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HARDY TYPE ESTIMATES FOR COMMUTATORS OF FRACTIONAL INTEGRALS

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ABSTRACT. Let I_α be the fractional integral, $0 < \alpha < n$. In this paper we will consider commutator operators $[b, I_\alpha](f)(x) = b(x)I_\alpha(f)(x) - I_\alpha(bf)(x)$ for $b \in BMO(\mathbf{R}^n)$. The boundedness of these operators from local Hardy spaces $h^p(\mathbf{R}^n)$ to spaces $h^p(\mathbf{R}^n) + L^q(\mathbf{R}^n)$ will be obtained.

1. Introduction and main theorem.

Let $0 < \alpha < n$; the fractional integral operator I_α is defined by

$$I_\alpha f(x) = \int_{\mathbf{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

Consider the commutator associated with the fractional integrals I_α and locally integrable function b,

$$[b, I_\alpha]f(x) = b(x)I_\alpha f(x) - I_\alpha(bf)(x). \tag{1.1}$$

When $b \in BMO(\mathbf{R}^n)$, Chanillo proved in [**2**] that $[b, I_\alpha]$ was bounded from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$ with $1/q = 1/p - \alpha/n$, $1 < p < n/\alpha$. Consider the case $p = 1$. It is natural to substitute $L^1(\mathbf{R}^n)$ with Hardy space $H^1(\mathbf{R}^n)$. We know that the fractional integral operator maps $H^1(\mathbf{R}^n)$ into $L^{n/(n-\alpha)}(\mathbf{R}^n)$. However, it was observed in [**5**] that the corresponding result for $[b, I_\alpha]$ is false when b is a $BMO(\mathbf{R}^n)$ function.

Definition 1.1. Let $\mathcal{S}(\mathbf{R}^n)$ be the Schwartz function space and $\mathcal{S}'(\mathbf{R}^n)$ its dual, the tempered distribution. Suppose that $\varphi \in \mathcal{S}(\mathbf{R}^n)$, $\int \varphi = 1$, $f \in \mathcal{S}'(\mathbf{R}^n)$. Define

$$M_\varphi f(x) = \sup_{t>0} |\varphi_t * f(x)|, \qquad m_\varphi f(x) = \sup_{0<t<1} |\varphi_t * f(x)|,$$

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where $\varphi_t(\cdot) = t^{-n}\varphi(\cdot/t)$. For a fixed p, $0 < p \le 1$, Hardy space $H^p(\mathbf{R}^n)$ denotes the subspace of $\mathcal{S}'(\mathbf{R}^n)$ such that $M_\varphi f \in L^p(\mathbf{R}^n)$, equipped with norm $\|f\|_{H^p(\mathbf{R}^n)} = \|M_\varphi f\|_{L^p(\mathbf{R}^n)}$; local Hardy space $h^p(\mathbf{R}^n)$ denotes the subspace of $\mathcal{S}'(\mathbf{R}^n)$ such that $m_\varphi f \in L^p(\mathbf{R}^n)$, equipped with norm $\|f\|_{h^p(\mathbf{R}^n)} = \|m_\varphi f\|_{L^p(\mathbf{R}^n)}$.

Obviously, $H^p(\mathbf{R}^n) \subset h^p(\mathbf{R}^n)$. The local Hardy space theory is more suited to the problems associated with partial differential equations, it was first introduced by Goldberg in [**4**].

Recently, Sun and Su in [**11**] considered the boundedness of commutators on local Hardy spaces $h^1(\mathbf{R}^n)$. They proved that these commutators associated with the Calderon-Zygmund singular integral and $LMO(\mathbf{R}^n)$, a subspace of $BMO(\mathbf{R}^n)$ (see Definition 2.5), are bounded from $h^1(\mathbf{R}^n)$ into spaces $h^1_q(\mathbf{R}^n) = h^1(\mathbf{R}^n) + L^q(\mathbf{R}^n)$, $q > 1$. As an application, they obtained interior h^1-estimates for second order elliptic equations with vanishing LMO coefficients.

Inspired by the work in [**11**], in this paper, we are interested in the boundedness of $[b, I_\alpha]$ from local Hardy spaces $h^p(\mathbf{R}^n)$ into spaces $h^p_q(\mathbf{R}^n) = h^p(\mathbf{R}^n) + L^q(\mathbf{R}^n)$ when b belongs to $BMO(\mathbf{R}^n)$ or $LMO(\mathbf{R}^n)$. Our main result is stated as follows:

Theorem 1.1. *Suppose* $0 < \alpha < n$, $q > n/(n-\alpha)$. *If* $b \in BMO(\mathbf{R}^n)$, *then* $[b, I_\alpha]$ *is bounded from* $h^p(\mathbf{R}^n)$ *to* $h^p_q(\mathbf{R}^n)$ *for any* $n/(n+\alpha) < p < 1$. *Moreover, there exists a constant* C *independent of* f *such that*

$$\|[b, I_\alpha]f\|_{h^p_q(\mathbf{R}^n)} \le C[b]_*\|f\|_{h^p(\mathbf{R}^n)}. \tag{1.2}$$

