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SOCLE FINITENESS OF THE LOCAL COHOMOLOGY

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ABSTRACT. Let R be a Gorenstein local ring and I an ideal of R. Denote by $\mathcal{H}(I,R)$ the assertion: "The socle of the local cohomology module $H_I^n(R)$ is finitely generated for each $n \geq 0$." We translate $\mathcal{H}(I,R)$ into a property of the minimal injective coresolution of R, and then use this translation to prove $\mathcal{H}(I,R)$ for all regular local rings R and all ideals I with level prime avoidance.

Finiteness properties of local cohomology modules over commutative local noetherian rings are among the basic topics of local cohomology theory. Of particular interest is the Artinianness of $H_I^n(R)$. It is well-known to hold for $I = \mathfrak{m}$ (= the maximal ideal of R) for each $n \in \mathbf{N}$ [**2**, 7.1.3]. The general case splits into two questions (cf. [**6**]):

1. Is the support of $H_I^n(R)$ contained in $\mathfrak{m}$?, and
2. Is the socle of $H_I^n(R)$ finitely generated?

A positive answer to the second question for all regular local rings was conjectured by Huneke (see [**3**, Conjecture, p. 200]). This conjecture was proved in a number of cases: for positive characteristic in [**7**], for equicharacteristic rings in [**9, 10**] (where [**9**] does the characteristic 0 subcase while [**10**] gives an 'almost characteristic free' proof), and for unramified rings of mixed characteristic in [**11**]. However, the conjecture remains open in the case of ramified (regular local) rings of mixed characteristic. By contrast, examples of non-regular (even Gorenstein) rings R such that $H_I^n(R)$ has infinitely generated socle for some I and n are known, cf. [**5, 12**].

In [**7**] the Frobenius homomorphism was essential while the papers [**9, 10, 11**] employed D-modules as the key tool. In the present paper,

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we use level prime avoidance (defined below) and divisible modules as our main tools.

Most of our results are proved in the setting of Gorenstein local rings R. For such an R, we denote by $\mathcal{H}(I,R)$ the assertion: "The socle of the local cohomology module $H_I^n(R)$ is finitely generated for each $n \geq 0$." In Lemma 2, we translate $\mathcal{H}(I,R)$ into a property of the minimal injective coresolution of R. Though this property may fail for Gorenstein rings, we use it to prove in Theorem 6 that $\mathcal{H}(I,R)$ holds for all regular local rings R and all ideals I with *level prime avoidance*.

The latter property means that for each $1 < i < \dim R$, I has the *ith level prime avoidance*, that is, $p \nsubseteq \cup_{q \in Q_i'} q$ whenever $p \in Q_i$ and $Q_i \neq \varnothing \neq Q_i'$, where Q_i (Q_i') denotes the set of all prime ideals of height i that do (do not) contain I.

Note that I has the ith level prime avoidance whenever the set Q_i' is finite (by the classical prime avoidance [**13**, Example 1.6]). Clearly, I has level prime avoidance whenever I is a principal ideal of R.

Throughout the paper, R is a commutative noetherian ring with unit, I is an ideal of R, and M, N are R-modules (not necessarily finitely generated).

For each $i \geq 0$, the *ith local cohomology module* of M with respect to the ideal I is defined by

$$H_I^i(M) = \varinjlim_{n} \operatorname{Ext}_R^i(R/I^n, M)$$

and the *ith generalized local cohomology module* of (M, N) with respect to I by

$$H_I^i(M,N) = \varinjlim_{n \in \mathbf{N}} \operatorname{Ext}_R^i(M/I^n M, N).$$

Basic properties of local cohomology modules can be found in [**2**], and of the generalized local cohomology modules in [**14**]. We refer to [**4, 13**] for basic facts from commutative and homological algebra.

Our first result is inspired by a recent work of Dibaei and Yassemi on local cohomology modules in [**3**, Section 2], but we deal here with the generalized local cohomology.

For an arbitrary R-module N, we fix the minimal injective coresolution of N:

$$0 \to N \to E_1 \to E_2 \to \cdots .$$

Let $\Omega_i N$ denote the ith cosyzygy of N in this coresolution for each $i \in \mathbf{N}$, and put $E_0 = \Omega_0 N (= N)$.

