ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 40, Number 1, 2010

GENERALIZED EIGENFUNCTION EXPANSIONS FOR SPECTRAL MULTIPLICITY ONE AND APPLICATION IN ANALYTIC NUMBER THEORY

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ABSTRACT. We study generalized eigenfunction expansions of multiplicity one, obtaining precise convergence estimates. We apply the theory to the expansion for the Laplacian and Hecke operators on the fundamental domain for the modular group, where the convergence estimates are shown to be optimal.

1. Introduction. In this paper, abstract convergence results are obtained for generalized eigenfunction expansions for a commuting family of operators which is of multiplicity one, in the sense (defined below) that there exists at most a one-dimensional space of generalized eigenfunctions for every value of the spectral parameter. The spectral parameter we use is the joint spectrum for the generators of the commuting family. The expansion is then applied to an eigenfunction expansion of fundamental significance in analytic number theory, obtaining new convergence results which are in a sense optimal.

In a previous paper [**11**] by one of the authors, a simple abstract formalism for the theory of generalized eigenfunction expansions for a C^* algebra of commuting operators was developed. The theory constructs "generalized eigenprojections" which are elements of the space $C(W, W')$ of continuous conjugate linear operators from a locally convex topological vector space W into its dual W', which contains W. Operators are then expanded in terms of these, with an integral expansion which converges in $C(W, W')$, and hence, in a sense, uniformly. The eigenprojection, acting upon some element $\phi \in W$, produces the appropriate eigenfunction needed to expand $A\phi$, for any member A of the algebra.

2010 AMS *Mathematics subject classification.* Primary 46L10, 47E05, 47F05, 47B25, 11F25, 11F03.
Received by the editors on March 24, 2006, and in revised form on August 22, 2007.

DOI:10.1216/RMJ-2010-40-1-243

In this paper, we first note that in the multiplicity one case, where the range of any generalized eigenprojection has dimension at most one, one can obtain better estimates than in the infinite multiplicity case. As an application, (and, as the primary motivation for the paper) we study the generalized eigenfunction expansion for the algebra generated by the Laplacian on the usual fundamental domain for the modular group and the Hecke operators, which is known to have multiplicity one in our sense. We recover from the abstract theory the known behavior of the generalized eigenfunctions. In the convergence estimates, the target space is the space where the eigenfunctions lie. In this sense, the convergence estimates are optimal, though the domain space Z below, such that the convergence is uniform from the unit ball of Z to the target space, can probably be improved with more work.

For the general theory, we assume that the algebra is a countably generated algebra of operators on a separable Hilbert space; that is, it is the smallest von Neumann algebra containing the spectral projections of a countable family of commuting normal operators. Since some of the operators A_i are unbounded normal operators, some care must be taken to define what it means for them to commute: for us, this means that the entire family $\{A_i\}$ is affiliated with a commutative von Neumann algebra $\mathcal{A}$; the minimal such $\mathcal{A}$ is the algebra we use in the paper. To say that A is *affiliated* with $\mathcal{A}$ means that for any unitary operator U in the commutant of $\mathcal{A}$,

$$UA = AU$$

in the strong sense that they have the same domain and the above relation holds on it. Note that, by the double commutant theorem, bounded operators affiliated with a von Neumann algebra are actually in the algebra.

The *joint spectrum* of $\{A_i\}$ is a natural closed subset of the set of all sequences of complex numbers $\{\lambda_i\}$ such that λ_i is in the spectrum of A_i; it is natural in the sense that if all operators were discrete and the family $\{A_i\}$ has multiplicity one, this subset would be the set of eigenvalues corresponding to the same eigenfunction. A more precise definition is given below. In the situation of this paper, all but a finite number of $\{A_i\}$ are bounded: picking the bounded operators to be of norm one, we see that the joint spectrum is a metric space It turns out to be a closed subset of the product space above.

Using the uniform convergence result contained in [**11**], applied to the multiplicity one situation, with some additional estimates, we obtain the following general result for the multiplicity one case: here Z and B are Banach spaces with certain properties below, where the generalized eigenfunctions turn out to lie in B;

Corollary 1. *For every $\varepsilon > 0$, there exists a compact subset K of the joint spectrum, and a positive δ, such that for every δ-net $\{\lambda_i\}$ of K there exists a set of generalized simultaneous eigenfunctions F_i such that*

$$A_n^t F_{\lambda_i} = (\lambda_i)_n F_{\lambda_i}$$

and a set of complex constants c_i such that, for every ϕ in the unit ball of Z,

$$\left\| \phi - \sum_{i=1}^{n(\varepsilon)} c_i F_{\lambda_i}(\phi) F_{\lambda_i} \right\|_B < \varepsilon.$$

Here, F_i and c_i are independent of ϕ. c_i and $n(\varepsilon)$ in general depend upon the choice of $\{F_i\}$, though once $\{\lambda_i\}$ is chosen, $\{F_i\}$ is of course unique up to a constant.

This is the basic abstract result of this paper. It will be a consequence of Theorem 31 below, which can be considered the main abstract result of the paper. Corollary 1 is then restated as Corollary 33. Note that the above result suggests that the continuous, point continuous and discrete spectrum may be treated, in the sense of approximation, by exactly the same formalism. Since in the discrete spectrum case, we cannot obtain a δ net in general without picking the exact points of the spectrum, it seems that as a general approximation procedure the above result cannot be improved. In addition, if the space B is (within the category of spaces studied) the smallest space where the eigenfunctions lie, then B is the smallest target space where the convergence estimates can hold. In the application to number theory below, B may be chosen to be

$$y^{-(1/2)-\varepsilon} L_\infty$$

for any $\varepsilon > 0$. In terms of a weighted L_∞ norm, with weight a power of y, this is optimal (Section 5).

In the example from analytic number theory, with $\mathbf{D}$ denoting the usual fundamental domain for the modular group in the upper half plane with the hyperbolic metric, the algebra is generated by the Hecke operators (defined below) and the self-adjoint Laplacian H on the complete Riemannian manifold $\mathbf{D}$. H is the closure of its restriction to the smooth compact support functions by Cordes's theorem, which states that on any complete Riemannian manifold, the restriction of any power of the Laplacian to C_0^∞ has the property that the closure of this restriction is self-adjoint. The expansion converges uniformly on the unit ball of the space Z below, in the space

$$B = \left\{F : y^{-((1+\delta)/2)} F \in L^\infty(\mathbf{D})\right\}$$

with the obvious norm. δ may be chosen to be any positive real number. Hence, as an application of the above corollary we obtain, letting $\{T_n\}$ be the family of Hecke operators, normalized to have norm one,

Corollary 2. *Let B be as above. Let $\beta > 0$, $A_0 = I$,*

$$A_1 = H$$
$$A_n = T_n$$
$$Z = \left\{\phi = y^{-\beta}\theta : \theta \in D\left((H+1)^3\right)\right\}$$

with norm

$$\|\phi\|_Z = \left\|(H+1)^3 \theta\right\|_2 .$$

Then, for every $\varepsilon > 0$, there exists a compact subset K of the joint spectrum, and a positive δ, such that for every δ-net $\{\lambda_i\}$ of K there exists a set of generalized simultaneous eigenfunctions $F_i \in C^\infty(\mathbf{D})$ such that

$$A_n^t F_{\lambda_i} = (\lambda_i)_n F_{\lambda_i}$$

and a set of complex constants c_i such that, for every ϕ in the unit ball of Z,

$$\left\|\phi - \sum_{i=1}^{n(\varepsilon)} c_i F_{\lambda_i}(\phi)(\lambda_i)_n F_{\lambda_i}\right\|_B < \varepsilon;$$

here, F_i and c_i are independent of ϕ. c_i and $n(\varepsilon)$ in general depend upon the choice of $\{F_i\}$.

This corollary will be restated in Section 5 as Corollary 64 to Theorem 62.

