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DOUBLE EULER SUMS ON HURWITZ ZETA FUNCTIONS

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ABSTRACT. In an attempt to derive explicit evaluation of the double Euler sum with Dirichlet characters defined by

$$S^{\chi}_{p,q} := \sum_{k=1}^{\infty} \frac{\chi(k)}{k^q} \sum_{j=1}^{k} \frac{1}{j^p},$$

we decompose it into double Euler sums on Hurwitz zeta functions. In this paper, we give explicit evaluation of these Euler sums on Hurwitz zeta functions in terms of Hurwitz zeta values and the digamma function.

1. Introduction. For a pair of positive integers p and q with $q \geq 2$, the classical (double) Euler sum is defined as

$$S_{p,q} := \sum_{k=1}^{\infty} \frac{1}{k^q} \sum_{j=1}^{k} \frac{1}{j^p}. \tag{1.1}$$

It is well known that $S_{p,q}$ can be evaluated in terms of values at positive integers of the Riemann zeta function when $p = 1$ or $(p,q) = (2,4)$ or $(p,q) = (4,2)$ or $p = q$ or $p + q$ is odd. For explicit evaluations of the classical Euler sums or related alternating sums, readers may consult [**2, 3, 6, 11, 16, 18**]. Note that our $S_{p,q}$ is referred to as the double Harmonic series and denoted by $S(q,p)$ in [**12, 13**] but is called double zeta-star value and denoted by $\zeta^*(q,p)$ in [**1**]. It now becomes more customary to denote the double zeta values by $\zeta(s_1, s_2)$, which unfortunately conflicts with equally standard $\zeta(s,x)$ notation for the

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Hurwitz zeta function. Since our focus is more on representation of the double Euler sums in terms of the Hurwitz zeta values, we shall reserve the ζ notation throughout for either the Riemann or Hurwitz zeta functions.

As a generalization of double Euler sums, we replace the role of the Riemann zeta function by the more general Hurwitz zeta function,

$$\zeta(s,x) := \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}, \quad \operatorname{Re} s > 1, \quad x > 0. \tag{1.2}$$

This leads to the consideration of *double Euler sums on Hurwitz zeta functions*.

For a pair of positive integers p and q with $q \geq 2$ and positive real numbers x and y with $x, y \leq 1$, we define

$$H_{p,q}(x,y) := \sum_{k=0}^{\infty} \frac{1}{(k+y)^q} \sum_{j=0}^{\widetilde{k}} \frac{1}{(j+x)^p}, \tag{1.3}$$

where

$$\widetilde{k} = \begin{cases} k & \text{if } x \leq y; \\ k-1, & \text{if } x > y. \end{cases}$$

The following reflection formulae are immediate: For $p, q \geq 2$,

$$H_{p,q}(x,y) + H_{q,p}(y,x) = \zeta(p,x)\zeta(q,y), \tag{1.4}$$

if $x \neq y$, and

$$H_{p,q}(x,x) + H_{q,p}(x,x) = \zeta(p,x)\zeta(q,x) + \zeta(p+q,x). \tag{1.5}$$

Such new Euler sums provide a continuous version of, and actually building blocks for, the classical Euler sums or various kinds of generalized Euler sums [**5, 7, 8, 9, 10, 18**].

Example 1. For the classical Euler sums, we have

$$H_{p,q}(1,1) = S_{p,q}, \tag{1.6}$$

and, for any positive integer N,

$$S_{p,q} = \frac{1}{N^{p+q}} \sum_{a=1}^{N} \sum_{b=1}^{N} H_{p,q}\left(\frac{a}{N}, \frac{b}{N}\right). \tag{1.7}$$

Example 2. Let $r = a/b$ with a and b being relatively prime positive integers. Then the extended Euler sum considered in [**5**],

$$E_{p,q}^{(r)} := \sum_{k=1}^{\infty} \frac{1}{k^q} \sum_{j=1}^{[kr]} \frac{1}{j^p}, \tag{1.8}$$

can be written as

$$E_{p,q}^{(r)} = \frac{1}{a^p b^q N^{p+q}} \sum_{u=1}^{aN} \sum_{v=1}^{bN} H_{p,q}\left(\frac{u}{aN}, \frac{v}{bN}\right). \tag{1.9}$$