Lemma 1. *Let $s \geq 0$ and $I \subseteq J$ be ideals in a commutative noetherian ring R. Let M and N be R-modules.*

For $s = 0$, assume $\mathrm{Hom}_R(R/J, \mathrm{Hom}_R(M,N))$ is finitely generated. For $s \geq 1$, assume that $\mathrm{Ext}_R^s(M,N) = 0$, and that the modules $\mathrm{Ext}_R^1(R/J, \mathrm{Hom}_R(M, E_{s-1}))$ and $\mathrm{Ext}_R^2(R/J, H_I^0(M, \Omega_{s-1}N))$ are finitely generated.

Then $\mathrm{Hom}_R(F, H_I^s(M,N))$ is finitely generated for each finitely generated module F such that $\mathrm{Supp}_R F \subseteq V(J)$.

Proof. Step I. First we prove the claim in the particular case of $F = R/J$ (and of arbitrary s, M, and N) by induction on s.

If $s = 0$, then, by assumption, $\mathrm{Hom}_R(R/J, \mathrm{Hom}_R(M,N))$ is finitely generated. Since for each $n \in \mathbf{N}$ there is the canonical inclusion $\mathrm{Hom}_R(M/I^nM, N) \subseteq \mathrm{Hom}_R(M,N)$, also $H_I^0(M,N) \subseteq \mathrm{Hom}_R(M,N)$. So $\mathrm{Hom}_R(R/J, H_I^0(M,N))$ is a finitely generated R-module.

If $s = 1$, then $\mathrm{Ext}_R^1(R/J, \mathrm{Hom}_R(M,N))$ and $\mathrm{Ext}_R^2(R/J, H_I^0(M,N))$ are finitely generated by assumption. We will prove that $\mathrm{Hom}_R(R/J, H_I^1(M,N))$ is finitely generated. Consider the exact sequence

$$0 \to H_I^0(M,N) \subseteq \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_R(M,N)/H_I^0(M,N) \to 0.$$

Then we have the long exact sequence

$$\cdots \to \mathrm{Ext}_R^1(R/J, \mathrm{Hom}_R(M,N)) \to \mathrm{Ext}_R^1\left(\frac{R}{J}, \frac{\mathrm{Hom}_R(M,N)}{H_I^0(M,N)}\right) \to \mathrm{Ext}_R^2\left(\frac{R}{J}, H_I^0(M,N)\right) \to \cdots,$$

so $\mathrm{Ext}_R^1(R/J, \mathrm{Hom}_R(M,N)/H_I^0(M,N))$ is finitely generated. Note that

$$0 \to \mathrm{Hom}_R(M,N)/H_I^0(M,N) \to D_I(M,N) \to H_I^1(M,N) \to 0$$
$$= \mathrm{Ext}_R^1(M,N)$$

is exact where

$$D_I(M,-) \cong \varinjlim_{n\in\mathbf{N}} \operatorname{Hom}_R(I^n M,-)$$

is the generalized ideal transform functor (cf. [**2**, 2.2.4]). Then the sequence

$$\cdots \to \operatorname{Hom}_R(R/J, D_I(M,N)) \to \operatorname{Hom}_R(R/J, H_I^1(M,N)) \\ \to \operatorname{Ext}_R^1(R/J, \operatorname{Hom}_R(M,N)/H_I^0(M,N)) \to \cdots$$

is exact. Since $I \subseteq J$, $D_I(M,N)$ contains no non-zero elements annihilated by J, so the module $\operatorname{Hom}_R(R/J, D_I(M,N))$ is zero. We have already proved that $\operatorname{Ext}_R^1(R/J, \operatorname{Hom}_R(M,N)/H_I^0(M,N))$ is finitely generated, hence the same is true of the module $\operatorname{Hom}_R(R/J, H_I^1(M,N))$.

Now suppose that $s > 1$, and assume the claim is true for $s-1$. We will prove it for s. Since

$$H_I^s(M,N) \cong \varinjlim_{n\in\mathbf{N}} \operatorname{Ext}_R^{s-1}(M/I^n M, \Omega_1 N) = H_I^{s-1}(M, \Omega_1 N),$$

it suffices to prove that $\operatorname{Hom}_R(R/J, H_I^{s-1}(M,\Omega_1 N))$ is finitely generated.

For this purpose, we only have to verify the assumptions of our lemma for $s-1$ and the pair $(M, \Omega_1 N)$.

