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CENTER CONFIGURATIONS OF HAMILTONIAN CUBIC SYSTEMS

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ABSTRACT. Second order eyes of Hamiltonian cubic systems are classified into seven classes based on the orientation of the cycles within these eyes. They are further categorized into nine Conti classes based on the structure of the separatrix cycles that bound them. Examples of systems of each type are presented, and examples of third and fourth order eyes are also given. The classification is connected to part of Hilbert's sixteenth problem which asks for the possible relative positions of cycles in an autonomous polynomial system in the plane.

1. Introduction. Hilbert's sixteenth problem poses two questions for systems of differential equations of the form

$$\begin{cases} dx/dt = P(x,y) \\ dy/dt = Q(x,y), \end{cases} \tag{1.1}$$

where P and Q are polynomials of degree at most n in x and y. The first is to determine the maximum number of limit cycles, $H(n)$, that such a system may have. The second is to determine all the possible relative positions of limit cycles for each given degree n. Here we consider a related problem for the case $n = 3$, but rather than limit cycles we consider the possible relative positions of the continuous bands of cycles that occur in Hamiltonian systems.

This is connected to the *weak Hilbert problem* or *Arnol'd problem* [**1**] which asks for a bound, $A(n)$, on the number of limit cycles that can be produced by perturbation having degree less than or equal to n from a Hamiltonian system of the same degree n. It is natural to also ask for the possible relative positions of limit cycles produced in this way. For quadratic systems ($n = 2$), the Arnol'd problem has been answered by Horozov and Iliev [**8**] in the case of generic Hamiltonian systems which have a single center, and three saddles whose separatrices do not

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connect. They show that two is the maximum number in this case. Li and Huang [**9**] have given an example of a cubic system which shows that $A(3) \geq 11$. The Hamiltonian system used by Li and Huang has the phase portrait shown in Figure 1.1.

$$H(x,y) = \frac{1}{2}x^2 + y^2 - \frac{1}{4}(2x^4 + y^4).$$

Li and Huang's example has a *global eye of order three* according to the definition of Cima and Llibre [**3**]:

Definition. A concentric set of closed orbits about a single equilibrium point is called a *simple eye* or an *eye of order one*. A compound structure of cycles is an *eye of order n* if inside the outermost fan of cycles there are two or more distinct eyes of cycles, at least one of which is an eye of order $n-1$.

In fact, Li and Huang's example has an eye of order three, two eyes of order two and five eyes of order one. In notation to be explained later, both eyes of order two are of type $\{\{++\}+\}$ and the eye of order three is of type $\{\{\{\{++\}+\}\{\{++\}+\}-\}+\}$.

In this paper, eyes of order two in Hamiltonian cubic systems are the main focus. In this setting, an eye of order two or second order eye refers to the complex structure in which a continuous band of cycles encircles two or more centers. Interest lies in determining all possible configurations of eyes of order two. It is shown that Hamiltonian cubic systems have exactly seven different types of second order eyes, and that these occur in nine different Conti classes as defined below. Examples are given of Hamilton cubic systems in each of the nine Conti classes.

2. Preliminary results. Cima, Gasull and Manosas [**2**] have proved the following result:

Theorem 2.1. *A Hamiltonian cubic system has at most five centers.*

From this, it is immediately clear that an eye of order two will consist of a continuous band of cycles that surrounds two, three, four or five centers. Thus, there are 4 possible configurations if the orientations

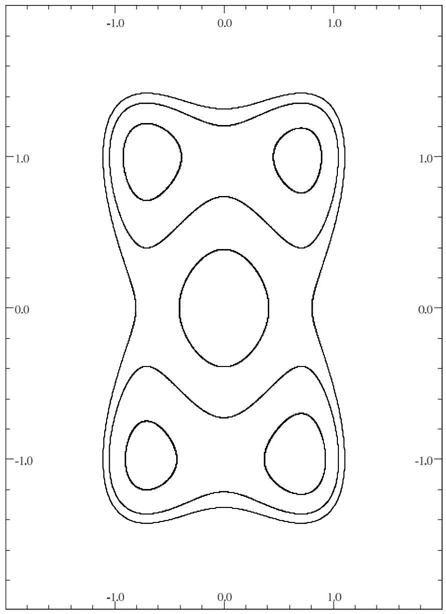


FIGURE 1.1. A Hamiltonian cubic system with an eye of order three.

of these cycles are not taken into consideration, and 18 possibilities if orientation is considered; for example, with two centers there are three possibilities:

$$\{\{++\}+\},\ \{\{+-\}+\},\ \{\{--\}+\}.$$

In this notation, the inner bracket indicates the orientation of the centers, and the plus sign outside that bracket indicates that the outermost band of cycles is positively oriented (this may be assumed by reversing time if necessary). We shall show that just 7 of these 18 possibilities can occur. The following is an extension of a well-known result for quadratic systems, see Coppel [**6**]; its proof is straightforward.