If $b \in LMO(\mathbf{R}^n)$, *then* $[b, I_\alpha]$ *is bounded from* $h^1(\mathbf{R}^n)$ *to* $h^1_q(\mathbf{R}^n)$. *Moreover, there exists a constant* C *independent of* f *such that*

$$\|[b, I_\alpha]f\|_{h^1_q(\mathbf{R}^n)} \le C[b]_{LMO(\mathbf{R}^n)}\|f\|_{h^1(\mathbf{R}^n)}. \tag{1.3}$$

The paper is organized as follows. In Section 2 we introduce some notations and definitions, and recall some preliminary results. Our theorem is proved in Section 3.

In what follows, we always omit the sign of the base space $\mathbf{R}^n$ in the notation of function spaces. For example, L^p means $L^p(\mathbf{R}^n)$.

2. Definitions and some preliminary results. In this section we will give some basic definitions and results which are needed in this paper. The most important property of Hardy type spaces is its atomic decomposition. First we introduce the concept of an *atom*.

Definition 2.1. Suppose $0 < p \le 1$, $s = [n((1/p) - 1)]$. A bounded measurable function a is called a (p, ∞, s)-atom provided that it satisfies the following conditions:

(i) there exists a ball B such that $\operatorname{supp} a \subset B = B_r(x_0)$;

(ii) $|a(x)| \le |B|^{-1/p}$ for almost every $x \in \mathbf{R}^n$;

(iii) $\int a(x)(x - x_0)^\nu dx = 0$, $0 \le |\nu| \le s$.

Here and in what follows, for any $t \in \mathbf{R}$, $[t]$ is the largest integer no more than t. We call B the supporting ball of atom a. For a (p, ∞, s)-atom, if there exists a supporting ball of a with radius less than 1, we call a a type-(a) atom. A measurable function b is called a type-(b) atom, if it satisfies (i) and (ii) and the radius of any supporting ball of b is greater or equal to 1. Type-(a) and type-(b) atoms are both called local (p, ∞, s)-atoms.

It is easy to see that a (p, ∞, s)-atom belongs to H^p and a local (p, ∞, s)-atom belongs to h^p. The following classical results of Hardy space theory can be found, for example, in [**7, 8**].

Theorem 2.1. *A distribution f is in H^p if and only if there exists a sequence $\{\lambda_j\}_{j=1}^\infty \in l^p$ and (p, ∞, s)-atoms $\{a_j\}_{j=1}^\infty$, $j = 1, 2, \dots$, such that*

$$f(x) = \lim_{N \to +\infty} \sum_{j=1}^N \lambda_j a_j(x) = \sum_{j=1}^{+\infty} \lambda_j a_j(x).$$

Moreover,

$$\|f\|_{H^p}^p \sim \inf \left\{ \sum |\lambda_j|^p \right\};$$

here the infimum is taken over all possible decompositions.

For the h^p distribution, there is a similar atomic decomposition of f (see [**4**]).

Theorem 2.2. *A distribution f is in h^p if and only if there exists a sequence $\{\lambda_j\}_{j=1}^{\infty}$, $\{\mu_j\}_{j=1}^{\infty} \in l^p$, type-(a) atoms $\{a_j\}_{j=1}^{\infty}$, and type-(b) atoms $\{b_j\}_{j=1}^{\infty}$ such that*

$$\begin{aligned} f(x) &= \lim_{N\to+\infty} \sum_{j=1}^{N} \lambda_j a_j(x) + \lim_{N\to+\infty} \sum_{j=1}^{N} \mu_j b_j(x) \\ &= \sum_{j=1}^{+\infty} \lambda_j a_j(x) + \sum_{j=1}^{+\infty} \mu_j b_j(x). \end{aligned}$$

Moreover,

$$\|f\|_{h^p}^p \sim \inf\left\{ \sum |\lambda_j|^p + \sum |\mu_j|^p \right\};$$

here the infimum is taken over all possible decompositions.

Definition 2.3. For $q \in [1,\infty)$ we define

$$h_q^p = h^p + L^q = \{f|\ f = h + g,\ h \in h^p,\ g \in L^q\},$$

with the usual norm

$$\|f\|_{h_q^p} = \inf_{f=h+g} (\|h\|_{h^p} + \|g\|_{L^q}).$$

Also we can define $H_q^p = H^p + L^q$ with the norm $\|\cdot\|_{H_q^p}$.