The idea of using finitely many generalized eigenfunctions for a single self-adjoint operator to get finite approximation inequalities of the type developed above was introduced by Kauffman in [**7**]; it was applied to some nontrivial situations in ordinary differential equations by Hinton and Kauffman in [**4**]. A formal theory of generalized eigenfunction expansions for families of commuting normal operators, concentrating on the convergence of the integrals involved, was given by Kauffman in [**8**]. A much simpler version of this theory, which uses minimal assumptions, was given by Sakata in her dissertation and [**11**]. That general theory is the basis for this paper.

Other authors concentrate on a single self-adjoint operator and use a different notion of convergence. The first modern paper on the foundations of generalized eigenfunction expansions was a portion of the article *Schrödinger semigroups*, by Simon [**12**], where a self-contained and readable account of the theory was given. A clear recent approach, with a systematic apparatus for calculating asymptotics of the eigenfunctions, was given (in the case of a single self-adjoint operator) by Poerschke, Stolz and Weidmann in [**10**]. The classical literature on the subject is anchored by Gelfand and Vilenkin in [**3**], and Berezhanskii in [**1**]; the paper of Poerschke, Stolz and Weidmann was designed to give a very applicable, streamlined and simple version of the theory. This paper was then applied by Poerschke and Stolz [**9**] to three different types of expansions: Titchmarsh-Weyl-Kodaira expansions for ordinary differential operators, BGK expansions for elliptic differential operators with smooth coefficients and Ikebe-Povsner expansions from scattering theory. Poerschke and Stolz obtain all these expansions from a single abstract result. In both these papers, a result analogous to the L_2 Fourier transform is proved: that is, the spectral theorem is implemented by using generalized eigenfunctions to give the Fourier coefficients of the given function. The integrals converge in the usual sense of the L_2 transform; they do not converge pointwise.

More recently, Boutet de Monvel and Stollmann [**2**] applied the approach of Poerschke, Stolz and Weidmann to operators generated by Dirichlet forms. This allows a treatment of most operators which appear in mathematical physics, including those where the coefficients are not smooth. A very interesting feature of this work is a bound on the eigenfunctions using an intrinsic metric (defined there) generated by the Dirichlet form.

The easiest way to compare the approach of the authors to convergence with the work of the authors just cited is to think of Fourier series. The Fourier series of any function converges in L_2, which is the type of convergence studied by other authors. However, if the function being expanded lies, for example, in the Sobolev space W_2^1, the series converges uniformly and error estimates may be given, which apply on the entire unit ball of W_2^1. The convergence results of this paper are completely analogous and are obtained at the price of additional hypotheses exactly in the same way as differentiability hypotheses are needed to guarantee uniform convergence of usual Fourier series.

In terms of obtaining the correct asymptotic behavior of the eigenfunctions, as a general method it appears that the techniques of Boutet de Monvel and Stollmann may well yield optimal results for the case of a single operator, provided it is generated by a Dirichlet form. Probably that method could be extended to the present situation, since the eigenfunctions we consider are eigenfunctions of every member of a given family, so if the single operator is a member of the family, the estimates will hold. It should be noted, however, that the example of Fourier series on a finite interval shows that uniform convergence estimates of the type studied here depend not only upon the asymptotics of the eigenfunctions, but also upon other considerations. The problem of convergence and of the asymptotics of the eigenfunctions are therefore related, but in general different, even for the case of a single self-adjoint operator.

2. Background. Throughout this paper we assume the following:

- $\{A_i\}$ is a possibly infinite family of closed operators in a Hilbert space $\mathfrak{h}$;
- $\mathcal{A}$ is the smallest von Neumann algebra with which $\{A_i\}_{i\in S}$ is affiliated, where S is either the positive integers or the subset of positive integers less than or equal to some N;
- $\mathcal{A}$ is commutative (hence each A_i is normal);
- there exists an N such that A_i is bounded for $i \geq N$.

Definition 3. Throughout the paper, let W be a locally convex topological vector space and $\widehat{W}$ a Banach space and $\mathfrak{h}$ a separable Hilbert space, such that

$$W \subset \widehat{W} \subset \mathfrak{h} \subset \widehat{W}' \subset W'$$

where the embeddings are assumed to be continuous, each space is dense in the next, and where embedding from $\mathfrak{h}$ into $\widehat{W}'$ is conjugate linear. Assume that W is separable and that each A_i takes W continuously into itself. Give W' the topology of uniform convergence on bounded subsets of W. Let $C(W, W')$ denote the continuous conjugate linear mappings from W into W', topologized by letting a sub-base of open sets about a fixed operator A_0 be given as follows: specify bounded convex subset B of V and a given open set $\Theta \subset V'$, and define a sub-basic open set containing zero to be $\Phi_{B\Theta}$ where

$$\Phi_{B\Theta} = \{A : A(x) \in \Theta \text{ for all } x \in B\}.$$

• We review some background material on commutative von Neumann algebras $\mathcal{A}$. Each A_i affiliated with $\mathcal{A}$ corresponds under the Gelfand transform G to a continuous function f_i on the maximal ideal space X of $\mathcal{A}$ which is defined on the complement of a meagre set; that is, a set whose closure has empty interior. (See [**6**].) Furthermore, each meager set has spectral measure zero with respect to any cyclic vector e for $\mathcal{A}'$; (this is probably well known to operator algebraists; a proof is given in Kauffman [**8**]). Each bounded A_i corresponds to a continuous function on the entire maximal ideal space, which is a compact Hausdorff space.

Definition 4. The joint spectrum J is defined to be the closure in the product space of the spectra of the A_i of

$$\{\lambda(x) : (\lambda(x))_i = G(A_i)(x)\}_{x \in X \setminus M}$$

where X is the maximal ideal space of $\mathcal{A}$ and M is the meagre set on the complement of which the Gelfand transform $G(A_i)$ of each A_i is continuous.

• We introduce the notation

$$\|x : B\|$$

to denote the norm of x in the Banach space B.

• Let J be the joint spectrum of $\{A_i\}$. Let K_N be a clopen subset of the complement of the meagre set above (the set where the functions $G(A_i)$ are not defined), such that for each i,

$$|G(A_i)|(x) \leq N \text{ for all } x \in K_N.$$

Note that the characteristic function of K_N corresponds to a projection operator $P(K_N)$ in $\mathcal{A}$. Each $G(A_i)$, restricted to K_N, is continuous and has range contained in the compact subset $\widehat{K}_N$ of J, where

$$\widehat{K}_N = \{x \in J : |x_i| \leq N \text{ for all } i\}.$$

Let $(\lambda)_i$ denote the ith component of $\lambda \in J$. It is clear that the infinity norm of any polynomial P in $(\lambda)_i$ on $\widehat{K}_N$ is the norm of $P(K_N)P(A_i)$. Hence, the C^* algebra of functions on $\widehat{K}_N$ formed by taking the closure in supremum norm on $\widehat{K}_N$ of all such polynomials is isometric to the C^* algebra $\mathcal{C}_N$ on the range of $P(K_N)$, where C_N is the smallest C^* algebra containing the restriction of each A_i to the range of $P(K_N)$. Noting that the polynomials in $\{\lambda_i\}$ separate points in $\widehat{K}_N$, we see by the Stone-Weierstrass theorem that the polynomials in $\{\lambda_i\}$ are dense in $C(\widehat{K}_N)$. Hence, $\mathcal{C}_N$ is isometric to $C(\widehat{K}_N)$ under a mapping we call $\widehat{G}$, the extended Gelfand transform. Now, if e is a cyclic vector for $\mathcal{A}'$ and

$$\mu_{e,N}(A) = [Ae, e],$$

we see that $\mu_{e,N}$ is a Radon measure on $C(\widehat{K}_N)$. Since the $\widehat{K}_N$ are a tower, this induces a positive finite regular Borel measure on J; we call this measure μ_e. Now, on $L_2(J, \mu_e)$, consider the map T such that, for any $A \in \mathcal{C}_N$,

$$T(Ae) = \chi\left(\widehat{K}_N\right)\widehat{G}(A).$$

Note that the linear extension of this map takes the smallest C^* algebra $\mathcal{C}$ containing each $\mathcal{C}_N$ into $L_\infty(J, \mu_e) \subset L_2(J, \mu_e)$. It is not difficult to see that

$$\|TAe : L_2(J, \mu_e)\| = \|Ae : \mathfrak{h}\|.$$

It follows that T extends to a mapping from $\mathcal{A}(e)$ onto $L_2(J, \mu_e)$. Elements of the range of L_∞ act as multiplication operators on $L_2(J, \mu_e)$, and the operator norm is the L_∞ norm. It now follows from standard arguments that the commutant of the range of $\widehat{G}$ on the union of the algebras $\mathcal{C}_N$ is contained in the algebra of operators of multiplication by elements of $L_\infty(J, \mu_e)$; since T is an isometry, we see that, finally, $\widehat{G}$ extends to an isometry taking $\mathcal{A}$ onto $L_\infty(J, \mu_e)$.