Theorem 2.2 (Tung's lemma for cubic systems). (i) *Four critical points are never collinear*;

(ii) *If* l *is a straight line that is not invariant, then the total number of critical points and contacts on* l *is at most* 3.

The following corollary shows, for example, that the configuration $\{\{--\}+\}$ cannot occur in Hamiltonian cubic systems.

Corollary 2.3. *If a cubic system has cycles of the same orientation occurring in different eyes, then they cannot be encircled by cycles of opposite orientation.*

Proof. Let P_1 and P_2 be critical points inside the distinct eyes, and let l be the line through P_1 and P_2. If there are cycles surrounding both centers which are oppositely oriented to the centers themselves then there must be at least 5 critical points or contacts on l. □

Theorem 2.4. *Consider a cubic Hamiltonian system with Hamiltonian $H(x,y)$. All saddles inside the outermost band of cycles of an eye of order two lie on a single branch of some level curve of H.*

Proof. Since the saddles are enclosed by cycles, their separatrices cannot go to infinity and hence must all connect. Moreover, each homoclinic or heteroclinic loop must contain a single center. In particular, no saddles lie within such loops, and consequently must all connect. □

The proof of the following theorem uses proofs by contradiction to Bezout's theorem: that a curve of degree m and a curve of degree n can intersect in at most mn points counting multiplicity. Recall that a point on an algebraic curve has order $n > 1$ if the curve has a transverse self-intersection at that point with n distinct tangent lines.

Theorem 2.5. *Let $H(x,y)$ be an irreducible polynomial of degree four, and consider the level curve C given by $H(x,y) = h$. Then*

(i) *C cannot have a point of order four or more*;

(ii) *If C has a point of order three, then it has no other points of order greater than* 1.

(iii) *C has at most three double points.*

Proof. (i) If C has a point of order ≥ 4, consider the line, l, through this point and any other point on C. Clearly, l and C intersect at ≥ 5 points including multiplicity.

(ii) Let p_1 be a point on C of order three, and suppose there is a point p_2 on C of order ≥ 2. Then the line through p_1 and p_2 intersects C at ≥ 5 points including multiplicity.

(iii) Suppose C has more than 3 points of order ≥ 2 and consider the conic through 4 of these points and a simple point of C. This conic intersects C in ≥ 9 points. □

The above two theorems imply that an eye of order two of an *irreducible* Hamiltonian cubic system cannot contain 5 centers. If so, the level curve containing the saddles has 4 double points contrary to (iii). But the case in which $H(x,y)$ is reducible must also be considered . In light of Theorem 2.4, consider the level curve containing the saddles. This is clearly bounded, and so $H(x,y)$ must consist of the product of two ellipses in general position [**2**], giving a second order eye with five centers, one of which is oriented differently to the other four.

The following theorem is also a consequence of Bezout's theorem.

Theorem 2.6. *Let $H(x,y)$ be an irreducible polynomial of degree four, and suppose the level curve C given by $H(x,y) = h$ contains one separatrix loop inside another. Then the level curve $H(x,y) = h$ consists entirely of the union of these two loops.*

Proof. Consider any straight line drawn through the center which lies inside the inner loop. This line intersects C at four points–two on each loop. If $H(x,y) = h$ contains point(s) other than the two loops, then a line could be drawn meeting C in five points contrary to Bezout's theorem. □

The facts given in the theorems of this section limit the number of configurations of Hamiltonian cubic systems to just seven. In the next section we show that all seven are indeed possible.