The notation of h_q^p was first introduced in [**12**]; it is a slightly larger space than h^p. By the atomic decomposition it is easy to show

Lemma 2.1. *For any $q \in [1,+\infty)$, $h^p \subset H_q^p$, hence $H_q^p = h_q^p$.*

Definition 2.4. A locally integrable function $f \in BMO$, if

$$[f]_* = \sup_B \frac{1}{|B|} \int_B |f(x) - f_B|\, dx < \infty,$$

where B runs over all balls in $\mathbf{R}^n$, $f_B = 1/|B| \int_B f(x)\, dx$ and $|B|$ is the Lebesgue measure of B.

The space of BMO was first introduced in [**6**] and there the following result was obtained:

Lemma 2.2. *For any $q \in [1, +\infty)$, there exists a constant C depending only on n, q such that*

$$\left(\frac{1}{|B|}\int_B |b(x) - b_B|^q dx\right)^{1/q} \le C[b]_*. \tag{2.1}$$

As a corollary, we have

Lemma 2.3. *For any $q \ge 1$ and $\beta > 0$, there exists a constant C depending only on n, q and β such that*

$$\int_{\mathbf{R}^n \setminus B} |f(y) - f_B|^q |x - y|^{-n-\beta} dy \le C r^{-\beta} [f]_*^q, \tag{2.2}$$

where $B = B_r(x)$.

The proof of Lemma 2.3 can be found in [**3**].

Lemma 2.4. *Suppose $0 < \alpha < n$, $1 < p < (n/\alpha)$. If $1/q = 1/p - \alpha/n$, then there exists a constant C such that*

$$\|I_\alpha f\|_{L^q} \le C\|f\|_{L^p}. \tag{2.3}$$

For the proof of this lemma , see Chapter 5 in [**10**].

Definition 2.5. LMO is a subspace of BMO, equipped with the semi-norm

$$\begin{aligned} [f]_{LMO} = &\sup_{r<1} \frac{1 + |lnr|}{|B_r|} \int_{B_r} |f(x) - f_{B_r}|\, dx \\ &+ \sup_{r \ge 1} \frac{1}{|B_r|} \int_{B_r} |f(x) - f_{B_r}|\, dx. \end{aligned} \tag{2.4}$$

LMO is essentially a special case of a kind of function space introduced by Spanne in [**9**]. For an LMO function, there are some properties similar to those of a BMO function. We introduce only a property needed in the following contest. For $q \in [1, \infty)$, define

$$[f]_{LMO^q} = \sup_{r<1/2} (1 + |lnr|) \left(\frac{1}{|B_r|} \int_{B_r} |f(x) - f_{B_r}|^q dx \right)^{1/q}.$$

Lemma 2.5. *Suppose $f \in LMO$. Then for any $q \in [1, +\infty)$ we have*

$$[f]_{LMO^q} \le C[f]_{LMO}. \tag{2.5}$$

The proof of this lemma can be found in [**1**].

3. Proof of the Theorem. By the atomic decomposition of local Hardy spaces, it is sufficient to show that there exists a constant C such that for each local (p, ∞, s)-atom a with $n/(n + \alpha) < p < 1$,

$$\|[b, I_\alpha](a)\|_{h^p_q} \le C[b]_* \tag{3.1}$$

and

$$\|[b, I_\alpha](a)\|_{h^1_q} \le C[b]_{LMO} \tag{3.2}$$

hold. To do this, let us fix a local (p, ∞, s)-atom a with $\operatorname{supp} a \subset B = B_r(x_0)$.

Since $q > n/(n - \alpha)$, we can choose $q_1 > 1$ such that $1/q = 1/q_1 - \alpha/n$. If $r \ge 1/8$, then by the (L^{q_1}, L^q) boundedness of $[b, I_\alpha]$ (see [**2**]),

$$\|[b, I_\alpha](a)\|_{L^q} \le C[b]_* \|a\|_{L^{q_1}} \le C[b]_* |B|^{(1/q_1)-(1/p)} \le C[b]_* \tag{3.3}$$

holds for $n/(n + \alpha) < p \le 1$. Especially, by Definition 2.5 we have

$$\|[b, I_\alpha](a)\|_{L^q} \le C[b]_* \le C[b]_{LMO} \tag{3.4}$$

for $p = 1$.

Assume $r < 1/8$. Decompose $[b, I_\alpha](a)$ as follows:

$$\begin{aligned}
[b, I_\alpha](a)(x) &= (b(x) - b_B)I_\alpha(a)(x) + I_\alpha((b - b_B)a)(x) \\
&= J_1(x) + J_2(x). \\
J_1(x) &= (b(x) - b_B)I_\alpha(a)(x)\chi_{4B}(x) \\
&\quad + (b(x) - b_B)I_\alpha(a)(x)\chi_{B_0\setminus 4B}(x) \\
&\quad + (b(x) - b_B)I_\alpha(a)(x)\chi_{B_0^c}(x) \\
&= K_1(x) + K_2(x) + K_3(x).
\end{aligned}$$

Here $B_0 = B_1(x_0)$. We will prove that $K_1, K_2 \in h^p$, $K_3 \in L^q$ while $J_2(x) \in h^p_q$.