Example 3. For Euler sums with Dirichlet character χ modulo N, as considered in [**7**],

$$S_{p,q}^{\chi} := \sum_{k=1}^{\infty} \frac{\chi(k)}{k^q} \sum_{j=1}^{k} \frac{1}{j^p}, \tag{1.10}$$

we note that

$$S_{p,q}^{\chi} = \frac{1}{N^{p+q}} \sum_{a=1}^{N} \sum_{b=1}^{N-1} \chi(b) H_{p,q}\left(\frac{a}{N}, \frac{b}{N}\right). \tag{1.11}$$

We may rewrite $S_{p,q}^{\chi}$ as

(1.12)

$$\begin{aligned} S_{p,q}^{\chi} = &\frac{1}{2N^{p+q}} \sum_{b=1}^{N-1} \chi(b) \\ &\times \sum_{a=1}^{N-1} \left\{ H_{p,q}\left(\frac{a}{N}, \frac{b}{N}\right) + \chi(-1) H_{p,q}\left(\frac{N-a}{N}, \frac{N-b}{N}\right) \right\} \\ &+ \frac{1}{2N^{p+q}} \sum_{b=1}^{N-1} \chi(b) \left\{ H_{p,q}\left(1, \frac{b}{N}\right) + \chi(-1) H_{p,q}\left(1, \frac{N-b}{N}\right) \right\}. \end{aligned}$$

Consequently, $S^{\chi}_{p,q}$ can be explicitly evaluated provided that the quantities in the braces of (1.12) can be evaluated, which is the case if $\chi(-1) = (-1)^{p+q-1}$. This was proved, without giving explicit evaluations, in [**7**].

This last example motivates us to consider, for $0 < x, y < 1$,

$$T_{p,q}(x) := H_{p,q}(1, x) + \varepsilon H_{p,q}(1, 1-x) \tag{1.13}$$

and

$$G_{p,q}(x, y) := H_{p,q}(x, y) + \varepsilon H_{p,q}(1-x, 1-y), \tag{1.14}$$

where

$$\varepsilon := (-1)^{w-1} = (-1)^{p+q-1}, \tag{1.15}$$

so that $\varepsilon = 1$ if the weight $w = p + q$ is odd and $\varepsilon = -1$ when w is even.

The exact purpose of the paper is to give explicit evaluations of $T_{p,q}(x)$ and $G_{p,q}(x, y)$. It will be easier to work with their companions, which we now define. For $0 < x < 1$ and a pair of positive integers p and q with $q \geq 2$, define

$$E_{p,q}(x) := H_{p,q}(x, 1) + \varepsilon H_{p,q}(1-x, 1). \tag{1.16}$$

In view of the reflection formula, which follows from (1.4),

$$E_{p,q}(x) + T_{q,p}(x) = \zeta(p, x)\zeta(q) + \varepsilon\zeta(p, 1-x)\zeta(q) \tag{1.17}$$

and the differentiation formula

$$E_{p,q}(x) = \frac{(-1)^{p-1}}{(p-1)!}\left(\frac{d}{dx}\right)^{p-1} E_{1,q}(x), \tag{1.18}$$

Theorem 1 and Corollary 1 below suffice to effect explicit evaluation of $T_{p,q}(x)$.

Theorem 1. *Let $E_{p,q}(x)$ be defined by* (1.16). *For positive integers $q \geq 2$, let $w = 1 + q$ and $\varepsilon = (-1)^q$. Then*

$$\begin{aligned} E_{1,q}(x) &= \frac{1}{2}\left\{\zeta(w-1, 1-x) - \varepsilon\zeta(w-1, x)\right\}\pi\cot\pi x \\ &\quad - \sum_{\ell=0}^{[(q-1)/2]} \zeta(2\ell)\left\{\zeta(w-2\ell, 1-x) + \varepsilon\zeta(w-2\ell, x)\right\}, \end{aligned} \tag{1.19}$$

where, as is customary, $\zeta(0) = -1/2$.

Corollary 1. *For positive integers p and q with $q \geq 2$, and $w = p+q$ and $\varepsilon = (-1)^{p+q-1}$, we have*

$$\begin{aligned} E_{p,q}(x) = (-1)^{p-1}\Bigg\{ &\frac{1}{2}\binom{w-2}{q-1} \\ &\times \left\{\zeta(w-1, 1-x) - \varepsilon\zeta(w-1, x)\right\}\pi\cot\pi x \\ &- \frac{1}{2}\sum_{\ell=1}^{p-1}\binom{w-\ell-2}{q-1}\left\{\zeta(\ell+1, 1-x) + (-1)^{\ell+1}\zeta(\ell+1, x)\right\} \\ &\times \left\{\zeta(w-\ell-1, 1-x) + \varepsilon(-1)^{\ell+1}\zeta(w-\ell-1, x)\right\} \\ &- \sum_{\ell=0}^{[(q-1)/2]}\binom{w-2\ell-1}{p-1}\zeta(2\ell) \\ &\times \left\{\zeta(w-2\ell, 1-x) + \varepsilon\zeta(w-2\ell, x)\right\}\Bigg\}. \end{aligned} \tag{1.20}$$