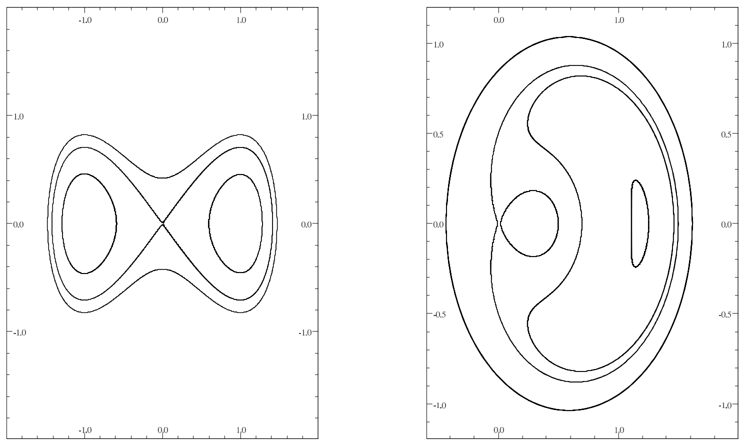
3. Configurations of second order eyes. The results of the last section show that a second order eye can contain at most five centers–with at most four having the same orientation, and that at most one of these centers can be oriented oppositely to the outermost band of cycles in the eye. This leads to seven possible configurations. In fact, we will make a finer classification using Conti's scheme [**6**] leading to nine possible configurations. All nine are shown to be possible with concrete examples.

Definition. A center is of class D_{mn} if the continuous band of closed orbits in the center are bounded by a separatrix cycle containing m critical points and n orbits which do not reduce to a point.

Conti's notation may be extended to classify second order eyes. As an example, $D_{pq} + D_{mn}$ is used to classify a second order eye with two centers, one of class D_{pq} and the other of class D_{mn}. If $m = p$ and $n = q$, we will abbreviate this to $2D_{mn}$.

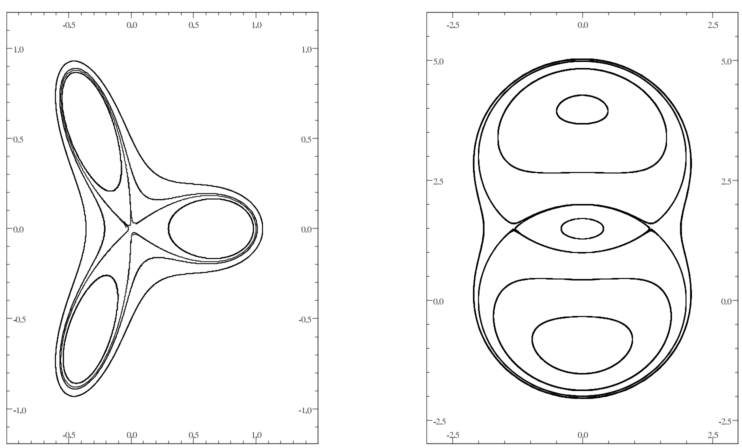
We begin by considering eyes of order two which contain two centers and one saddle. Since the level curve containing the saddle is bounded, then the four separatrices of the saddle must in fact be two homoclinic loops. There are two possible configurations depending on whether these loops sit side-by-side or one inside the other. The first is of type $\{\{++\}+\}$ and class $2D_{11}$; the second has type $\{\{+-\}+\}$ and class $D_{11} + D_{12}$. Both types are represented by well-known examples–the first by Duffing's equation, while the second has the limaçon given by $r = (1/2) + \cos\theta$ as the level curve containing the saddle. These are shown in Figures 3.1 a) and b).

For second order eyes that have three centers, either there are two saddles or a single critical point of index 2. In the latter case, the critical point has six separatrices, and so $H(x, y)$ consists of three homoclinic loops that meet at the critical point. This is of type $\{\{+++\}+\}$ and class $3D_{11}$. An example of this type, given in Figure 3.2 a), has the level curve containing the nonelementary critical point given by the three-leaf rose $r = \cos 3\theta$. In the case of two saddles, by Theorem 2.4 these are connected, and so there are two cases, depending on whether there are homoclinic orbits or not. The first case gives an eye of order two of type $\{\{++-\}+\}$ and class $3D_{22}$ and the other of type $\{\{+++\}+\}$ and class $2D_{11} + D_{22}$. An example of the former is shown in Figure 3.2 b) and has the level curve containing the saddles given by the generic intersection of two circles. An example of the latter is shown in Figure 4.1 a).