Step 1: *Estimate of* $\|K_1\|_{h^p}$. Let $q_2 > q_1 > 1$ be such that $q_3 = q_1q_2/(q_2 - q_1) > n/(n - \alpha)$. Using the Hölder inequality and L^{q_1} boundedness of the maximal operator, we obtain

$$\begin{aligned}
\int_{8B} |m_\varphi K_1(x)|^p dx &\le |8B|^{1-(p/q_1)}\|m_\varphi K_1\|^p_{L^{q_1}} \\
&\le C|B|^{1-(p/q_1)}\left(\int_{4B} |b(y) - b_B|^{q_1}|I_\alpha(a)|^{q_1}dy\right)^{p/q_1}.
\end{aligned}$$

Again, by Hölder's inequality, the above expression is majorized by

$$C|B|^{1-(p/q_1)}\left(\int_{4B} |b(y) - b_B|^{q_2}dy\right)^{p/q_2}\left(\int_{4B} |I_\alpha(a)|^{q_3}dy\right)^{p/q_3}.$$

By Lemma 2.2 and Lemma 2.4 we can observe
(3.5)
$$\int_{8B} |m_\varphi K_1(x)|^p dx \le C[b]^p_*|B|^{1-(p/q_1)+(p/q_2)}\,\|a\|^p_{L^{q_3 n/(n+q_3\alpha)}} \le C[b]^p_*$$

for $n/(n + \alpha) < p \le 1$.

If $x \notin 8B$ and $y \in 4B$, since $t < 1$, then

$$|\varphi_t(x - y)| \le C\min\left\{|x - x_0|^{-n}, |x - x_0|^{-n-s-1}\right\}.$$

For $n/(n+\alpha) < p < 1$, notice $p(n+s+1) > n$, the following holds

$$\begin{aligned}\int_{\mathbf{R}^n\setminus 8B} \sup_{t<1} |\varphi_t * K_1(x)|^p\,dx &\leq \int_{B_0\setminus 8B} \sup_{t<1} |\varphi_t * K_1(x)|^p\,dx \\ &\quad + \int_{B_0^c} \sup_{t<1} |\varphi_t * K_1(x)|^p\,dx \\ &\leq C\bigg(\int_{B_0\setminus 8B} |x-x_0|^{-np}dx \\ &\qquad + \int_{B_0^c} |x-x_0|^{-(n+s+1)p}dx\bigg) \\ &\quad \times \bigg(\int_{4B} |(b(y)-b_B)I_\alpha(a)(y)|dy\bigg)^p \\ &\leq C\bigg(\int_{4B} |(b(y)-b_B)I_\alpha(a)(y)|dy\bigg)^p\end{aligned}$$

By Hölder inequality with the index $l > n/(n-\alpha)$ and (2.1),

$$\begin{aligned}(3.6)\quad &\int_{\mathbf{R}^n\setminus 8B} \sup_{t<1} |\varphi_t * K_1(x)|^p\,dx \\ &\leq C\bigg(\int_{4B} |I_\alpha(a)(y)|^l dy\bigg)^{p/l}\bigg(\int_{4B} |b(y)-b_B|^{l'}dy\bigg)^{p/l'} \\ &\leq C|B|^{p/l'}\,\|a\|^p_{L^{ln/(n+l\alpha)}}\bigg(\frac{1}{|B|}\int_{4B} |b(y)-b_B|^{l'}dy\bigg)^{p/l'} \\ &\leq Cr^{(n+\alpha)p-n}[b]_*^p \leq C[b]_*^p.\end{aligned}$$

Combining (3.5) and (3.6), we obtain

$$\|K_1\|_{h^p} \leq C[b]_* \tag{3.7}$$

for $n/(n+\alpha) < p < 1$.

For $p = 1$, by Hölder inequality and (2.5), it follows that

$$\begin{aligned}
(3.8)\qquad \int_{\mathbf{R}^n\setminus 8B} \sup_{t<1} |\varphi_t * K_1(x)|\, dx
&\le \int_{B_0\setminus 8B} \sup_{t<1} |\varphi_t * K_1(x)|\, dx \\
&\quad + \int_{B_0^c} \sup_{t<1} |\varphi_t * K_1(x)|\, dx \\
&\le C\bigg(\int_{B_0\setminus 8B} |x-x_0|^{-n} dx \\
&\qquad + \int_{B_0^c} |x-x_0|^{-n-1} dx \bigg) \\
&\quad \times \int_{4B} |(b(y)-b_B) I_\alpha(a)(y)|\, dy \\
&\le C|B|^{1/l'}(1+|lnr|)\left(\frac{1}{|B|}\int_{4B} |b(y)-b_B|^{l'} dy\right)^{1/l'} \\
&\quad \times \left(\int_{4B} |I_\alpha(a)(y)|^l dy\right)^{1/l} \\
&\le C[b]_{LMO}|B|^{1/l'}\|a\|_{ln/(n+l\alpha)} \le Cr^\alpha [b]_{LMO} \\
&\le C[b]_{LMO}.
\end{aligned}$$