Definition 5. If Δ is a bounded Borel subset of the joint spectrum J, and $\xi(\Delta)$ is the characteristic function of Δ, considered as an element of $L_\infty(J, \mu)$, then we define

$$P(\Delta) = \widehat{G}^{-1}(\xi(\Delta)).$$

The map $\Delta \to P(\Delta)$ is called a *spectral measure for* $\{A_i\}$ on J.

Definition 6. A *generalized eigenprojection* P_λ for $\{A_i\}$, corresponding to λ in the joint spectrum, and corresponding to the space W above, is an operator in $C(W, W)$ such that there exists a family $\{\Delta_i\}$ of Borel subsets of the joint spectrum such that

$$\Delta_i \subset \left\{ x \in J : d(x, \lambda) < \frac{1}{i} \right\}$$

and a sequence of complex numbers $\{b_n\}$ such that

$$b_n P(\Delta_n) \longrightarrow P_\lambda$$

in

$$C(W, W').$$

Remark 7. In the case where $\lambda \in J$ is not an element of the point spectrum of all A_i, then, as a sequence of operators from $\mathfrak{h}$ into $\mathfrak{h}$, $P(\Delta_n)$ converges strongly to zero; in this case the sequence b_n converges to infinity in absolute value. It usually turns out that

$$b_n = \frac{1}{\mu(\Delta_n)}$$

where μ is a positive measure on J.

Remark 8. It was shown in [**11**] that if $F \in \text{range}\,(P_\lambda)$, then for all i

$$A_i^t F = \lambda_i F,$$

where we have here identified F with an element of W', and where A_i^t is the operator A_i considered as a continuous mapping from W into W.

Definition 9. Let W be as above. A *generalized eigenfamily* on J, with respect to $W, \{K_n\}$, μ, where μ is a positive scalar measure on J, and K_n is a tower of compact subsets of J, such that the complement of $\cup K_n$ has measure zero, is a map

$$\xi : \lambda \longrightarrow P_\lambda,$$

where the map ξ takes $\cup K_n$ into $C(W, W')$, such that the following hold:

1. each P_λ is a generalized eigenprojection for each A_i, with eigenvalue λ_i where

$$\lambda = \{\lambda_i\}\,;$$

2. the restriction of ξ to each K_n is continuous from K_n into $C(W, W')$

3.

$$P\,(K_n) = \int_{K_n} P_\lambda \, d\mu;$$

here the integral converges in the sense of an operator valued integral, continuous with respect to the spectral parameter.

Definition 10. The *multiplicity* at λ of a generalized eigenprojection P_λ is the dimension of its range.

Definition 11. Let $\mathcal{A}$ be the smallest von Neumann algebra with which each A_i is affiliated. Let $\widehat{W}, V$ be Banach spaces such that $W \subset \widehat{W} \subset V \subset \mathfrak{h}$, where each space is dense and continuously embedded in the next. Suppose that the embedding from $\widehat{W}$ into V is two nuclear, and the embedding of V into $\mathfrak{h}$ is nuclear. We then call the scheme $W, \widehat{W}, V, \mathfrak{h}$ a *regular embedding scheme*.

Remark 12. If $W = C_0^\infty(M)$, and the operators A_i take W continuously into itself, then spaces $\widehat{W}$ and V may be easily constructed so that a regular embedding scheme holds. The argument is given in [**11**].

Definition 13. If W admits spaces $\widehat{W}, V$ such that a regular embedding scheme holds and each A_i takes W continuously into itself, we call W an $\{A_i\}$ -*admissable space.* The associated generalized eigenfamily

$$\lambda \longrightarrow P_\lambda$$

with

$$P_\lambda \in C\left(\widehat{W}, \widehat{W}'\right)$$

is called the generalized eigenfamily associated with W.

• The following is the main theorem in [**11**].

Theorem 14. *Suppose that a regular embedding scheme holds. Then a generalized eigenfamily associated with W exists, corresponding to each scalar measure μ which is absolutely continuous with respect to the measure μ_e induced by a cyclic vector e for $\mathcal{A}'$, defined by*

$$\mu_e(\Delta) = [P(\Delta)e, e].$$

This generalized eigenfamily is essentially unique. This means that if $\{K_n\}$ and μ are specified, P_λ is uniquely defined. Furthermore, each P_λ extends to an element of $C(\widehat{W}, \widehat{W}')$, and the map $\lambda \to P_\lambda$ is continuous from each K_n into $C(\widehat{W}, \widehat{W}')$. Finally, for every $\varepsilon > 0$, there exists a K_n such that if

$$P_{K_n} = \int_{K_n} P_\lambda \, d\mu,$$

then

$$\left\| P_{K_n} - \widehat{E} : C\left(\widehat{W}, \widehat{W}'\right) \right\| < \varepsilon,$$

where $\widehat{E}$ denotes the embedding from $\widehat{W}$ into $\widehat{W}'$, and where we use the notation $\|x : B\|$ to denote the norm of x in the Banach space B.

• The following theorem follows immediately from the previous theorem.

Corollary 15. *Assume that a regular embedding scheme holds. For any $\varepsilon > 0$, there exists a $\delta > 0$ and positive integer N such that for any δ-net $\{\lambda_j\}$ of $K_n : n > N$, and for any i, there exist constants c_j such that*

$$\left\| \sum c_j P_{\lambda_j} \theta - \theta : \widehat{W}' \right\| < \varepsilon$$

for all i and all

$$\theta \in \widehat{W} : \left\| \theta : \widehat{W} \right\| \leq 1.$$

In other words, the expansion converges uniformly to the operator on bounded sets, in the topology of $C(\widehat{W}, \widehat{W}')$.

Remark 16. The following theorem follows from the construction in the paper [**11**]. To define the eigenprojections for almost every λ in the joint spectrum, some observations are in order: picking a tower of disjoint compact sets, we see that it is possible to define the generalized eigenprojections almost everywhere, and the K_n may be chosen to be a tower.