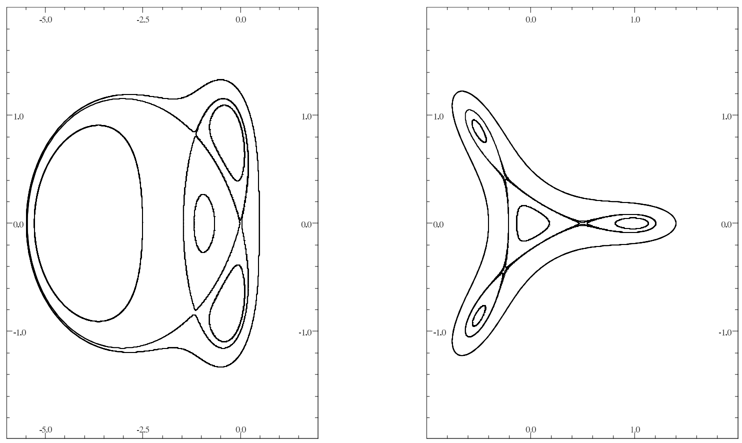
a) b)

FIGURE 3.1. Hamiltonian cubic systems with eyes of order two and two centers. a) $H(x,y) = (1/2)x^2 - y^2 - (1/4)x^4$; b) $H(x,y) = (1/4)x^2 + y^2 - x^2 + y^2 - x^2$.



a) b)

FIGURE 3.2. Hamiltonian cubic systems with eyes of order two and three centers. a) $H(x,y) = x^2 + y^2, -x(x^2 - 3y^2)$; b) $H(x,y) = x^2 + y^2 - 4x^2 + (y-3)^2 - 4$.



a) b)

FIGURE 3.3. Hamiltonian cubic systems with eyes of order two and four centers. a) $H(x,y) = x^4 + 5x^2y^2 + 4y^4 + 4\sqrt{3}x(x^2+y^2) + 8x^2 - 4y^2$ b) $H(x,y) = x^2 + y^2$, $-2x(x^2 - 3y^2) + (x^2+y^2)$.

For eyes of order two that have four centers, there must be three saddles as Theorem 2.5 disallows the possibility of nonelementary critical points in this case. If there are no homoclinic loops, each saddle shares two separatrices with each of the other saddles, giving an eye of order two of type $\{\{+++-\}+\}$ and class $3D_{22} + D_{33}$. An example is given in Figure 3.3 a). If there are two homoclinic loops, the saddle without homoclinic separatrices must share two separatrices with each of the other two saddles, resulting in an eye of order two of type $\{++++\}+\}$ and class $2D_{11} + 2D_{22}$. An example is given in Figure 4.1 b). If there are three homoclinic loops, each saddle shares a separatrix with each of the other saddles, giving an eye of order two of type $\{\{+++-\}+\}$ and class $3D_{11} + 3D_{33}$. An example is given in Figure 3.3 b). The case of a single homoclinic loop means that the two saddles without homoclinic separatrices would share three separatrices and be of type $\{\{+++-\}+\}$. But this case is an impossibility since the line joining the center inside the homoclinic loop with the center of opposite orientation would violate Theorem 2.2. Both examples are due to Conti [**5**].

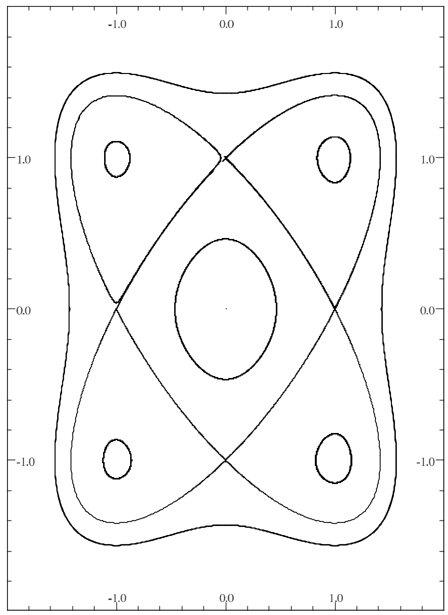
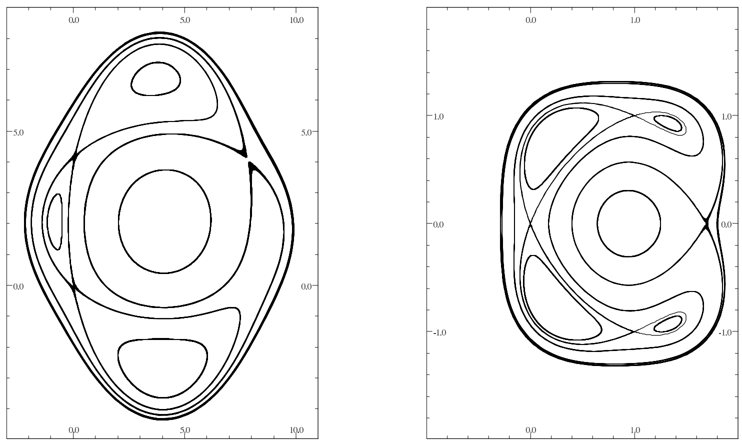