Combining (3.5) and (3.8), we have

$$\|K_1\|_{h^1} \le C[b]_{LMO}. \tag{3.9}$$

Step 2: *Estimate of* $\|K_2\|_{h^p}$. We introduce the following pointwise estimates

$$|I_\alpha(a)(x)| \le Cr^{1+s-n((1/p)-1)}|x-x_0|^{-n-s-1+\alpha} \tag{3.10}$$

for any $|x-x_0| > 2r$ and $n/(n+\alpha) < p \le 1$. It is an easy consequence of properties (iii) and (ii) of the local (p,∞,s)-atom and Taylor's formula. In fact, let $g(x) = |x|^{-(n-\alpha)}$, then

$$g(x-y) = P(x,y) + R(x,y).$$

Here,

$$P(x,y)=\sum_{|\nu|\le s}\frac{D^{\nu}(g)(x-x_0)}{\nu!}(-(y-x_0))^{\nu},$$

$$R(x,y)=\sum_{|\nu|=s}\frac{D^{\nu+1}(g)((x-x_0)-\theta(y-x_0))}{(\nu+1)!}(-(y-x_0))^{\nu+1},$$

$$0<\theta<1.$$

If $y\in B$ and $x\in(2B)^c$, then

$$|R(x,y)|\le c\frac{|y-x_0|^{s+1}}{|x-x_0|^{n+s+1-\alpha}}.$$

Hence,

$$\begin{aligned}|I_\alpha(a)(x)|&=\left|\int_B\left(\frac{1}{|x-y|^{n-\alpha}}-P(x,y)\right)a(y)\,dy\right|\\&\le C\int_B\frac{|a(y)||y-x_0|^{s+1}}{|x-x_0|^{n+s+1-\alpha}}dy\\&\le Cr^{1+s-n((1/p)-1)}|x-x_0|^{-n-s-1+\alpha}.\end{aligned}$$

By (3.10),

$$\int_{B_0\setminus 4B}|b(y)-b_B||I_\alpha(a)(y)|\,dy\le Cr^{1+s-n((1/p)-1)}\int_{4r<|y-x_0|\le 1}\frac{|b(y)-b_B|}{|y-x_0|^{n+s+1-\alpha}}dy.$$

Assume k is a nonnegative integer such that $k<\alpha\le k+1$. If $n/(n+k+1)\le n/(n+\alpha)<p\le n/(n+k)$, then $s+1=[n((1/p)-1)]+1=k+1>\alpha$. Taking notice of $\alpha>n((1/p)-1)$, and using (2.2), we see

$$\begin{aligned}\int_{B_0\setminus 4B}|b(y)-b_B||I_\alpha(a)(y)|\,dy&\le Cr^{1+s-n((1/p)-1)}\int_{|y-x_0|>r}\frac{|b(y)-b_B|}{|y-x_0|^{n+s+1-\alpha}}dy\\&\le Cr^{\alpha-n(\frac{1}{p}-1)}\|b\|_*.\end{aligned}$$

If $n/(n+l+1) < p \le n/(n+l)$, where l is a nonnegative integer less than k, then $s = [n((1/p)-1)] = l$. Choosing $l < \beta < l+1$ such that $0 < \beta - n((1/p)-1) < 1$, then

$$\begin{aligned}
&\int_{B_0\backslash 4B} |b(y) - b_B||I_\alpha(a)(y)|\,dy \\
&\qquad \le Cr^{1+l-n((1/p)-1)} \int_{4r<|y-x_0|\le 1} \frac{|b(y)-b_B|}{|y-x_0|^{n+l+1-\alpha}} dy \\
&\qquad \le r^{1+l-n((1/p)-1)} \int_{4r<|y-x_0|\le 1} \frac{|b(y)-b_B||y-x_0|^{\alpha-\beta}}{|y-x_0|^{n+l+1-\beta}} dy \\
&\qquad \le r^{1+l-n((1/p)-1)} \int_{|y-x_0|>r} \frac{|b(y)-b_B|}{|y-x_0|^{n+l+1-\beta}} dy \\
&\qquad \le Cr^{\beta-n((1/p)-1)}\|b\|_*.
\end{aligned}$$

Then for all $0 < \alpha < n$, there exists $0 < \delta < 1$ such that

$$\int_{B_0\backslash 4B} |b(y) - b_B||I_\alpha(a)(y)|\,dy \le Cr^\delta \|b\|_*. \tag{3.11}$$