Theorem 17. *If a regular embedding scheme holds, with e a cyclic vector for $\mathcal{A}'$, and in addition $\mu_e(\Delta)$ is the finite positive measure*

$$\mu_e(\Delta) = [P(\Delta)e, e],$$

with μ any measure on the joint spectrum which is mutually absolutely continuous with respect to μ_e, then there exists a sequence K_n with the above properties and uniquely defined generalized eigenfamily for $\{A_i\}, W, \mu, \{K_n\}$; the generalized eigenprojections may be defined almost everywhere and the sets K_n may be taken to be a tower. Furthermore, if

$$\mathfrak{h} = \oplus_{j=1}^{N} \mathfrak{h}_j$$

where

$$N \in \mathbf{N} \cup \infty$$

is a direct sum decomposition of $\mathfrak{h}$ in terms of reducing subspaces $\mathfrak{h}_j$ of each A_i, such that $\mathfrak{h}_j$ is the range of the orthogonal projection Q_j, if

$$W_j = Q_j W,$$

then each W_j *has a regular embedding scheme in* $\mathfrak{h}_j$ *and a corresponding generalized eigenfamily*

$$\beta \longrightarrow P_{\beta j} : \beta \in M_j,$$

each one with multiplicity one, where M_j *is the joint spectrum of the restriction of* $\{A_i\}$ *to* $\mathfrak{h}_j$*, there exists a unitary operator* U *taking* $\mathfrak{h}$ *onto*

$$\mathcal{H} = \oplus L_2 (M_j, \mu)$$

such that

$$P_\beta = \sum P_{\beta j}$$

with convergence in $C(\widehat{W}, \widehat{W}')$ *and such that, for any* $\phi \in W$*,* $\beta \in K_n$*,*

$$(U\phi)_j (\beta) = P_{\beta j} \phi (\phi) .$$

Under this isometry, each A_i *goes into multiplication by* λ_i*, and* $\mathcal{A}$ *goes to* $L_\infty(J, \mu_e)$*.*

Remark 18. In case the set $\{A_i\}$ consists of a single self-adjoint operator H, and W is a core of H, this is an eigenfunction expansion of the same type as the Poerschke-Stolz-Weidmann expansion. The uniform convergence obtained above, together with the existence of the eigenprojection P_β, follows from the stronger hypotheses assumed here.

Theorem 19. *The expansion above is unique, in the following sense: given two different towers of compact sets, and two different mutually absolutely continuous measures* μ_1 *and* μ_2*, such that*

$$f (\lambda)\, d\mu_1 = d\mu_2,$$

then, for almost every λ*,*

$$P_\lambda = f (\lambda)\, Q_\lambda$$

where P_λ *is the expansion using* μ_1 *and* Q_λ *is the expansion using* μ_2*.*

Proof. The continuity of the maps $\lambda \to P_\lambda$ and $\lambda \to Q_\lambda$ on their respective towers of compact sets imply directly that the generalized eigenfamilies are uniquely defined on them. The conclusion is immediate. □

Remark 20. Note that two different orthogonal decompositions in terms of reducing subspaces nevertheless produce the same eigenfamily, for a given tower of compact sets.

• We have defined the multiplicity of a single eigenprojection; we now define the multiplicity of a family of operators.

Definition 21. If a regular embedding scheme holds, the multiplicity of $\{A_i\}$ is the least upper bound over n of the maximum multiplicity of

$$\{P_\lambda : \lambda \in K_n\}\,.$$

Theorem 22. *$\{A_i\}$ has multiplicity one if and only there exists a cyclic vector for $\mathcal{A}$.*

Proof. If there exists a cyclic vector for $\mathcal{A}$, then the multiplicity is automatically one: this follows from the construction used in [**11**]. If the multiplicity is one, the original Hilbert space is isometric to $L_2(\mu)$, where μ is the measure on the joint spectrum induced by the measure on any cyclic vector e for $\mathcal{A}'$, under a map U which takes each A_i to multiplication by λ_i. The conclusion follows. □

Theorem 23. *All generalized eigenfamilies have the same multiplicity, independent of the choice of compact sets K_n; this multiplicity is the smallest cardinality $\mathfrak{n}$ of any family of orthogonal cyclic reducing subspaces $\{\mathfrak{h}_j\}$ for $\mathcal{A}$ such that*

$$\mathfrak{h} = \oplus \mathfrak{h}_j\,.$$

This multiplicity is the L_∞ norm of the multiplicity function

$$\mathfrak{n}_\lambda = \dim\left(\text{range }(P_\lambda)\right).$$

Proof. For any K_n, the characteristic function of K_n corresponds to a projection operator $L_2(J,\mu)$, where J is the joint spectrum.

This clearly strongly commutes with multiplication by any λ_i, so it corresponds to an element P_n of $\mathcal{A}'$, since the original Hilbert space is isometric to $L_2(J, \mu)$. Hence, the restriction of $\mathcal{A}$ to the range of P_n is another von Neumann algebra, which clearly has a generalized eigenfamily for $P_n W = W_n$, $P_n V = V_n$; these spaces have the same properties relative to the range of P_n as the spaces W, V do relative to $\mathfrak{h}$. However, for this space, the generalized eigenprojections are continuous functions of λ, as λ moves through the joint spectrum. Since each P_λ is a limit of spectral projections, it follows from the construction of [**11**] plus the uniqueness theorem, that our multiplicity is less than or equal to the usual multiplicity, which is the number $\mathfrak{n}$ above. □

Theorem 24. *With μ given, we may find another positive measure μ_1 which also admits a generalized eigenfamily, with the same sets $\{K_n\}$, and with for each λ*

$$\left\| P_\lambda : C\left(\widehat{W}, \widehat{W}'\right) \right\| = 1.$$

We call the unique eigenfamily

$$\lambda \to P_\lambda$$

with this property $\widehat{W}$ normalized.

Proof. Let

$$d\mu_1 = \|P_\lambda\| \, d\mu\left(\lambda\right).$$

The generalized eigenprojections for the measure μ_1 are of the form

$$\frac{P_\lambda}{\|P_\lambda\|}. \qquad \square$$

Theorem 25. *If a generalized eigenprojection has multiplicity one, and each P_λ is $\widehat{W}$ normalized, and $F \in \operatorname{range}(P_\lambda)$ with*

$$\|F : V'\| = 1,$$

then

$$P_\lambda\left(\theta\right) = F\left(\theta\right) F.$$

Proof. It is clear that P_λ must be a multiple of $F_\lambda(\theta)F_\lambda$. Since P_λ is a limit of positive multiples of spectral projections, it must be a positive real multiple of $F_\lambda(\theta)F_\lambda$. Since P_λ is $\hat{W}$ normalized, the only possible multiple is 1. □

3. General expansions of spectral multiplicity one. In case the multiplicity of the expansion is one, better estimates on convergence may be obtained. The reason is that part of the problem of convergence is to sum over the cyclic subspaces to obtain the generalized eigenprojections; in the multiplicity one case, that problem no longer exists. A technical point is that, to apply the result in concrete situations, it is usually necessary to use our definition of multiplicity in terms of the dimension of the range of the generalized eigenprojections. For this to make sense, we need a base space W where the expansion exists; in the general situation we assume this, but in practice it is often possible to take

$$W = C_0^\infty(M)$$

where M is a Riemannian manifold.

To get multiplicity one in the number theory example below, we use an expansion for the Laplacian and the Hecke operators on the fundamental domain, not just the Laplacian. Hence, for this application, it is essential to use an algebra of commuting operators, rather than just a single operator. Note that, in this case, we are able to reduce our hypotheses to the Hilbert-Schmidt level, analogous to the theory of Poerschke-Stolz-Weidmann in the case of a single operator. In what follows, then, the original generalized eigenprojections lie in $C(W, W')$.

Remark 26. In what follows, we are attempting to set up an abstract scheme which can deal with actual uniform convergence; in other words, convergence with respect to a weighted L_∞ norm. In order to do this in the dual space formalism we have been using, it turns out that the space we want as a target space for convergence is not the dual space of our space of test functions, but that an embedding exists from a subspace of this dual space into the target space. This explains the need for the hypotheses below.

Definition 27. In what follows, W is a space which admits a regular embedding scheme, Z is a Hilbert space containing W, such that

$$W \subset Z \subset \mathfrak{h} = \mathfrak{h}' \subset Z' \subset W' :$$

the embedding of each space into the next is continuous, and each space is dense in the next one. Let B be a Banach space containing Z. We say that a Z, B *spectral embedding* exists if following properties hold:

1.
$$Z \subset B \subset Z';$$

2. the embedding of Z into $\mathfrak{h}$ is two-nuclear;

3. the embedding of Z into B is nuclear;

4. each P_λ in the generalized eigenfamily associated with W extends to a bounded linear transformation

$$\widehat{P}_\lambda \in C\left(Z, Z'\right);$$

5. for some i, there exists a positive constant C and a positive integer n such that if $F \in Z' \subset W'$, and

$$\left(A_i^t\right)' F \in Z',$$

then $F \in B$;

6. for the i in the previous hypothesis, $Z \subset D(A_i^2)$.