As regards the evaluation of $G_{p,q}(x, y)$, it is convenient to employ the notation for the digamma function:

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}. \tag{1.21}$$

Theorem 2. *For $0 < x < y < 1$ and positive integers $q \geq 2$, let $\varepsilon = (-1)^q$. Then*

(1.22)
$$\begin{aligned}2\,G_{1,q}(x,y) &= \{\zeta(q,y-x)-\varepsilon\zeta(q,1+x-y)\}\,\{\pi\cot\pi x-\pi\cot\pi y\}\\ &\quad+\{\zeta(q,y)+\varepsilon\zeta(q,1-y)\}\\ &\quad\times\{-\psi(x)-\psi(1-x)+\psi(y-x)+\psi(1+x-y)\}\\ &\quad-\sum_{\ell=2}^{q-1}\big\{\zeta(q+1-\ell,y-x)+\varepsilon(-1)^{\ell}\zeta(q+1-\ell,1+x-y)\big\}\\ &\qquad\quad\times\big\{\zeta(\ell,y)+(-1)^{\ell}\zeta(\ell,1-y)\big\}.\end{aligned}$$

Remark 1. Since we have the following differentiation formulae,

$$G_{p,q}(x,y)=\frac{(-1)^{p-1}}{(p-1)!}\left(\frac{\partial}{\partial x}\right)^{p-1}G_{1,q}(x,y) \tag{1.23}$$

and

$$G_{p,q+1}(x,y)=\frac{-1}{q}\left(\frac{\partial}{\partial y}\right)G_{p,q}(x,y), \tag{1.24}$$

it is not difficult to derive explicit evaluation of $E_{p,q}(x,y)+H_{p,q}(1-x,1-y)$ for odd w and of $H_{p,q}(x,y)-H_{p,q}(1-x,1-y)$ for even w from Theorem 2 if one so wishes.

2. The $p=1$ case. A key step to evaluating $E_{p,q}(x)$ or $G_{p,q}(x,y)$ is the explicit evaluation when the index p is 1. As $H_{1,q}(x,y)$ cannot be evaluated directly, we shall start with obtaining explicit evaluation of suitable combinations of $H_{1,q}(x,y)$ for various arguments x and y. We need the Kronecker limit formula for the Hurwitz zeta function, see e.g., [**17**, page 271, Section 13.21],

$$\lim_{s\to 1^+}\left\{\zeta(s,x)-\frac{1}{s-1}\right\}=-\psi(x), \tag{2.1}$$

and its consequence

$$\sum_{j=0}^{\infty}\left\{\frac{1}{j+x}-\frac{1}{j+y}\right\}=-\psi(x)+\psi(y). \tag{2.2}$$

We shall also need

$$\psi(x) - \psi(1-x) = -\pi \cot \pi x \tag{2.3}$$

and the following partial fraction decomposition, see [**3, 14**]:

$$\begin{aligned}\frac{1}{(X+\alpha)^q X^p} &= \sum_{\ell=2}^{p} (-1)^{p+\ell} \binom{w-1-\ell}{q-1} \frac{1}{\alpha^{w-\ell} X^\ell} \\ &\quad + \sum_{\ell=2}^{q} (-1)^{p} \binom{w-1-\ell}{p-1} \frac{1}{\alpha^{w-\ell} (X+\alpha)^\ell} \\ &\quad + (-1)^{p+1} \binom{w-2}{q-1} \frac{1}{\alpha^{w-1}} \left\{ \frac{1}{X} - \frac{1}{X+\alpha} \right\}.\end{aligned} \tag{2.4}$$