If $|y-x_0| \ge 4r$ and $|x-x_0| \le 2r$, then $|x-y| \ge |y-x_0| - |x-x_0| \ge |y-x_0|/2$. Thus, by (3.11),

$$\begin{aligned}
|\varphi_t * K_2(x)| &\le C\int_{4r<|y-x_0|\le 1} |x-y|^{-n}|b(y)-b_B||I_\alpha(a)(y)|\,dy \\
&\le C\int_{4r<|y-x_0|\le 1} |y-x_0|^{-n}|b(y)-b_B||I_\alpha(a)(y)|\,dy \\
&\le Cr^{-n}\int_{4r<|y-x_0|\le 1} |b(y)-b_B||I_\alpha(a)(y)|\,dy \\
&\le Cr^{-n+\delta}[b]_*.
\end{aligned}$$

Hence

$$\int_{2B} \sup_{t<1} |\varphi_t * K_2(x)|^p dx \le Cr^{n-np+\delta p}[b]_*^p \le C[b]_*^p. \tag{3.12}$$

Denote $B_i(x) = B_{2^i r}(x)$; then

$$\begin{aligned}
|\varphi_t * K_2(x)| &= \int_{B_0\setminus 4B} \varphi_t(x-y)|b(y)-b_B||I_\alpha(a)(y)|\,dy \\
&\le \int_{(B_0\setminus 4B)\cap B_{i-1}(x)} \varphi_t(x-y)|b(y)-b_B||I_\alpha(a)(y)|\,dy \\
&\quad + \int_{(B_0\setminus 4B)\cap B_{i-1}^c(x)} \varphi_t(x-y)|b(y)-b_B||I_\alpha(a)(y)|\,dy \\
&= M_{t,1}(x) + M_{t,2}(x).
\end{aligned}$$

Observe that if $x \in B_{i+1} \setminus B_i$, $y \in B_{i-1}(x)$, then $y \in B_{i+2}$ and $|x - x_0| \sim |y - x_0|$. Hence, for $x \in B_{i+1} \setminus B_i$,

$$\begin{aligned}
M_{t,1}(x) &\le \int_{(B_0\setminus 4B)\cap B_{i-1}(x)} \varphi_t(x-y)|b(y)-b_B||I_\alpha(a)(y)|\,dy \\
&\le Cr^{1+s-n((1/p)-1)} \\
&\quad \times \int_{B_{i-1}(x)} |y-x_0|^{-n-s-1+\alpha}\varphi_t(x-y)|b(y)-b_B|\chi_{B_{i+2}}dy \\
&\le Cr^{1+s-n((1/p)-1)}|x-x_0|^{-n-s-1+\alpha} \\
&\quad \times \int_{B_{i-1}(x)} \varphi_t(x-y)|b(y)-b_B|\chi_{B_{i+2}}dy \\
&\le Cr^{1+s-n((1/p)-1)}|x-x_0|^{-n-s-1+\alpha} m_\varphi \\
&\quad \times (|b-b_B|\chi_{B_{i+2}})(x).
\end{aligned}$$

Thus, for $n/(n+\alpha) < p \le 1$,

$$\begin{aligned}
&\int_{2B_0\setminus 2B} |M_{t,1}(x)|^p dx \\
&\quad \le Cr^{p(1+s-n((1/p)-1))} \sum_{i=1}^{[\log_2 2/r]+1} \\
&\qquad \times \int_{B_{i+1}\setminus B_i} \Big(|x-x_0|^{-n-s-1+\alpha} m_\varphi(|b-b_B|\chi_{B_{i+2}})(x)\Big)^p dx \\
&\quad \le Cr^{p(1+s-n((1/p)-1))} \sum_{i=1}^{[\log_2 2/r]+1}
\end{aligned} \tag{3.13}$$

$$
\begin{aligned}
&\quad \times \frac{(2^{ir})^{\alpha}}{(2^i r)^{(n+s+1)p}} \int_{B_{i+1}} \big(m_\varphi(|b-b_B|\chi_{B_{i+2}})(x)\big)^p\,dx \\
&\leq C \sum_{i=1}^{[\log_2 2/r]+1} \frac{(2^i r)^{\alpha p}}{2^{i((n+s+1)p-n)}} \frac{1}{|B_{i+1}|} \\
&\quad \times \left(\int_{B_{i+1}} (m_\varphi(|b-b_B|\chi_{B_{i+2}})(x))^2 dx\right)^{p/2} |B_{i+1}|^{(2-p)/2} \\
&\leq C \sum_{i=1}^{[\log_2 2/r]+1} \frac{(2^i r)^{\alpha p}}{2^{i((n+s+1)p-n)}} \\
&\qquad \times \left(\frac{1}{|B_{j+1}|}\int_{B_{i+1}} (m_\varphi(|b-b_B|\chi_{B_{i+2}})(x))^2 dx\right)^{p/2} \\
&\leq C \sum_{i=1}^{[\log_2 2/r]+1} \frac{(2^i r)^{\alpha p}}{2^{i((n+s+1)p-n)}} \\
&\qquad\qquad \times \left(\frac{1}{|B_{i+2}|}\int_{B_{i+2}} |b-b_B|^2 dy\right)^{p/2} \\
&\leq C \sum_{i=1}^{[\log_2 2/r]+1} \frac{(2^i r)^{\alpha p} i^p}{2^{i((n+s+1)p-n)}} [b]_*^p \leq C[b]_*^p.
\end{aligned}
$$

The last inequality comes from $2^i r \leq 4, i = 1, 2, \dots, [\log_2 2/r] + 1$.

