $j = 1, 2, \dots, t$, are all nonzero integers. It is easily verified that

$$\begin{aligned} \sum_{i=1}^{6} x_{ij}^{-1} &= 0, & \sum_{i=1}^{6} x_{ij} &= 6M, \\ \sum_{i=1}^{6} x_{ij}^2 &= 10M^2, & \sum_{i=1}^{6} x_{ij}^3 &= 18M^3, \\ \sum_{i=1}^{6} x_{ij}^4 &= 34M^4, & \sum_{i=1}^{6} x_{ij}^5 &= 66M^5. \end{aligned} \tag{62}$$

Since for each of the exponents $r = -1,\ 1,\ 2,\ 3,\ 4,\ 5$, the sum $\sum_{i=1}^{6} x_{ij}^r$ is the same for each j, it follows that the integers $x_{1j},\ x_{2j},\ x_{3j},\ x_{4j},\ x_{5j},\ x_{6j}$, $j = 1,\ 2, \dots, t$, provide a solution of the multigrade chain (60). Hence, using (8) we obtain, for arbitrary $t \geq 2$,

$$N_t(-1,\ 1,\ 2,\ 3,\ 4,\ 5) = 6. \tag{63}$$

As a numerical example, when $t = 4$, taking $(a_1, b_1) = (2, 1)$, $(a_2, b_2) = (3, 1)$, $(a_3, b_3) = (5, 1)$, $(a_4, b_4) = (5, -3)$, we get the multigrade chain,

$$\begin{aligned}(-7657)^r &+ 15314^r + 30628^r + 76570^r + 91884^r + 114855^r \\ &= (-8246)^r + 20615^r + 24738^r + 82460^r + 86583^r + 115444^r \\ &= (-6916)^r + 12103^r + 34580^r + 72618^r + 95095^r + 114114^r \\ &= (-5642)^r + 8463^r + 39494^r + 67704^r + 98735^r + 112840^r,\end{aligned}$$

where the equality holds for $r = -1,\ 1,\ 2,\ 3,\ 4,\ 5$.

5.4. Multigrade chains for exponents $r = -2,\ -1, 1, 3, \dots, 2k+1$**.** In this section we give a method of generating multigrade chains of the type

$$\sum_{i=1}^{2k+3} x_{i1}^r = \sum_{i=1}^{2k+3} x_{i2}^r = \cdots = \sum_{i=1}^{2k+3} x_{it}^r, \qquad r = -2,\ -1,\ 1,\ 3, \dots, 2k+1, \tag{64}$$

by using symmetric ideal solutions of the Tarry-Escott problem of degree $2k + 2$. We know parametric symmetric ideal solutions of the Tarry-Escott problem of degrees 4 and 6 [**5**] and two numerical solutions of the Tarry-Escott problem of degree 8 [**2**]. These solutions lead to parametric solutions of the chains (64) when $k = 1$ or 2 and t is any integer > 2, and a numerical solution of (64) when $k = 3$ and $t = 2$.

We note that every symmetric ideal solution of the Tarry-Escott problem of degree $2k + 2$ is reducible to the form

$$\sum_{i=1}^{2k+3} x_i^r = \sum_{i=1}^{2k+3} y_i^r, \quad r = 1,\ 2,\ 3, \dots, 2k+2, \tag{65}$$

where $y_i = -x_i$, $i = 1, 2, \ldots, 2k+3$. Thus, each such symmetric ideal solution gives a solution of the following diophantine system:

$$\sum_{i=1}^{2k+3} x_i^r = 0, \quad r = 1, 3, 5, \ldots, 2k+1. \tag{66}$$

We will use t such solutions to generate the multigrade chain (64). We first prove a couple of preliminary lemmas. We will use these lemmas to obtain solutions of (64) for $k = 1$, 2 and 3 as mentioned above.