Remark 28. If the eigenfamily P_λ has multiplicity one, there is always a Z, B spectral embedding when $B = Z'$. For properties 5 and 6, we may take $i = 0$ and A_i to be the identity. The existence of the generalized eigenfamily follows from Theorem 3.11 together with Theorem 3.15 of Sakata [**11**], which is stated below. That the embedding of Z into Z' is nuclear follows from the fact that if E is the two nuclear embedding from Z into $\mathfrak{h}$, then A^* is a two nuclear embedding from $\mathfrak{h}$ into Z', so that A^*A is a product of two nuclear (or, in other language, Hilbert-Schmidt) operators between Hilbert spaces, and is therefore nuclear (or, in other language, trace class).

• In order to deal with the multiplicity one case, we need the following theorem, which combines Theorems 3.11 and 3.15 of Sakata [**11**].

Theorem 29. *Suppose that $\widehat{W}$ is either a Hilbert space with a two nuclear embedding into $\mathfrak{h}$ or a Banach space with a nuclear embedding into $\mathfrak{h}$. Let S_e be a cyclic subspace, generated by a single vector e. Then a generalized eigenfamily exists on S_e, corresponding to $P(S_e)W$, $P(S_e)\widehat{W}$, where the algebra $\mathcal{A}_e$ is the smallest von Neumann algebra in S_e generated by the restrictions of $\mathcal{A}$ to S_e.*

Remark 30. In the situation of this paper, $S_e = \mathfrak{h}$, since the generalized eigenfamily has multiplicity one.

Theorem 31. *Suppose that W admits a regular embedding scheme and that the associated expansion has multiplicity one. Therefore, by Theorem 17, there exists a unitary operator from $\mathfrak{h}$ onto $L_2(J,\mu)$, where μ is any positive measure which is mutually absolutely continuous with respect to the spectral measure generated by the cyclic vector, such that for $\phi \in W$,*

$$U\phi\,(x) = F_x\,(\phi)$$

for every

$$\phi \in W;$$

here

$$F_x \in W'$$

and the projections P_x obey

$$P_x\,(\phi) = F_x\,(\phi)\,F_x.$$

Suppose that a Z, B spectral embedding holds. Then

1. *for almost every $x \in J$,*

$$P_x \in C\,(Z, B)\,;$$

2. *let $\{K_n\}$ be any tower of compact sets satisfying the definition of a generalized eigenfamily: then for every K_n the mapping*

$$x \longrightarrow P_x$$

is continuous from K_n into

$$C\,(Z, B)\,;$$

3. *for every* $\varepsilon > 0$ *there exists set* K_n *such that for any* $\phi \in Z$ *such that* $\|\phi\| = 1$,

$$\left\| \left(\theta - \int_{K_n} P_\lambda \, d\mu\left(\lambda\right) \right) : B \right\| < \varepsilon.$$

Proof. That each P_λ extends to an element of $C(Z, Z')$, and that the mapping

$$\lambda \to P_\lambda \in C\left(Z, Z'\right)$$

is continuous on each K_n follows from [**11**, Theorem 3.15]. That each $P_x \in C(Z, B)$ follows from the definition of a spectral embedding and the closed graph theorem. Continuity from each K_n of the mapping

$$\lambda \longrightarrow P_\lambda \in C\left(Z, B\right)$$

follows from the continuity on K_n into $C(Z, Z')$ plus the fact that for some C_n

$$\left\| A_i^t : C\left(Z', Z'\right) \right\| \leq C_n$$

on each K_n; this fact follows directly from the fact that $\lambda \to P_\lambda$ is a generalized eigenfamily.

We need only prove the second property. However, this follows from the fact that the element $P(K_n)$ is in $C(Z, B)$, and that if E is the identity embedding from Z into $\mathfrak{h}$, the norm of $E - P(K_n)$ in $C(Z, \mathfrak{h})$ goes to zero as $n \to \infty$, as may be seen from [**8**, Lemma 230, page 72], together with Theorem 17. Therefore, it follows that the same projection goes to zero in norm in $C(Z, D(A_i))$, since Z is in the domain of A_i^2. Finally, property 5 in the definition of a spectral embedding shows that if E_{ZB} is the embedding from Z into B,

$$\left\| \left(E_{ZB}\left(E - P\left(K_n\right)\right)\right) : C\left(Z, B\right) \right\| \longrightarrow 0. \qquad \square$$

Remark 32. Corollary 1 follows immediately. We restate it here.

Corollary 33. *Under the hypotheses of the previous theorem, for every* $\varepsilon > 0$, *there exists a compact subset* K *of the joint spectrum, and* *a positive δ, such that for every δ-net $\{\lambda_i\}$ of K there exists a set of generalized simultaneous eigenfunctions F_i such that*

$$A_n^t F_{\lambda_i} = (\lambda_j)_n F_{\lambda_j}$$

and a set of complex constants c_i such that, for every ϕ in the unit ball of Z,

$$\left\| \phi - \sum_{i=1}^{n(\varepsilon)} c_j F_{\lambda_j}(\phi) F_{\lambda_j} \right\|_B < \varepsilon.$$

Here, F_i and c_i are independent of ϕ. c_i and $n(\varepsilon)$ in general depend upon the choice of $\{F_{\lambda_j}\}$, though once $\{\lambda_j\}$ is chosen, $\{F_{\lambda_j}\}$ is of course unique up to a constant.

4. Estimates on nonpositively curved two dimensional Riemannian manifolds. In order to apply the above theory to situations which are of interest in geometry and number theory, we need estimates which guarantee that a space Z has an embedding into L_∞.

Definition 34. Let M be a complete simply connected two-dimensional Riemannian manifold of nonpositive curvature. Let $\{E_i\}$ be a local frame field constructed by parallel translating an orthonormal basis at p. Let α be multi-index, or in other words a finite sequence

$$\{\alpha(i)\}_{i=1}^n$$

where each $\alpha(i)$ is one or two. Define

$$D^\alpha = D_{E_{\alpha(1)}} \cdots D_{E_{\alpha(n)}}$$

where D_{E_i} denotes covariant differentiation. We say that M has *bounded geometry* if, for every multi-index α, and every i,

$$\sup_{p\in M} \sup_{d(x,p)\leq 1} |D^\alpha E_i| \leq M_\alpha$$

where M_α is a constant depending on α.

Remark 35. An inequality of the above type does not depend upon which orthonormal basis is used at p.

Remark 36. The upper half-plane, with the Poincarè metric, which we call *hyperbolic space*, has bounded geometry. For any space with bounded geometry in the above sense, its Riemannian curvature tensor, together with all covariant derivatives, is a bounded multilinear form.

Definition 37. On a complete Riemannian manifold M, let $\mathcal{L}^n$ denote the nth power of the Laplacian, considered as a differential expression. Let $(\mathcal{L}^n)_0$ denote the closure of the restriction of $\mathcal{L}^n$ to $C_0^\infty(M)$.

• The following theorem is fundamental in the study of spectral geometry.