Proposition 1. *For $0 < x < 1$ and a positive integer q with $q \geq 2$, we have*

$$\begin{aligned}H_{1,q}(1,x) &= \frac{q}{2}\zeta(q+1,x) + \zeta(q,x)\{\psi(x)+\gamma\} \\ &\quad - \frac{1}{2}\sum_{\ell=2}^{q-1} \zeta(\ell,x)\zeta(q+1-\ell,x),\end{aligned} \tag{2.5}$$

where γ is Euler's constant defined by

$$\gamma := \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right).$$

Proof. By definition, we have

$$\begin{aligned}H_{1,q}(1,x) &= \sum_{k=0}^{\infty} \frac{1}{(k+x)^q} \sum_{j=0}^{k-1} \frac{1}{j+1} \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{(k+j+1+x)^q (j+1)}.\end{aligned}$$

If we let $p = 1$ in the partial fraction decomposition (2.4), then

$$\frac{1}{(X+\alpha)^q X} = \frac{1}{\alpha^q} \left\{ \frac{1}{X} - \frac{1}{X+\alpha} \right\} - \sum_{\ell=1}^{q-1} \frac{1}{\alpha^\ell (X+\alpha)^{q+1-\ell}}. \tag{2.6}$$

Using this we rewrite $H_{1,q}(1,x)$ as

$$H_{1,q}(1,x) = \sum_{k=0}^{\infty} \frac{1}{(k+x)^q} \sum_{j=0}^{\infty} \left\{ \frac{1}{j+1} - \frac{1}{k+j+1+x} \right\} - \sum_{\ell=1}^{q-1} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{(k+x)^{\ell}(k+j+1+x)^{q+1-\ell}},$$

which, in view of (2.2) and the fact that $\psi(1) = -\gamma$, is equal to

(2.7)
$$H_{1,q}(1,x) = \zeta(q,x)\{\gamma + \psi(x)\} + H_{1,q}(x,x) + (q-1)\zeta(q+1,x) - \sum_{\ell=1}^{q-1} H_{\ell,q+1-\ell}(x,x).$$

In the sum of the righthand side, the first term corresponding to $\ell = 1$ cancels with $H_{1,q}(x,x)$ and, by (1.5), the remaining sum is

$$\begin{aligned} \sum_{\ell=2}^{q-1} H_{\ell,q+1-\ell}(x,x) &= \frac{1}{2}\sum_{\ell=2}^{q-1}(H_{\ell,q+1-\ell}(x,x) + H_{q+1-\ell,\ell}(x,x)) \\ &= \frac{1}{2}\sum_{\ell=2}^{q-1}(\zeta(\ell,x)\zeta(q+1-\ell,x) + \zeta(q+1,x)) \\ &= \frac{q-2}{2}\zeta(q+1,x) + \frac{1}{2}\sum_{\ell=2}^{q-1}\zeta(\ell,x)\zeta(q+1-\ell,x). \end{aligned}$$

Substituting this in (2.7), we finish the proof. □

The following propositions can be derived in a similar manner, and so we omit the proof.

Proposition 2. *For $0 < x < 1$ and a positive integer $q \geq 2$, we have*

(2.8)
$$H_{1,q}(x,x) + H_{1,q}(1-x,1) = \zeta(q+1,x) - \zeta(q,x)\{\psi(1-x) + \gamma\} - \sum_{\ell=2}^{q-1} \zeta(\ell)\zeta(q+1-\ell,x).$$

Proposition 3. *For $0 < x < y < 1$ and a positive integer $q \geq 2$, we have*

$$\begin{aligned} H_{1,q}(x,y) &+ H_{1,q}(1-x,y-x) \\ &= \zeta(q,y-x)\{-\psi(x)+\psi(y)\} + \zeta(q,y)\{-\psi(1-x)+\psi(y-x)\} \\ &\quad - \sum_{\ell=2}^{q-1} \zeta(\ell,y)\zeta(q+1-\ell,y-x). \end{aligned} \tag{2.9}$$

3. The ETD system and proof of Theorem 1. As suggested by (1.17) and (1.18), it is more natural to start our evaluation with $E_{1,q}(x)$, than those of $E_{p,q}(x)$, and hence the evaluation of $T_{p,q}(x)$ follows at once.