Lemma 4. *If α_i, $i = 1, 2, \ldots, n$, are the roots of the polynomial equation*

$$\begin{gathered} a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-3}x^3 + a_{n-1}x + a_n = 0, \\ a_0a_n \neq 0, \end{gathered} \tag{67}$$

that is, the equation $\sum_{j=0}^{n} a_jx^{n-j} = 0$ where $a_{n-2} = 0$, $a_0a_n \neq 0$, then

$$\sum_{i=1}^{n} \alpha_i^{-2} = \left(\sum_{i=1}^{n} \alpha_i^{-1}\right)^2. \tag{68}$$

Proof. If β_i, $i = 1, 2, \ldots, n$, are the roots of the polynomial equation

$$\begin{gathered} b_0x^n + b_1x^{n-1} + b_3x^{n-3} + \cdots + b_{n-1}x + b_n = 0, \\ b_0b_n \neq 0, \end{gathered} \tag{69}$$

that is, the equation $\sum_{j=0}^{n} b_jx^{n-j} = 0$ where $b_2 = 0$, $b_0b_n \neq 0$, then, by the theory of equations,

$$\sum_{\substack{i=1,\\ i\neq j}}^{n} \sum_{j=1}^{n} \beta_i\beta_j = b_2/b_0 = 0,$$

and hence

$$\left(\sum_{i=1}^{n} \beta_i\right)^2 = \sum_{i=1}^{n} \beta_i^2 + 2\sum_{\substack{i=1,\\ i\neq j}}^{n} \sum_{j=1}^{n} \beta_i\beta_j = \sum_{i=1}^{n} \beta_i^2. \tag{70}$$

Since α_i, $i = 1, 2, \ldots, n$, are the roots of equation (67), it follows that α_i^{-1}, $i = 1, 2, \ldots, n$, are the roots of the polynomial equation

$$a_n x^n + a_{n-1}x^{n-1} + a_{n-3}x^{n-3} + \cdots + a_1 x + a_0 = 0, \tag{71}$$

which is of type (69), and the result of the lemma follows from (70). □

Lemma 5. *If for each value of* j, $j = 1, 2, \ldots, t$, *there exist nonzero integers* α_{ij}, $i = 1, 2, \ldots, 2k+3$, *such that*

$$\sum_{i=1}^{2k+3} \alpha_{ij}^r = 0, \quad r = 1, 3, \ldots, 2k+1, \tag{72}$$

then a solution in integers of the multigrade chain (64) *is given by*

$$x_{ij} = M\left(\sum_{i=1}^{2k+3} \alpha_{ij}^{-1}\right)\alpha_{ij}, \quad i = 1, 2, \ldots, 2k+3, \; j = 1, 2, \ldots, t, \tag{73}$$

where M *is an arbitrary nonzero integer such that* x_{ij} *is an integer for* $i = 1, 2, \ldots, 2k+3$, $j = 1, 2, \ldots, t$.

Proof. For any fixed value of j, let the nonzero integers α_{ij}, $i = 1, 2, \ldots, 2k+3$, satisfying the simultaneous diophantine equations (72), be the roots of the polynomial equation

$$x^{2k+3} + a_1 x^{2k+2} + a_2 x^{2k+1} + \cdots + a_{2k}x^3 + a_{2k+1}x^2 + a_{2k+2}x + a_{2k+3} = 0, \tag{74}$$

where we necessarily have $a_{2k+3} \neq 0$. If we write $s_r = \sum_{i=1}^{2k+3} \alpha_{ij}^r$, Newton's theorem [**1**, page 297] gives the following relations between the coefficients of (74) and s_r:

$$s_r + a_1 s_{r-1} + a_2 s_{r-2} + \cdots + a_{r-1}s_1 + r a_r = 0, \quad (r < 2k+3). \tag{75}$$

It follows from (72) that $s_r = 0$ for $r = 1, 3, \ldots, 2k+1$, and applying successively the relations (75) for $r = 1, 3, \ldots, 2k+1$, we obtain

$$a_1 = 0, \quad a_3 = 0, \quad a_5 = 0, \ldots, a_{2k+1} = 0. \tag{76}$$

Since $a_{2k+1}=0$, $a_{2k+3}\neq 0$, we may apply Lemma 4 to equation (74), and hence we obtain

$$\sum_{i=1}^{2k+3}\alpha_{ij}^{-2}=\Big(\sum_{i=1}^{2k+3}\alpha_{ij}^{-1}\Big)^2. \tag{77}$$