The first assumption for $(M,\Omega_1 N)$ says that $\operatorname{Ext}_R^{s-1}(M,\Omega_1 N) = 0$. This holds as $\operatorname{Ext}_R^{s-1}(M,\Omega_1 N) \cong \operatorname{Ext}_R^s(M,N) = 0$. The second assumption for $(M,\Omega_1 N)$ says that the module $\operatorname{Ext}_R^1(R/J, \operatorname{Hom}_R(M, E_{1+s-2}))$ is finitely generated; but this is just the second assumption for (M,N). Also the third assumptions for $(M,\Omega_1 N)$ and (M,N) coincide because $\Omega_{s-2}\Omega_1 N = \Omega_{s-1} N$.

This finishes the proof of Step I.

Step II. For the general case, we first recall a classical result of Gruson saying that F contains a finite chain of submodules $0 = F_0 \subseteq \cdots \subseteq F_n = F$ such that F_i/F_{i-1} is a homomorphic image of a finite direct sum of copies of R/J for each $0 < i \leq n$. So there is an epimorphism $(R/J)^{m_i} \to F_i/F_{i-1}$ for some $m_i > 0$. Then $\operatorname{Hom}_R(F_i/F_{i-1}, H_I^s(M,N)) \subseteq \operatorname{Hom}_R(R/J, H_I^s(M,N))^{m_i}$, and the latter module is finitely generated by Step I. By induction n, we obtain that $\operatorname{Hom}_R(F, H_I^s(M,N))$ is finitely generated. □

From now on we will restrict ourselves to the particular case of Gorenstein local rings. We will fix further notation for this case:

Let R be a Gorenstein local ring of Krull dimension $\dim R = k$ and with the maximal ideal $\mathfrak{m}$. By a classical result of Bass [**1**], the minimal injective coresolution of R has the form

$$0 \to R \to G_0 \to \cdots \to G_{k-1} \to G_k \to 0$$

where $G_i = \oplus_{p \in P_i} E(R/p)$, and P_i denotes the set of all prime ideals of height i for each $i \leq k$ (so in particular, $G_k = E(R/\mathfrak{m})$).

Let I be an ideal of R. For each $1 \leq i \leq k$, we let $Q_i = P_i \cap V(I)$, $K_i = \oplus_{p \in Q_i} E(R/p)$, and $S_i = \Omega_i R + K_i$. Notice that G_i is the injective envelope of S_i, and G_i/S_i is a factor of the $(i+1)$-st cosyzygy $\Omega_{i+1} R = G_i/\Omega_i R$ for each $1 \leq i < k$.

Finally, let $K_i' = \oplus_{p \in Q_i'} E(R/p)$ where $Q_i' = P_i \setminus Q_i$, so $G_i = K_i \oplus K_i'$ and $S_i = K_i \oplus (S_i \cap K_i')$.

Lemma 1 can be applied to rephrasing the socle finiteness of local cohomology in terms of properties of the modules S_i $(i < k)$:

Lemma 2. *Let R be a Gorenstein local ring of Krull dimension k, and let I be an ideal of R. Then the following conditions are equivalent*:

(1) $\mathrm{Soc}_R(G_i/S_i)$ *is finitely generated for each* $1 \leq i < k$.

(2) *The socle of* $H_I^n(R)$ *is finitely generated for each* $n \geq 0$.

Proof. Clearly, both (1) and (2) hold for $I = R$, so we may assume that $I \subseteq \mathfrak{m}$.

Assume (1). In order to prove (2), we only have to verify the assumptions of Lemma 1 in the given setting, that is, for $s = n$, $M = N = R$, $J = \mathfrak{m}$, $E_i = G_{i-1}$ for $i \geq 1$ (where $G_i = 0$ for $i \geq k+1$), and $F = R/\mathfrak{m}$, where $\mathfrak{m}$ denotes the maximal ideal of R.