Taking note of

$$
\begin{aligned}
M_{t,2}(x) &= \left| \int_{(B_0\setminus 4B)\cap B^c_{j-1}(x)} \varphi_t(x-y)(b(y)-b_B)I_\alpha(a)(y)\,dy \right| \\
&\leq C \int_{(B_0\setminus 4B)\cap\{|y-x|\geq 2^{j-1}r\}} |x-y|^{-n}|b(y)-b_B||I_\alpha(a)(y)|\,dy \\
&\leq C|B_j|^{-1}\int_{B_0\setminus 4B} |b(y)-b_B||I_\alpha(a)(y)|\,dy \leq\ C|B_j|^{-1}r^\delta[b]_*,
\end{aligned}
$$

then for $p < 1$,

(3.14)

$$
\int_{2B_0\setminus 2B} |M_{t,2}(x)|^p dx \leq \sum_{j=1}^{[\log_2 2/r]+1} \int_{B_{j+1}\setminus B_j} |M_{t,2}(x)|^p dx
$$

$$\begin{aligned}&\leq C[b]_*^p \sum_{j=1}^{[\log_2 2/r]+1} |B_j|^{1-p}\\&= Cr^{n-np}[b]_*^p \sum_{j=1}^{[\log_2 2/r]+1} (2^{n-np})^j\\&\leq C[b]_*^p.\end{aligned}$$

For $p = 1$, since $0 < r < 1/8$ and $0 < \delta < 1$, then

$$\int_{2B_0\setminus 2B} |M_{t,2}(x)|\, dx \leq r^\delta log_2 \frac{4}{r}\|f\|_* \leq \frac{1}{\delta ln2} 2^{2-(1/ln2)}[f]_{LMO}. \tag{3.15}$$

If $x \notin 2B_0$, then

$$\begin{aligned}&\int_{\mathbf{R}^n\setminus 2B_0} |m_\varphi * K_2(x)|^p\, dx \qquad (3.16)\\&= \int_{\mathbf{R}^n\setminus 2B_0} \left(\sup_{t<1}\left|\int_{B_0\setminus 4B} \varphi_t(x-y)(b(y)-b_B)I_\alpha(a)(y)\, dy\right|\right)^p dx\\&\leq C\int_{\mathbf{R}^n\setminus 2B_0} |x-x_0|^{-(n+s+1)p} dx\\&\qquad \left(\int_{B_0\setminus 4B} |b(y)-b_B|||I_\alpha(a)(y)|\, dy\right)^p\\&\leq C\left(\int_{B_0\setminus 4B} |b(y)-b_B||I_\alpha(a)(y)|\, dy\right)^p \leq C[b]_*^p.\end{aligned}$$

Combining (3.12)–(3.14) and (3.16), we obtain

$$\|K_2\|_{h^p} \leq C[b]_*. \tag{3.17}$$

for $n/(n+\alpha) < p < 1$. By (3.12), (3.13), (3.15) and (3.16), we have

$$\|K_2\|_{h^1} \leq C[b]_{LMO}. \tag{3.18}$$

Step 3: *Estimate of* $\|K_3\|_{L^q}$. By (3.10), for any $x \in B_0^c$ we have

$$|K_3(x)| \leq Cr^{1+s-n((1/p)-1)}|b(x)-b_B||x-x_0|^{-n-s-1+\alpha}.$$

Since $q > n/(n-\alpha) > n/((n/p)-\alpha)$, then $(q-1)n + qs + q - q\alpha > q(1+s-n((1/p)-1))$. Hence by Lemma 2.3,

$$\begin{aligned}\|K_3\|_q^q &\le Cr^{q(1+s-n((1/p)-1))} \\ &\quad \times \int_{|x-x_0|>1} \frac{|b(x)-b_B|^q}{|x-x_0|^{q(n+s+1-\alpha)}}\,dx \\ &\le Cr^{q(1+s-n((1/p)-1))} \\ &\quad \times \int_{|x-x_0|>1} \frac{|b-b_B|^q}{|x-x_0|^{n+q(1+s-n((1/p)-1))}}\,dx \\ &\le Cr^{q(1+s-n((1/p)-1))} \\ &\quad \times \int_{|x-x_0|>r} \frac{|b-b_B|^q}{|x-x_0|^{n+q(1+s-n((1/p)-1))}}\,dx \\ &\le C[b]_*^q. \end{aligned} \tag{3.19}$$