Theorem 38 (Cordes). *Let M be a complete Riemannian manifold. Then $(\mathcal{L}^n)_0$ is self-adjoint for any n. Furthermore, if*

$$H = \mathcal{L}_0,$$

then

$$H^n = (\mathcal{L}^n)_0 .$$

Corollary 39. *If $f \in L_2(M)$ and $(\mathcal{L}^n)_0 f \in L_2(M)$, where the derivatives are taken in the distribution sense, then*

$$f \in D(H^n) .$$

Corollary 40. *H is identical to the Friedrichs extension of the restriction of $(\mathcal{L})_0$ to $C_0^\infty(M)$.*

Lemma 41. *Let H be the self-adjoint Laplacian on the universal cover of M, where M is a complete simply connected two-dimensional bounded geometry Riemannian manifold of nonpositive curvature. Let $\delta > 0$. Let U be any ball of radius 1 about p in M. Then $\exp_p^{-1} U$ is a ball of radius 2 about 0 in $T_p(M)$, where $\exp_p$ is the exponential map taking (in this situation, where M is Hadamard manifold) the tangent space $T_p(M)$ onto M. Let*

$$f \in C_0^\infty(U) .$$

Let N be a positive integer. Then there exist constants C, ε such that

$$\varepsilon\left(\|f\|+\left\|H^{N} f\right\|\right) \leq\left\|f \circ \exp_{p}\right\|+\left\|H_{1}^{N}\left(f \circ \exp_{p}\right)\right\| \leq C\left(\|f\|+\left\|H^{N} f\right\|\right),$$

where H_1 is the self-adjoint Euclidean Laplacian on T_pM.

Remark 42. The hypothesis that M is simply connected is needed only to assure that such balls exist.

Proof. About any point p, coordinate the open unit ball by using the exponential map in the following way: select an orthonormal basis $\{X_1, X_2\}$ for $T_p(M)$. If $|X| = 1$, and

$$y = \exp_p(Xt)$$

with $\{c_1, c_2\}$ a set of real numbers such that

$$X = c_1 X_1 + c_2 X_2,$$

then

$$\Phi(y) = (t, c_1, c_2).$$

This map is smooth. Now

$$Hf = E_1^2 f + E_2^2 f + (D_{E_1} E_1) f + (D_{E_2} E_2) f,$$

where $\{E_j\}$ is the local frame field formed by parallel translating the basis $\{X_j\}$ on the unique geodesic from p.

Note that the differential of the exponential map is given by solving the Jacobi equation, which is an ordinary differential equation which depends only upon the Riemannian curvature tensor. It follows that the exponential map and its inverse are uniformly Lipschitz; the same is true for its differentials of any order.

The conclusion of the lemma now follows from elliptic regularity in $T_p(M)$. □

Lemma 43. *Let W be a ball of radius one about some point $p \in M$. Suppose that*

$$f \in L_2(W)$$

and

$$H^j f \in L_2(W).$$

Then, if U is a ball of the same center and radius $1/2$, f agrees on U with an element g of $C_0^\infty(W)$; furthermore, there is a universal constant C such that we may find such a function g with

$$\|g\| + \left\|H^j g\right\| \leq C\left(\|f\| + \left\|H^j f\right\|\right).$$

Proof. This is a standard cutoff function argument. □

Theorem 44. *Let $f \in D(H)$. Then $f \in L_\infty(M)$.*

Proof. Restrict f to a ball of radius 1, and use the previous two lemmas and the Fourier transform on the Euclidean Laplacian on $T_p(M)$. □

5. An example from number theory. In this section, we apply the above theory to the algebra generated by the Hecke operators and any fundamental domain for the modular group. A fundamental domain, defined for a subgroup G of the modular group Γ, is an open subset of the upper half plane with the following properties: (a) no two distinct points of the subset are equivalent under G; (b) if z is in the upper half plane, there is a point z' in the closure of the subset such that z' is equivalent under G. Recall that two points z and z' are said to be equivalent under G if $z' = Az$ for some A in G. Here we may assume that it is the usual fundamental domain which is given by $|z| > 1$ and $|z + \bar{z}| < 1$, since all such are isometric. Any fundamental domain is a complete Riemannian manifold, because it is the quotient of hyperbolic space by a discrete group of isometries. We include some fundamental material on the injectivity radius, for the convenience of the reader. Two nuclear mappings are sometimes called Hilbert-Schmidt; the terms mean the same thing, in a Hilbert space.

We summarize known facts about this expansion [**5**]:

- It is an integral expansion of type

$$\theta(z) = \frac{1}{4\pi}\int_{t=-\infty, s=(1/2)+it}^{\infty} c_s(\theta)\, E(z,s)\, dt + \sum_{j=1}^{\infty} [\theta, f_j]\, f_j,$$

where

$$\theta \in C_0^\infty(\mathbf{D})$$

and

$$c_s(\theta) = \int_{\mathbf{D}} \theta(z) E(z,s)\ dz,$$

and where $E(z,s)$ is a C^∞ function which is a simultaneous eigenfunction of the Laplacian $\mathcal{L}$ on the fundamental domain and all Hecke operators, meaning that as a distribution, and therefore as a C^∞ function,

$$\mathcal{L}(E(z,x)) = s(s-1)E(z,s),$$
$$T_n(E(z,s)) = \lambda_n E(Z,s)$$

for each Hecke operator, where $\lambda = \{\lambda_n\}$. It is well known that there exists at most a one-dimensional simultaneous eigensolution for these operators. The equality in the above expansion is in the sense of the Plancherel theorem: the map from θ to

$$c_s(\theta), \{[\theta, f_i]\}$$

is unitary, from C_0^∞ to $L_2(\mathbf{R}) \times \ell_2$. (This is analogous to the L_2 Fourier transform). The $E(z,s)$ are functions given by the Eisenstein series, and they are known to have the property that for any $\varepsilon > 0$

$$y^{-(1/2)-\varepsilon} E(z,s) \in L_\infty(\mathbf{D}).$$

We ask several questions here:

• What is the relationship of this expansion to one derived from our theory?

• Is the convergence of this expansion uniform in the sense discussed above (analogous to the theory of the Fourier transform where the transformed function lies in L_1), or to Fourier series for a function in, say, $W_2^1[0,1]$?

• If the expansion is uniformly convergent, is the convergence obtained by our theory optimal? (This means that, for nice functions at least, it should converge uniformly after being multiplied by $y^{-(1/2)-\varepsilon}$.)

The answer to the first question is that a simple transformation turns this expansion into one of the type above, and the answers to all of the other questions are "yes."

Definition 45. Let Γ be the modular group, considered as a group of transformations taking the upper half plane $\mathbf{H}$, with the hyperbolic metric, into itself. Hence, if $\gamma \in \Gamma$,

$$\gamma(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbf{Z}$, and

$$ad - bc = 1.$$

Note that, if we regard γ as a matrix

$$\widehat{\gamma} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the map from $\widehat{\gamma}$ to γ is a homomorphism with kernel $\pm I$. Let $\mathbf{D}$ denote the fundamental domain of Γ; select the usual one above the unit circle and symmetric with respect to the y axis. Note that $\mathbf{D}$ is a complete Riemannian manifold under the hyperbolic metric, with finite volume.

Definition 46. Let n be a positive integer. For $f \in L_2(\mathbf{D})$, let

$$T_n f(z) = \frac{1}{n} \sum_{d|n} \sum_{b=1}^{d-1} f\left(\frac{nz+bd}{d^2}\right).$$

Note that f extends to an element of $L_2^{\mathrm{loc}}(\mathbf{H})$ which is invariant under the modular group, and then $T_n f$ has the same property, so that $T_n f$ corresponds also to an element of $L_2(\mathbf{D})$. T_n is called the Hecke operator corresponding to n.