First, $\mathrm{Hom}_R(R/\mathfrak{m}, \mathrm{Hom}_R(R, R)) \cong \mathrm{Hom}_R(R/\mathfrak{m}, R) \cong \mathrm{Soc}(R)$ is finitely generated. Clearly $\mathrm{Ext}_R^n(R, R) = 0$ for $n \geq 1$ since R is projective. For $n = 1$ we see that $\mathrm{Ext}_R^1(R/\mathfrak{m}, R)$ is finitely generated because $R/\mathfrak{m}$ and R are such. If $n \geq 2$, then

$$\mathrm{Ext}_R^1(R/\mathfrak{m}, \mathrm{Hom}_R(R, G_{n-2})) = \mathrm{Ext}_R^1(R/\mathfrak{m}, G_{n-2}) = 0$$

because G_{n-2} is injective.

It remains to prove that $\mathrm{Ext}^2_R(R/\mathfrak{m}, H^0_I(\Omega_{n-1}R))$ is finitely generated for each $n \geq 1$. We recall that for a module M, $H^0_I(M) \cong \Gamma_I(M)$ where

$$\Gamma_I(M) = \{x \in M \mid \exists k \geq 0 : I^k x = 0\}.$$

In particular, $\Gamma_I(E(R/p)) = E(R/p)$ when $p \in V(I)$, and $\Gamma_I(E(R/p)) = 0$ when $p \notin V(I)$.

For $n = 1$, we have $H^0_I(\Omega_0 R) = H^0_I(R) \subseteq \mathrm{Hom}_R(R,R) \cong R$, hence $H^0_I(\Omega_0 R)$ is finitely generated, and so is $\mathrm{Ext}^2_R(R/\mathfrak{m}, H^0_I(\Omega_0 R))$. If $n > k$, then $\Gamma_I(\Omega_{n-1}R) = \Omega_{n-1}R$ is injective, so we have $\mathrm{Ext}^2_R(R/\mathfrak{m}, H^0_I(\Omega_{n-1}R)) = 0$.

Assume $1 < n \leq k$. Then $\Omega_{n-1}R$ is an essential submodule of G_{n-1}. So $\Gamma_I(\Omega_{n-1}R) = \Omega_{n-1}R \cap K_{n-1}$, and K_{n-1} is the injective envelope of $\Gamma_I(\Omega_{n-1}R)$. We have

$$K_{n-1}/\Gamma_I(\Omega_{n-1}R) \cong S_{n-1}/\Omega_{n-1}R \subseteq G_{n-1}/\Omega_{n-1}R = \Omega_n R \subseteq G_n,$$

so the exact sequence $0 \to \Gamma_I(\Omega_{n-1}R) \to K_{n-1} \to K_{n-1}/\Gamma_I(\Omega_{n-1}R) \to 0$ yields

$$\begin{aligned}\mathrm{Ext}^2_R(R/\mathfrak{m}, H^0_I(\Omega_{n-1}R)) &= \mathrm{Ext}^2_R(R/\mathfrak{m}, \Gamma_I(\Omega_{n-1}R)) \\ &\cong \mathrm{Ext}^1_R(R/\mathfrak{m}, S_{n-1}/\Omega_{n-1}R).\end{aligned}$$

First, consider the case of $n = k$. Then $G_n = E(R/\mathfrak{m}) = \Omega_n R$ is an injective module containing $S_{n-1}/\Omega_{n-1}R$, so in order to prove that $\mathrm{Ext}^2_R(R/\mathfrak{m}, H^0_I(\Omega_{n-1}R))$ is finitely generated, it is enough to show that $\mathrm{Hom}_R(R/\mathfrak{m}, G_n/(S_{n-1}/\Omega_{n-1}R))$ is such. But this is clear because the module G_n, and hence also $G_n/(S_{n-1}/\Omega_{n-1}R)$, is artinian.

Now, assume $1 < n < k$. In order to prove that $\mathrm{Ext}^1_R(R/\mathfrak{m}, S_{n-1}/\Omega_{n-1}R)$ is finitely generated, it again suffices to show that $\mathrm{Hom}_R(R/\mathfrak{m}, G_n/(S_{n-1}/\Omega_{n-1}R))$ is such.

We have the exact sequence

$$\begin{aligned}0 \to (G_{n-1}/\Omega_{n-1}R)/(S_{n-1}/\Omega_{n-1}R) \subseteq G_n/(S_{n-1}/\Omega_{n-1}R) \\ \to G_n/\Omega_n R \to 0\end{aligned}$$

where $G_n/(G_{n-1}/\Omega_{n-1}R) = G_n/\Omega_n R = \Omega_{n+1}R \subseteq G_{n+1}$.