It now follows that, for each j, $j=1,\,2,\dots,t$,

$$\sum_{i=1}^{2k+3}x_{ij}^r=M^r\Big(\sum_{i=1}^{2k+3}\alpha_{ij}^{-1}\Big)^r\Big(\sum_{i=1}^{2k+3}\alpha_{ij}^r\Big)=0,$$
$$r=1,\,3,\dots,2k+1,$$
$$\sum_{i=1}^{2k+3}x_{ij}^{-1}=M^{-1}\Big(\sum_{i=1}^{2k+3}\alpha_{ij}^{-1}\Big)^{-1}\Big(\sum_{i=1}^{2k+3}\alpha_{ij}^{-1}\Big)=M^{-1},$$
$$\sum_{i=1}^{2k+3}x_{ij}^{-2}=M^{-2}\Big(\sum_{i=1}^{2k+3}\alpha_{ij}^{-1}\Big)^{-2}\Big(\sum_{i=1}^{2k+3}\alpha_{ij}^{-2}\Big)=M^{-2},$$
in view of (77).

Since for each of the exponents $r=-2,\,-1,\,1,\,3,\dots,2k+1$, the sum $\sum_{i=1}^{2k+3}x_{ij}^r$ is the same for each j, it follows that the integers x_{ij}, $i=1,\,2,\dots,2k+3$, $j=1,\,2,\dots,t$, provide a solution of the multigrade chain (64). □

We will now apply Lemma 5 to obtain specific multigrade chains of type (64). To obtain a solution of (64) with $k=1$, we use the following solution of the diophantine equations

$$\sum_{i=1}^{5}\alpha_i^r=0,\quad r=1,\,3, \tag{78}$$

given in [**7**, pages 315–317]:

$$\begin{aligned}
\alpha_1(p,\,q,\,r,\,s)&=pq(r^2-s^2)+q^2r^2,\\
\alpha_2(p,\,q,\,r,\,s)&=-\{p^2s(r+s)-q^2rs\},\\
\alpha_3(p,\,q,\,r,\,s)&=p^2r(r+s)+pqr^2-q^2rs,\\
\alpha_4(p,\,q,\,r,\,s)&=-\{p^2r(r+s)+pq(r^2-s^2)\},\\
\alpha_5(p,\,q,\,r,\,s)&=p^2s(r+s)-pqr^2-q^2r^2,
\end{aligned} \tag{79}$$

where p, q, r, s, are arbitrary parameters. Following Lemma 5, we define $\alpha_{ij} = \alpha_i(p_j, q_j, r_j, s_j)$, $i = 1, 2, \ldots, 5$, $j = 1, 2, \ldots, t$, where p_j, q_j, r_j, s_j, are arbitrary integer parameters, and

$$x_{ij} = M\left(\sum_{i=1}^{5} \alpha_{ij}^{-1}\right)\alpha_{ij}, \quad i = 1, 2, \ldots, 5, \ j = 1, 2, \ldots, t, \tag{80}$$

when, as in Lemma 5, we get,

$$\begin{aligned} \sum_{i=1}^{5} x_{ij} &= 0, & \sum_{i=1}^{5} x_{ij}^3 &= 0, \\ \sum_{i=1}^{5} x_{ij}^{-1} &= M^{-1}, & \sum_{i=1}^{5} x_{ij}^{-2} &= M^{-2}, \end{aligned} \tag{81}$$

and hence x_{ij}, $i = 1, 2, \ldots, 5$, $j = 1, 2, \ldots, t$, provide a parametric solution of the multigrade chain

$$\sum_{i=1}^{5} x_{i1}^r = \sum_{i=1}^{5} x_{i2}^r = \cdots = \sum_{i=1}^{5} x_{it}^r, \quad r = -2, -1, 1, 3, \tag{82}$$