Lemma 47. *The Hecke operators have the following properties, for any $n, m \in \mathbf{Z}$:*

1. $$T_m T_n = T_n T_m;$$

2. *if m and n are relatively prime,*

$$T_m T_n = T_{m+n};$$

3. *each* T_n *is bounded and self-adjoint*;

4. *Let* H *denote the self-adjoint Laplacian on* $\mathbf{D}$; *that is,* H *is the restriction of the Laplacian* $\mathcal{L}$ *to the set of all* $f \in L_2(\mathbf{D})$ *such that* $\mathcal{L}(f) \in L_2(\mathbf{D})$; *then for each* T_n, T_n *takes* $D(H)$ *into* $D(H)$, *and for* $f \in D(H)$,

$$T_n\,(Hf) = H\,(T_n f)\,.$$

Remark 48. The Hecke operators are of fundamental significance in number theory. They are associated with subgroups of the modular group, and correspond to "hidden symmetries" of the fundamental domains of these subgroups, that is, rotational symmetries which are not members of the modular group. In a sense, the Hecke operator averages over these symmetries in order to produce a function which is invariant under the modular group. For mathematical physicists, though, the Hecke operators may be considered as an extra family of observables which is needed to produce a complete (maximal) commuting family, that is, a family with multiplicity one. (This is well known and may be shown by Fourier series techniques.) A further comment is in order: the Hecke operators are often considered in number theory as acting on Hilbert spaces of modular forms. However, here, they are acting only on functions of weight zero (that is, functions which are invariant under the modular group), and these are not assumed to have any smoothness. Nonetheless, the expansion we develop here is well known to be important in number theory, and is useful for expanding number theoretic functions such as the Selberg trace function. Thus, the expansion we develop is well known, although looking at it from a functional analytic point of view is nonetheless helpful; our hope is that the convergence properties given in this paper, which appear to be new, might prove to be useful in number theory in some of the same ways that, for example, uniform convergence of Fourier series for differentiable functions is useful in mathematical physics.

Theorem 49. *Let* $\mathcal{A}$ *be the von Neumann algebra in* $L_2(\mathbf{D})$ *generated by the spectral projections for the self-adjoint Laplacian on* $L_2(\mathbf{D})$ *and the Hecke operators. Then* $\mathcal{A}$ *is commutative.*

Proof. It is easy to see that the Hecke operators are a commuting family of bounded self-adjoint operators, and is known (and not difficult to see) that for any Hecke operator T, and any

$$\phi \in C_0^\infty(\mathbf{D}),$$

if $\mathcal{L}$ is the Laplacian on this domain

$$\mathcal{L}(T(\phi)) = T(\mathcal{L}(\phi)).$$

The self-adjoint operator H on $L_2(\mathbf{D})$, formed by taking the Friedrichs extension of the restriction of $\mathcal{L}$ to $C_0^\infty(\mathbf{D})$, is known by Cordes's theorem to be the closure of its restriction to C_0^∞, since by that theorem this is true for any complete Riemannian manifold. It is obvious that any Hecke operator T takes $C_0^\infty(\mathbf{D})$ to itself. Hence, given a sequence $\{\theta_n\}$ of continuous functions converging in $L_2(\mathbf{D})$ to $f \in D(H)$, such that

$$\mathcal{L}(\theta_n) \longrightarrow Hf,$$

it follows that

$$T\mathcal{L}(\theta_n) \longrightarrow THf$$

and therefore

$$Tf \in D(H),$$
$$H(Tf) = T(Hf).$$

By the spectral theorem, T commutes with the spectral projections for H. The conclusion follows immediately. □

• In the next lemma, we assume that $y \geq 3$ because that is where problems occur. The rest of $\mathbf{D}$ is compact and offers no analytic challenges.

Theorem 50. *Let H be the self-adjoint Laplacian on M. Let*

$$U_a = \left\{(x,y) : \frac{a}{3} \leq y \leq 3a\right\}.$$

Let

$$U_{a,0} = \left\{\left\{(x,y) : \frac{2a}{3} \leq y \leq \frac{3a}{2}\right\}\right\}$$

If $\beta > 1/2$ and $f \in C_0^\infty(\mathbf{D})$, there exists a universal constant C such that

$$\|f : L_\infty(U_{a,0})\| \leq Ca^\beta \left(\|(H+1)f : L_2(U_a)\| + \|f : L_2(U_a)\|\right).$$

Proof. We may assume without loss of generality that f is supported on U, using a standard partition of unity argument. Note that f extends to a smooth function on hyperbolic space which is invariant under the modular group; this extension is periodic in x with period 1. The set of all (x, y) in hyperbolic space such that

$$(x, y) \in U \cap V$$

where

$$V = \{(x, y) : y \in U_a,\ |x| < [3a+1]\},$$

where

$$[3a+1]$$

denotes the greatest integer function, contains a ball about $(0, a)$ of hyperbolic radius greater than one independent of a. To see this, recall that geodesics are semicircles perpendicular to the x axis and perform an elementary calculation. However, in hyperbolic space, the norm of the extension $E(f)$ satisfies

$$\|E(f)\| \leq \sqrt{3a+1}\,\|f\|.$$

Since the Laplacian $\mathcal{L}$ on hyperbolic space commutes with the modular group

$$\mathcal{L}(Ef) = E(\mathcal{L}(f)),$$

the result follows from Theorem 44. □

Corollary 51. *In the fundamental domain, the map $y^{-\varepsilon}S^{-1}$ is two nuclear, where*

$$S = H + 1.$$

Proof. For any positive δ, $y^{(1/2)-\delta} \in L_2$. A Hilbert space mapping which factors into a mapping from L_2 into L_∞ composed with multiplication by an L_2 function is well known to be Hilbert-Schmidt, which is the Hilbert space case of a two-nuclear mapping. □

Lemma 52. *Let $\varepsilon > 0$. If W is the domain of $y^{-\varepsilon}S^{-3}$, with norm*

$$\|g : W\| = \|h : L_2(\mathbf{D})\|,$$

where

$$g = y^{-\varepsilon}S^{-3}h,$$

then

$$W' = \left\{(\mathcal{L}+1)^3(y^{\varepsilon}g) : g \in L_2\right\},$$

equipped with norm

$$\left\|(\mathcal{L}+1)^3(y^{\varepsilon}g) : W'\right\| = \|g : L_2\|;$$

the derivatives are taken in the distribution sense.

Proof. This is an elementary calculation. □

Lemma 53. *If $\mathcal{L}(F) = \lambda F$ in the distribution sense, and*

$$F = \mathcal{L}^3(y^{\varepsilon}g);$$

$g \in L_2(\mathbf{D})$, then $F \in C^{\infty}(\mathbf{D})$, and

$$\mathcal{L}\left(\mathcal{L}^3(y^{\varepsilon}g)\right) = \lambda\mathcal{L}^3(y^{\varepsilon}g),$$

so that for every $\varepsilon > 0$,

$$y^{-\varepsilon}\mathcal{L}^4(y^{\varepsilon}g) = \lambda y^{-\varepsilon}\mathcal{L}^3 y^{\varepsilon}g.$$

Proof. The only thing to check is that $F \in C^{\infty}$, which is just elliptic regularity. □

Definition 54. Let $\langle g, h\rangle$ be a closable sesquilinear form with domain $C_0^{\infty}(M)$. Let Q be the m-sectorial operator which generates the form (if the form is, for example, positive semi-definite, and $\langle g, h\rangle = [Ag, h]$, where $[\ ,\ ]$ is the usual inner product in $L_2(M)$, then Q is the Friedrichs extension of A). We say that Q is *essentially m-sectorial* if Q is the smallest closed operator containing its restriction to $C_0^\infty(M)$.

Lemma 55. *Let $\varepsilon > 0$. The m-sectorial operator Q associated with the form*

$$\left[y^{-\varepsilon}\mathcal{L}^4\left(y^\varepsilon g\right), h\right]$$

on $C_0^\infty(\mathbf{D})$ has the property that Q is essentially m-sectorial. Furthermore, Q has domain $D(H^4)$, where H is the self-adjoint Laplacian.