Note that if $n < k-1$ then $\mathrm{Hom}_R(R/\mathfrak{m}, G_{n+1}) = 0$ by [**4**, Theorem 3.3.8(5)], and if $n = k-1$ then $\mathrm{Hom}_R(R/\mathfrak{m}, G_{n+1}) = \mathrm{Hom}_R(R/\mathfrak{m}, E(R/\mathfrak{m})) \cong \mathrm{Soc}_R(E(R/\mathfrak{m})) \cong R/\mathfrak{m}$. So in either case $\mathrm{Hom}_R(R/\mathfrak{m}, G_n/(G_{n-1}/\Omega_{n-1}R))$ is finitely generated. It remains to show that $\mathrm{Hom}_R(R/\mathfrak{m}, G_{n-1}/S_{n-1})$ is finitely generated. But this is exactly our assumption in (1). This finishes the proof of (2).

Assume (2). First, note that $\mathrm{Hom}_R(R/\mathfrak{m}, H_I^1(\Omega_i R))$ is finitely generated for each $i \geq 1$, because $H_I^n(N) \cong H_I^{n-1}(\Omega_1 N)$ for each module N and each $n > 1$.

We claim that $\mathrm{Ext}_R^1(R/\mathfrak{m}, D_I(\Omega_i R)) = 0$ for all $1 \leq i < k$ where $D_I(-)$ is the classical ideal transform functor [**2**, 2.2.1]. Since $D_I(-)$ is left exact, we have the exact sequence $0 \to D_I(\Omega_i R)) \to D_I(G_i) \to T_i \to 0$ where $T_i \subseteq D_I(G_i/\Omega_i R) = D_I(\Omega_{i+1}R)$.

Notice that $\mathrm{Hom}_R(R/\mathfrak{m}, T_i) \subseteq \mathrm{Hom}_R(R/\mathfrak{m}, D_I(\Omega_{i+1}R)) = 0$ because $D_I(\Omega_{i+1}R)$ contains no elements annihilated by $\mathfrak{m}\,(\supseteq I)$. On the other hand, [**2**, 2.2.7] and the injectivity of G_i yield the exact sequence

$$0 \to G_i/\Gamma_I(G_i) \to D_I(G_i) \to H_I^1(G_i) = 0$$

where $G_i/\Gamma_I(G_i)$ is injective by [**2**, 2.1.5], so $\mathrm{Ext}_R^1(R/\mathfrak{m}, D_I(G_i)) = 0$. Then our claim follows from the exactness of the sequence

$$0 = \mathrm{Hom}_R(R/\mathfrak{m}, T_i) \to \mathrm{Ext}_R^1(R/\mathfrak{m}, D_I(\Omega_i R)) \to \mathrm{Ext}_R^1(R/\mathfrak{m}, D_I(G_i)) = 0.$$

Now [**2**, 2.2.7] yields exactness of the sequence

$$\mathrm{Hom}_R(R/\mathfrak{m}, H_I^1(\Omega_i R)) \to \mathrm{Ext}_R^1(R/\mathfrak{m}, \Omega_i R/\Gamma_I(\Omega_i R)) \to \mathrm{Ext}_R^1(R/\mathfrak{m}, D_I(\Omega_i R)) = 0.$$

This shows that the module $\mathrm{Ext}_R^1(R/\mathfrak{m}, \Omega_i R/\Gamma_I(\Omega_i R))$ is finitely generated.

Finally, $\Omega_i R/\Gamma_I(\Omega_i R) \cong S_i/K_i \subseteq G_i/K_i \cong K_i'$. Since $i < k$, $\mathrm{Hom}_R(R\mathfrak{m}, K_i') = 0$, so $\mathrm{Hom}_R(R/\mathfrak{m}, G_i/S_i) \cong \mathrm{Ext}_R^1(R/\mathfrak{m}, S_i/K_i)$ is finitely generated, and (1) holds. □

As an immediate corollary of Lemma 2, we obtain the low dimensional case of the Huneke conjecture:

Corollary 3. *Let R be a Gorenstein local ring of Krull dimension ≤ 2 with the maximal ideal $\mathfrak{m}$, and let I be an ideal of R. Then the socle of $H^n_I(R)$ is finitely generated for each $n \geq 0$.*

Proof. Just note that $G_2/(S_1/\Omega_1(R))$ is artinian for $k = 2$ (see the proof that (1) implies (2) in Lemma 2 above), so $\mathrm{Soc}_R(G_1/S_1)$ is finitely generated. □

Let R be a Gorenstein local ring of Krull dimension k with the maximal ideal $\mathfrak{m}$, and let $x \in \mathfrak{m}$ be a non-zerodivisor on R.