where t is any positive integer > 2. Hence we obtain, for arbitrary $t \geq 2$,

$$N_t(-2, -1, 1, 3) \leq 5. \tag{83}$$

As a numerical example, taking

$$\begin{aligned} &(p_1, q_1, r_1, s_1) = (1, 1, 2, 1), && (p_2, q_2, r_2, s_2) = (1, 2, 2, 1), \\ &(p_3, q_3, r_3, s_3) = (1, -2, 2, -1), && (p_4, q_4, r_4, s_4) = (3, 5, 1, 1), \end{aligned}$$

we get the multigrade chain,

$$\begin{aligned} &(-1156760)^r + (-1012165)^r + 144595^r + 722975^r + 1301355^r \\ &\quad = (-818370)^r + (-467640)^r + 194850^r + 233820^r + 857340^r \\ &\quad = (-743490)^r + (-578270)^r + 165220^r + 330440^r + 826100^r \\ &\quad = (-630718)^r + (-516042)^r + 200683^r + 229352^r + 716725^r, \end{aligned}$$

where the equality holds for $r = -2, -1, 1, 3$.

When $k = 2$, we may similarly obtain a parametric solution of the multigrade chain (64) with arbitrary $t > 2$ by using the parametric symmetric ideal solution of the Tarry-Escott problem problem of degree 6 given in [**5**] and establish that for arbitrary $t \geq 2$,

$$N_t(-2,\,-1,\,1,\,3,\,5) \leq 7. \tag{84}$$

As the parametric solution is cumbersome, we restrict ourselves to giving a numerical example with $t = 3$ obtained from the following three solutions (quoted in [**2**, page 9] and [**5**, pages 631–632]) of the equations (72) with $k = 2$:

$$\begin{gathered}(-51)^r + (-33)^r + (-24)^r + 7^r + 13^r + 38^r + 50^r = 0,\\ (-134)^r + (-75)^r + (-66)^r + 8^r + 47^r + 87^r + 133^r = 0,\\ (-11907)^r+(-6001)^r+(-5893)^r+121^r+5200^r+6586^r+11894^r = 0,\end{gathered} \tag{85}$$

where, in each case, the equality holds for $r = 1, 3, 5$.

We accordingly write

$$\begin{array}{lll} x_1 = -51m_1, & y_1 = -134m_2, & z_1 = -11907m_3,\\ x_2 = -33m_1, & y_2 = -75m_2, & z_2 = -6001m_3,\\ x_3 = -24m_1, & y_3 = -66m_2, & z_3 = -5893m_3,\\ x_4 = 7m_1, & y_4 = 8m_2, & z_4 = 121m_3,\\ x_5 = 13m_1, & y_5 = 47m_2, & z_5 = 5200m_3,\\ x_6 = 38m_1, & y_6 = 87m_2, & z_6 = 6586m_3,\\ x_7 = 50m_1, & y_7 = 133m_2, & z_7 = 11894m_3,\end{array} \tag{86}$$

when, in view of (85), we have

$$\sum_{i=1}^{7} x_i^r = 0, \quad \sum_{i=1}^{7} y_i^r = 0, \quad \sum_{i=1}^{7} z_i^r = 0, \tag{87}$$

where, in each case, the equality holds for $r = 1, 3, 5$, and further, we obtain by direct computation,

$$\sum_{i=1}^{7} x_i^{-1} = \frac{11285167}{64664600m_1},$$
$$\sum_{i=1}^{7} y_i^{-1} = \frac{3456089047}{26720524600m_2}, \tag{88}$$
$$\sum_{i=1}^{7} z_i^{-1} = \frac{4292107975870279197444 1}{518848785688786141909320 0m_3}.$$