Step 4: *Estimate of* $\|J_2\|_{h_q^p}$. In fact, by the above steps we have proved the result

$$\|I_\alpha(f)\|_{h_q^p} \le C\|f\|_{h^p} \tag{3.20}$$

for $n/(n+\alpha) < p \le 1$. Then to prove

$$\|I_\alpha((b-b_B)a)\|_{h_q^p} \le C[b]_* \tag{3.21}$$

for $p < 1$ and

$$\|I_\alpha((b-b_B)a)\|_{h_q^1} \le C[b]_{LMO}, \tag{3.22}$$

it is enough to prove

$$\|(b-b_B)a\|_{h^p} \le C[b]_* \tag{3.23}$$

$$\text{for } p<1 \text{ and} \|(b-b_B)a\|_{h^1} \le C[b]_{LMO}. \tag{3.24}$$

We write

$$\begin{aligned}\|(b-b_B)a\|_{h^p} &\le \int_{2B} (m_\varphi((b-b_B)a)(x))^p dx \\ &\quad + \int_{R^n\setminus 2B} (m_\varphi((b-b_B)a)(x))^p dx \\ &= G_1 + G_2. \end{aligned}$$

In view of the L^2 boundedness of maximal function,

(3.25)
$$G_1 \leq \ C|B|^{p/2}\|m_\varphi(b-b_B)a)\|^p_{L^2} \leq C|B|^{p/2}\|(b-b_B)a)\|^p_{L^2} \leq C[b]^p_*.$$

Similar to Step 1, if $n/(n+\alpha) < p < 1$, then

$$(3.26)\quad \begin{aligned} G_2 &= \int_{R^n\backslash 2B}(m_\varphi((b-b_B)a)(x))^p dx \\ &= \int_{R^n\backslash 2B}\left(\sup_{t<1}\left|\int_B \varphi_t(x-y)(b(y)-b_B)a(y)\,dy\right|\right)^p dx \\ &\leq C\bigg(\int_{B_0\backslash 2B}|x-x_0|^{-np}dx + \int_{B_0^c}|x-x_0|^{-(n+s+1)p}dx\bigg) \\ &\quad \times\bigg(\int_B |(b(y)-b_B)a(y)|\,dy\bigg)^p \leq C[b]^p_*. \end{aligned}$$

By (3.25) and (3.26) we obtain (3.23).

If $p=1$,

$$(3.27)\quad \begin{aligned} G_2 &\leq C\bigg(\int_{B_0\backslash 2B}|x-x_0|^{-n}dx + \int_{B_0^c}|x-x_0|^{-(n+1)p}dx\bigg) \\ &\quad \times \int_B |(b(y)-b_B)a(y)|\,dy \\ &\leq C(1+|lnr|)\bigg(\frac{1}{|B|}\int_B |b(y)-b_B|\,dy\bigg) \\ &\leq C[b]_{LMO}. \end{aligned}$$

By (3.25) and (3.27) we proved (3.24). Then the proof of Step 4 is accomplished.

Summing up the above steps, the proof of the Theorem is completed.

REFERENCES

1. P. Acquistapace, *On BMO regularity for linear elliptic systems*, Ann. Mat. Pura Appl. **161** (4) (1992), 231–269.

2. S. Chanillo, *A note on commutator*, Indiana Univ. Math. J. **31** (1982), 7–16.

3. C.L. Fefferman and E.M. Stein, *H^p-spaces of several variable*, Acta Math. **129** (1972), 137–193.

4. D. Goldberg, *A local version of real Hardy spaces*, Duck Math. J. **46** (1979), 27–42.

5. E. Harboure, C. Segovia and J.L. Torrea, *Boundedness of commutators of fractional and singular integrals for the extreme values of p*, Illinois J. Math. **41** (1997), 676–700.

6. F. John and L. Nirenberg, *On function of bounded mean oscillation*, Comm. Pure. Appl.Math. **14** (1961), 451–426.

7. R.H. Latter, *A decomposition of $H^p(\mathbf{R}^n)$ in terms of atoms*, Studia Math. **62** (1978), 92–101.

8. S.Z. Lu, *Four lectures on real Hardy space*, World Scientific, Singapore, 1995.

9. S. Spanne, *Some function spaces defined using the mean oscillation over cubes*, Ann. Scuala Norm. Sup. Pisa. **19** (1965), 593–608.

10. E.M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, 1970.

11. Y.Z. Sun and W.Y. Su, *Interior h^1-estimates for second order elliptic equations with vanishing LMO coefficients*, J. Funct. Anal. **234** (2006), 235–260.

12. ———, *Hardy type estimates for commutators of singular integrals*, Sci. in China **48** (2005), 563–575.

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