Proof. Its restriction to C_0^∞ has the property that, for $\|g\| = 1$,

$$\left\|y^{-\varepsilon}\mathcal{L}^4\left(y^\varepsilon g\right) - \mathcal{L}^4 g\right\| = \sum_{n=0}^{7} W_n g,$$

where

$$\|W_n g\| \leq \varepsilon \left\|H^{7/2} g\right\| + K_n \|g\| .$$

To see this, use the extension technique of Theorem 50, noting that fractional powers of H and multiplication by powers of y commute with the extension operator E. In fact, recall that

$$\left\|\sqrt{H} f\right\|^2 = \int_M |\nabla f|^2 \, dx.$$

Note that, using cutoff functions as before, we may assume that the extension of g is supported in a ball of radius 1 in hyperbolic space. Hence, using elliptic regularity in this ball, we obtain the corresponding inequality, for the extended functions, where we have used the Leibniz rule to express the inverse image of

$$y^{-\varepsilon}\mathcal{L}^4\left(y^\varepsilon g\right) - \mathcal{L}^4 g$$

on the tangent space $T_p(\mathbf{H})$ in terms of multiples of lower order partial derivatives. To complete the argument, we pull the extension back into $\mathbf{D}$, taking the inverse image. Hence, it is not difficult to see that the range of the closure (with i denoting $\sqrt{-1}$)

$$\left(y^{-\varepsilon}\mathcal{L}^4 y^\varepsilon + N\right)_0$$

of the restriction of the expression

$$y^{-\varepsilon}\mathcal{L}^4 y^{\varepsilon} + Ni$$

to

$$C_0^\infty(\mathbf{D})$$

is closed, for $|N|$ large enough; this is because the norm of the inverse of $H^4 + Ni$ is less than or equal to $1/N$. It is also clear that the nullity of this operator is zero. It follows by Atkinson's theorem on the perturbation of the index of a Fredholm operator that the deficiency indices of

$$\left(y^{-\varepsilon}\mathcal{L}^4 y^{\varepsilon} + Ni\right)_0$$

are 0, because by Cordes's theorem the restriction of $\mathcal{L}^4$ to $C_0^\infty(\mathbf{D})$ is essentially self-adjoint, and hence the deficiency indices of the unperturbed operators are zero. The result now follows. □

Remark 56. The same proof, but somewhat simpler, yields the following

Lemma 57. *The m-sectorial operator Q_n associated with the form $[y^{-\varepsilon}\mathcal{L}^n y^{\varepsilon} g, h]$ on $C_0^\infty(\mathbf{D})$ is essentially m-sectorial and has domain $D(H^n)$ for $n = 1, 2, 3$.*

Corollary 58. *Suppose that $F \in \operatorname{range}(P_\lambda)$. Then $y^{-\varepsilon}F \in \operatorname{domain}(Q_2) = \operatorname{domain}(H)$. In particular, $y^{-\varepsilon}y^{-1/2}F \in L_\infty$. Furthermore,*

$$\left\| y^{-\varepsilon}y^{-1/2}F \right\|_\infty \leq C\lambda_1 \|F\|_{Z'}.$$

• We are now ready to state the main theorem of the paper and to apply it to prove the corollary stated in the introduction.

Definition 59. Let $\{A_n\}$ be the family such that

$$A_1 = H$$

where H is the self-adjoint Laplacian in $L_2(\mathbf{D})$; and

$$A_n : n > 1 = T_n$$

where T_n is the corresponding Hecke operator, normalized to have norm one.

Definition 60. Let

$$W = C_0^\infty(\mathbf{D});$$
$$Z = \left\{\phi = y^{-\varepsilon}\theta : \theta \in D\left(H^3\right)\right\}$$

with norm

$$\|\phi : Z\| = \|(H+1)\,\theta : L_2\|.$$

Let

$$B = \left\{F : y^{-(1/2)-\delta}F \in L_\infty(\mathbf{D})\right\};$$
$$\|F : B\| = \left\|y^{-(1/2)-\delta}F : L_\infty\right\|.$$

Theorem 61. *There exists a generalized eigenfamily of multiplicity one, for* $\{A_n\}$*, associated with the space* $W = C_0^\infty(M)$*.*

Proof. Existence of the generalized eigenfamily is shown by Theorem 17. Multiplicity one follows from the fact that there exists at most a one-dimensional simultaneous eigenspace for $\{A_i\}$ in $C^\infty(\mathbf{D})$. This is known and is shown by using Fourier series techniques, see Huxley [**5**]. □

Theorem 62. *Let*

$$\lambda \longrightarrow P_\lambda$$

be the generalized eigenfamily of Theorem 61*, for the family of operators in Definition* 59*, corresponding to the Laplacian and the Hecke operators. Let* Z, B *be as in Definition* 60*. For every* $\varepsilon > 0$ *there exists a compact subset* K *of the joint spectrum of* $\{A_n\}$ *such that the mapping*

$$\lambda \longrightarrow P_\lambda$$

is continuous from K into $C(Z,B)$, and such that for any $\phi \in Z$, such that

$$\|\phi : Z\| \le 1,$$
$$\left\| \left(\theta - \int_K P_\lambda \phi \, d\mu\,(\lambda) \right) : B \right\| < \varepsilon.$$

Proof. We use Theorem 31. We must show that a spectral embedding exists. Note that

$$Z \subset D\left(H^3\right),$$

because for

$$\phi \in Z,$$
$$\mathcal{L}^3 y^\varepsilon \phi \in L_2$$

and therefore

$$y^{-\varepsilon} \mathcal{L}^3 y^\varepsilon \phi \in L_2.$$

We know that the embedding from Z_0 into L_2 is two nuclear, where

$$Z_0 = \left\{ \phi = y^{-\varepsilon} S^{-1} \theta : \theta \in L_2 \right\},$$
$$\|\phi : Z_0\| = \|\theta : L_2\|.$$

This follows from Corollary 51. Iterating this, it is not difficult to show that the embedding from Z into the domain of H is nuclear. In fact, if

$$A\theta = y^{-\varepsilon} S^{-1} \theta,$$

then

$$AA^* = y^{-\varepsilon} \left(H+1\right)^{-2} y^{-\varepsilon}.$$

Clearly, AA^* is nuclear, since it is a product of two-nuclear operators. However, the range of AA^* is the set of all functions f such that

$$y^{-2\varepsilon} f \in D\left(H+1\right)^2.$$

In fact, as we saw above, the set of all

$$\theta = y^\varepsilon \left(H+1\right)^{-2} \left(y^{-\varepsilon} f\right) : f \in L_2$$

is the same as the domain of $(H+1)^2$, which is the same as the domain of H^2. Hence, the set of all $y^{-2\varepsilon}(H+1)^{-2} f$ is the same as the range of AA^*.

It is also easy to see that

$$Z \subset D\left(H^2\right).$$

Finally, since $H^2 Z \subset D(H) \subset B$, the conclusion follows. □

Remark 63. Corollary 2 stated in the introduction now follows immediately. Note that if λ_j is in the joint spectrum, $(\lambda_j)_n$ is the associated generalized eigenvalue of A_n.

Corollary 64. *For every $\varepsilon > 0$, there exists a compact subset K of the joint spectrum of $\{A_n\}$, and a positive δ, such that for every δ-net $\{\lambda_i\}$ of K there exists a set of generalized simultaneous eigenfunctions F_i such that*

$$\begin{aligned} \mathcal{L}F_{\lambda_j} &= \left(\lambda_j\right)_1 F_{\lambda_j} \\ T_n F_{\lambda_j} &= \left(\lambda_j\right)_n F_{\lambda_j} \end{aligned}$$

and a set of complex constants c_i such that, for every ϕ in the unit ball of Z,

$$\left\| \left(\phi - \sum_{j=1}^{n(\varepsilon)} c_j F_{\lambda_j}\left(\phi\right) F_{\lambda_j} \right) : B \right\| < \varepsilon.$$

Here, F_{λ_j} and c_j are independent of ϕ. c_j and $n(\varepsilon)$ in general depend upon the choice of $\{\lambda_j\}$, though clearly $n(\varepsilon)$ may be chosen to be minimal.

Afterword. Professor Robert M. Kauffman passed away on February 8, 2004. He was a great friend and mentor for many of us. He was also a fine mathematician and generous co-author and contributed half of the paper, which is one of his final ones.

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