For a module M, denote by $\overline{M}$ the R/xR-module $\{m \in M \mid m \cdot x = 0\}$. Then $\overline{M} \cong \mathrm{Hom}_R(R/xR, M)$.

We will need the following well-known fact (see [**8**, Lemma 8.12]):

Lemma 4. *The ring R/xR is a Gorenstein local ring of Krull dimension $k-1$, and R/xR has a minimal injective coresolution of the form*

$$0 \to R/xR \to \overline{G_1} \to \cdots \to \overline{G_k} \to 0 \quad (*)$$

where the ith cosyzygy of R/xR in $(*)$ is $\overline{\Omega_{i+1}R}$ for each $0 \leq i < k$.

Notice that since $\mathrm{Ass}(E(R/p)) = \{p\}$ for each prime ideal p, we have $\overline{G_i} = \oplus_{p \in P_i, x \in p} \overline{E(R/p)}$, $\overline{K_i} = \oplus_{p \in P_i, I \subseteq p, x \in p} \overline{E(R/p)}$, and $\overline{K_i'} = \oplus_{p \in P_i, I \nsubseteq p, x \in p} \overline{E(R/p)}$ $(1 \leq i \leq k)$. All these modules are injective as R/xR-modules by the well-known Hom-$\otimes$ relations.

Our next 'transfer lemma' says roughly that if for some $1 < i \leq k$, an ideal $I \subseteq \mathfrak{m}$ has ith level prime avoidance and G_i/S_i has infinitely generated socle, then the same holds for the ideal $(I + xR)/x$ of R/xR at $i - 1$ where $x \in \mathfrak{m}$ is any non-zerodivisor on R.

Before stating the lemma, we recall that a module D is *divisible* provided that $\mathrm{Ext}^1_R(R/rR, D) = 0$ (or, equivalently, $Dr = D$) for each non-zerodivisor $r \in R$. Any injective module is divisible, and so are (direct) sums and homomorphic images of divisible modules. In particular, all the ith cosyzygy modules for $i > 0$ are divisible, and so are the modules S_i $(1 \leq i \leq k)$ defined above.

Lemma 5. *Let $I \subseteq \mathfrak{m}$ be an ideal of R and $1 < i \leq k = \dim R$. Assume that G_i/S_i has infinitely generated socle (so clearly $Q_i \neq \varnothing \neq Q'_i$), and that I has ith level prime avoidance. Let $x \in \mathfrak{m}$ be a non-zerodivisor on R.*

Then $\overline{S_i} = \overline{\Omega_i R} + \overline{K_i}$ (so that $\overline{G_i}/\overline{S_i}$ is isomorphic to G_{i-1}/S_{i-1} for the ideal $(I + xR)/xR$ of the ring R/xR), and $\overline{G_i}/\overline{S_i}$ has infinitely generated socle as R/xR-module. Moreover, the ideal $(I + xR)/xR$ of R/xR has $(i-1)$st level prime avoidance.

Proof. First we prove that $\overline{S_i} = \overline{\Omega_i R} + \overline{K_i}$. Since $S_i = \Omega_i R + K_i$ by definition, the inclusion $\supseteq$ is clear. The direct sum decomposition $S_i = K_i \oplus (S_i \cap K'_i)$ yields $\overline{S_i} = \overline{K_i} \oplus \overline{S_i \cap K'_i}$.

Assume there exists $y \in \overline{S_i} \setminus (\overline{\Omega_i R} + \overline{K_i})$. Without loss of generality, $y = 0 + k'$ where $0 \neq k' \in \overline{S_i \cap K'_i}$. Moreover, $y = k + \omega$ for some $k \in K_i \setminus \overline{K_i}$ and $\omega \in \Omega_i R \setminus \overline{\Omega_i R}$, so $\omega = -k + k'$.