We take

$$\begin{aligned} M &= \operatorname{lcm}(64664600,\ 26720524600,\\ &\qquad\qquad 5188487856887861419093200),\\ &= 473817899578856392653010117200,\\ m_1 &= \frac{11285167M}{64664600},\\ m_2 &= \frac{3456089047M}{26720524600},\\ m_3 &= \frac{4292107975870279197444 1M}{5188487856887861419093200}, \end{aligned} \tag{89}$$

when

$$\sum_{i=1}^{7} x_i^{-1} = M^{-1}, \quad \sum_{i=1}^{7} y_i^{-1} = M^{-1}, \quad \sum_{i=1}^{7} z_i^{-2} = M^{-1}, \tag{90}$$

and in view of Lemma 5, we have

$$\sum_{i=1}^{7} x_i^{-2} = M^{-2}, \quad \sum_{i=1}^{7} y_i^{-2} = M^{-2}, \quad \sum_{i=1}^{7} z_i^{-2} = M^{-2}. \tag{91}$$

The relations (91) may also be verified by direct computation. It follows from (87), (90) and (91) that when we take m_1, m_2, m_3 as in (89), the integers x_i, y_i, z_i defined by (86) provide an example of the multigrade chain

$$\sum_{i=1}^{7} x_i^r = \sum_{i=1}^{7} y_i^r = \sum_{i=1}^{7} z_i^r, \quad r = -2,\ -1,\ 1,\ 3,\ 5. \tag{92}$$

Finally using the following two numerical solutions given in [**2**, page 9],

$$(-98)^r + (-82)^r + (-58)^r + (-34)^r + 13^r + 16^r + 69^r + 75^r + 99^r = 0,$$
$$(-169)^r + (-161)^r + (-119)^r + (-63)^r + 8^r + 50^r + 132^r + 148^r + 174^r = 0,$$

where, in both cases, the equality holds for $r = 1, 3, 5, 7$, we obtain a numerical solution of (64) with $k = 3$, $t = 2$, which is given below:

$$\begin{aligned}
&(-59866534997533082)^r + (-50092406834670538)^r \\
&\quad + (-35431214590376722)^r + (-20770022346082906)^r \\
&\quad + 7941479132325817^r + 9774128162862544^r + 42150927702344721^r \\
&\quad + 45816225763418175^r + 60477418007711991^r \\
&= (-122664510625104994)^r + (-116857906571845586)^r \\
&\quad + (-86373235292233694)^r + (-45727006919417838)^r \\
&\quad + 5806604053259408^r + 36291275332871300^r \\
&\quad + 95808966878780232^r + 107422174985299048^r \\
&\quad + 126293638158392124^r,
\end{aligned}$$

where the equality holds for $r = -2, -1, 1, 3, 5, 7$.

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REFERENCES

1. S. Barnard and J.M. Child, *Higher algebra*, Macmillan, London, 1936.

2. P. Borwein and C. Ingalls, *The Prouhet-Tarry-Escott problem revisited*, L'Enseignement Math. **40** (1994), 3–27.

3. A. Bremner, *A geometric approach to equal sums of sixth powers*, Proc. London Math. Soc. **43** (1981), 544–581.

4. ———, *A geometric approach to equal sums of fifth powers*, J. Number Theory **13** (1981), 337–354.

5. J. Chernick, *Ideal solutions of the Tarry-Escott problem*, Amer. Math. Month. **44** (1937), 626–633.

6. A. Choudhry, *On equal sums of sixth powers*, Rocky Mountain J. Math. **30** (2000), 843–848.

7. ———, *Ideal solutions of the Tarry-Escott problem of degree four and a related diophantine system*, L'Enseign. Math. **46** (2000), 313–323.

8. A. Choudhry, *Ideal solutions of the Tarry-Escott problem of degrees four and five and related diophantine systems*, L'Enseign. Math. **49** (2003), 101–108.

9. L.E. Dickson, *History of theory of numbers*, Vol. 2, Chelsea Publishing Company, New York, 1952, reprint.

10. H.L. Dorwart and O.E. Brown, *The Tarry-Escott problem*, Amer. Math. Month. **44** (1937), 613–626.

11. A. Gloden, *Mehrgradige Gleichungen*, Noordhoff, Groningen, 1944.

12. Chen Shuwen, Website at `http://go6.163.com/~eslp`, and mirror website at `http://euler.free.fr/eslp/`

13. E.M. Wright, *Equal sums of powers*, Bull. Amer. Math. Soc. **54** (1948), 755–757.

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