FIGURE 3.4. A Hamiltonian cubic system with an eye of order two and five centers. $H(x,y) = (1/2)x^2 + y^2 - (1/4)x^4 + y^4$.

For eyes of order two that have five centers, the only possibility is when $H(x,y)$ reduces to the product of two ellipses in general position. Such centers are of type $\{\{++++-\}+\}$ and of class $D_{44} + 4D_{22}$. An example is shown in Figure 3.4.

The above results are summarized in the following theorem. Note that all nine Conti classes are possible, as shown in Figure 3.1 a), b), Figure 3.2 a), b), Figure 3.3 a), b), Figure 3.4 and Figure 4.1 a), b).

Theorem 3.1. *There are seven types of second order eyes for Hamiltonian cubic systems, and these occur in nine Conti classes.*



a) b)

FIGURE 4.1. Hamiltonian cubic systems with eyes of order three. a) $H(2x^2+y^2-16x-4y)(x^2+3y^2-8x-12y)+(8x+4xy)$ b) $H(x,y)=y^2-x^2(x-1)(2x-3)-4(x^2-2x+y^2)^2$.

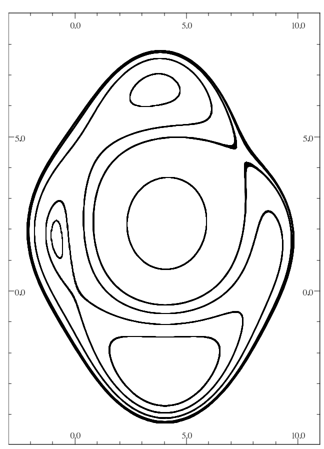


FIGURE 4.2. A Hamiltonian cubic system with an eye of order four. $H(x,y)=(2x^2+y^2-16x-4y)(x^2+3y^2-8x-12y)+(8x+5xy+6y^2)$.

4. Eyes of order three and four. All the examples given in Section 3 are such that the outermost band of cycles extends to infinity; as such, these second order eyes are *global* eyes. Extending the definition of Conti [**4**] from centers to include continuous bands of cycles, we say that the outermost bands of cycles are of *class A*. If the second order eyes are part of an eye of order three, then the outermost band of cycles will be of *class* D_{mn} for some natural numbers m and n.

The example of Figure 1 has second order eyes of type $\{\{++\}+\}$ which belong to an eye of order three (or third order eye). The third order eye is of type $\{\{\{\{++\}+\}\ \{\{++\}+\}-\}+\}$ and is itself of *Class A*.

The examples shown in Figure 4.1 also have third order eyes; that of Figure 4.1 a) has type $\{\{\{\{+++\}+\}-\}+\}$ and that of Figure 4.1 b) has type $\{\{\{\{+++++\}+\}-\}+\}$. Again the third order eye is of *Class A*.

The example of Figure 4.1 a) has a single second order eye of type $\{\{+++\}+\}$ whose outermost cycles are of class D_{12}. This is similar to Li and Huang's example shown in Figure 1.1. This has two second order eyes, both of which are of type $\{\{++\}+\}$ and whose outermost bands of cycles have class D_{22}. The example of Figure 4.1 b), due to Salmon [**10**], has a single second order eye of type $\{\{+++++\}+\}$ whose outermost bands have class D_{12}.

The example of Figure 4.2, due to Hilton [**7**], shows an example of a fourth order eye. A second order eye of type $\{\{++\}+\}$ lies within a third order eye of type $\{\{\{\{++\}+\}+\}+\}$ and *Class* D_{12} which forms part of a fourth order eye of type $\{\{\{\{\{\{++\}+\}+\}+\}-\}+\}$ and *Class A*.

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