We have $k \in \oplus_{p \in F} E(R/p) \subseteq K_i$ for a finite set $F \subseteq Q_i$. By ith level prime avoidance, for each $p \in F$ there is an $r \in p \setminus \cup_{q \in Q'_i} q$. By Matlis theorem [**13**, 18.4], it follows that there exists an $x_1 \in R$ such that $x_1 \cdot k = 0$, but the multiplication by x_1 is an automorphism of K'_i.

In particular, $x_1 \cdot \omega \in \overline{\Omega_i R}$. Since $i > 1$ and $\overline{\Omega_i R}$ is the $(i-1)$st cosyzygy of R/Rx in $(*)$ by Lemma 4, $\overline{\Omega_i R}$ is a divisible R/xR-module. So there exists an $\omega_1 \in \overline{\Omega_i R}$ such that $x_1 \cdot \omega_1 = x_1 \cdot \omega$. Since the multiplication by x_1 is an automorphism of K'_i, the decomposition of ω_1 in $K_i \oplus K'_i$ is $\omega_1 = k_1 + k'$ where $(xR + x_1 R) \cdot k_1 = 0$ because $\omega_1 \in \overline{\Omega_i R}$. So $k_1 \in \overline{K_i}$, and $y = \omega_1 - k_1 \in \overline{\Omega_i R} + \overline{K_i}$, a contradiction.

Since the socle of G_i/S_i is not finitely generated, there is a module $S_i \subseteq T_i \subseteq G_i$ such that $V_i = T_i/S_i$ is an infinite direct sum of copies of $R/\mathfrak{m}$. Applying the functor $\mathrm{Hom}_R(R/xR, -)$ we obtain $\overline{S_i} \subseteq \overline{T_i} \subseteq \overline{G_i}$, and also the exact sequence $0 \to \overline{S_i} \to \overline{T_i} \to \overline{V_i} \to \mathrm{Ext}^1_R(R/xR, S_i) = 0$ where the latter Ext is zero because S_i is divisible. However, $\overline{V_i} = V_i$ because $x \in \mathfrak{m}$, so the socle of $\overline{G_i}/\overline{S_i}$ is not finitely generated as R/xR-module.

The final claim follows from the fact that the prime ideals of R/xR of height $i-1$ containing (not containing) $(I + xR)/xR$ are exactly the ideals of R/xR of the form p/xR where $x \in p \in Q_i$ ($x \in p \in Q'_i$). □

Finally, we consider the case when R is regular:

Theorem 6. *Let R be a regular local ring, and let I be an ideal with level prime avoidance. Then the socle of $H_I^n(R)$ is finitely generated for each $n \geq 0$.*

Proof. Assume the claim fails and take a counterexample (R, I) with R of minimal Krull dimension k. By Corollary 3, $k > 2$. By Lemma 2, there is a $1 \leq i < k$ such that $\mathrm{Soc}_R(G_i/S_i)$ is not finitely generated (and hence $Q_i \neq \varnothing \neq Q'_i$).

By the Auslander-Buchsbaum theorem, R is a UFD, so all $p \in P_1$ are principal. By classical prime avoidance, there exists an $x \in \mathfrak{m}$ such that $x \notin \mathfrak{m}^2 \cup \cup_{p \in P_0} p$. Then $xR \in P_1$ and R/xR is a regular local ring of Krull dimension $k-1$ by [**13**, 14.2].

If $i > 1$, then Lemma 5 shows that the R/xR-module $G_{i-1}/S_{i-1} \cong \overline{G_i}/\overline{S_i}$ corresponding to the ideal $(I + xR)/xR$ of R/xR has infinitely generated socle, and $(I + xR)/xR$ has level prime avoidance. Thus, by Lemmas 2 and 4, $(R/xR, (I + xR)/xR)$ is a counterexample such that R/xR has Krull dimension $k - 1$, a contradiction.

So $i = 1$. Take $p \in Q_1$, so $p = xR$ for a non-zerodivisor of R. Note that $\overline{\Omega_1 R} \cong R/xR$ by Lemma 4. Then $\overline{G_1} = \overline{E(R/xR)} = \overline{S_1} = \overline{K_1}$ is just the quotient field of the regular local ring R/xR. But, as in the proof of Lemma 5, the divisibility of the R-module S_1 implies that the R/xR-module $\overline{G_1}/\overline{S_1}$ has infinitely generated socle, so certainly $\overline{G_1}/\overline{S_1} \neq 0$, a contradiction. □

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