For $0 < x < 1$ and a pair of positive integers p and q with $q \geq 2$, we recall

$$E_{p,q}(x) := H_{p,q}(x,1) + \varepsilon H_{p,q}(1-x,1), \tag{3.1}$$

$$T_{p,q}(x) := H_{p,q}(1,x) + \varepsilon H_{p,q}(1,1-x), \tag{3.2}$$

where $\varepsilon = (-1)^{p+q-1}$, and define

$$D_{p,q}(x) := H_{p,q}(1-x,1-x) + \varepsilon H_{p,q}(x,x). \tag{3.3}$$

Proposition 4. *For $0 < x < 1$ and each positive integer q with $q \geq 2$, and $\varepsilon = (-1)^q$, we have*

$$\begin{aligned} E_{1,q}(x) + D_{1,q}(x) &= \zeta(q+1,1-x) + \varepsilon\zeta(q+1,x) \\ &\quad - \zeta(q,1-x)\{\psi(x)+\gamma\} - \varepsilon\zeta(q,x)\{\psi(1-x)+\gamma\} \\ &\quad - \sum_{\ell=2}^{q-1} \zeta(\ell)\{\zeta(q+1-\ell,1-x) + \varepsilon\zeta(q+1-\ell,x)\}. \end{aligned} \tag{3.4}$$

Proof. This follows from

$$\begin{aligned} E_{1,q}(x) + D_{1,q}(x) &= \varepsilon\{H_{1,q}(x,x) + H_{1,q}(1-x,1)\} \\ &\quad + \{H_{1,q}(1-x,1-x) + H_{1,q}(x,1)\} \end{aligned}$$

and the evaluation obtained in Proposition 2. □

Proposition 5. *For $0 < x < 1$ and each positive integer q with $q \geq 2$, let $\varepsilon = (-1)^q$. Then*

$$E_{1,q}(x) - D_{1,q}(x) = \zeta(q, 1-x)\{\psi(1-x)+\gamma\} + \varepsilon\zeta(q,x)\{\psi(x)+\gamma\} + \sum_{\ell=2}^{q-1}(-1)^{\ell+1}\zeta(\ell)\{\zeta(q+1-\ell, 1-x) + \varepsilon\zeta(q+1-\ell, x)\}. \tag{3.5}$$

Proof. To ease the burden of carrying around the ε factor in our derivation, we shall deal only with the case $q = 2n$ throughout, the case of odd q being similar.

Specializing the parameters $(p, q) = (2n-1, 2)$ in (2.4), we get

$$\frac{1}{(X+\alpha)^2 X^{2n-1}} = \sum_{\ell=2}^{2n-1}(-1)^{\ell+1}(2n-\ell)\frac{1}{\alpha^{2n+1-\ell}X^{\ell}} - \frac{1}{\alpha^{2n-1}(X+\alpha)^2} + (2n-1)\frac{1}{\alpha^{2n}}\left\{\frac{1}{X} - \frac{1}{X+\alpha}\right\}. \tag{3.6}$$

It follows that

$$H_{2n-1,2}(x,x) = \zeta(2n+1,x) + \sum_{\ell=2}^{2n-1}(-1)^{\ell+1}(2n-\ell)\zeta(2n+1-\ell)\zeta(\ell,x) - H_{2n-1,2}(1,x) + (2n-1)H_{1,2n}(x,1).$$

Replacing x by $1-x$ and adding together, we get

$$T_{2n-1,2}(x) + D_{2n-1,2}(x) = \zeta(2n+1,x) + \zeta(2n+1,1-x) + (2n-1)E_{1,2n}(x) + \sum_{\ell=2}^{2n-1}(-1)^{\ell+1}(2n-\ell)\zeta(2n+1-\ell)\{\zeta(\ell,x) + \zeta(\ell,1-x)\}. \tag{3.7}$$

In the same way, we get

$$T_{2n-1,2}(x) + D_{2n-1,2}(x) = \zeta(2n+1,x) + \zeta(2n+1,1-x) + (2n-1)\zeta(2n,x)\{\gamma+\psi(x)\} + (2n-1)\zeta(2n,1-x)\{\gamma+\psi(1-x)\} + (2n-1)D_{1,2n}(x) + \sum_{\ell=2}^{2n-1}(-1)^{\ell+1}(2n-\ell)\{\zeta(2n+1-\ell,x) + \zeta(2n+1-\ell,1-x)\}\zeta(\ell) \tag{3.8}$$

if we begin with

$$H_{2n-1,2}(1,x)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{(k+j+1+x)^2(j+1)^{2n-1}}.$$

Subtraction of (3.8) from (3.7) then yields the result. □

Proof of Theorem 1 *and Corollary* 1. The evaluation of $E_{1,q}(x)$ follows from Propositions 4 and 5. Now Corollary 1 is an easy consequence by virtue of (2.3) and the facts that

$$\frac{d}{dx}\zeta(s,1-x)=s\zeta(s+1,1-x),$$
$$\frac{d}{dx}\zeta(s,x)=-s\zeta(s+1,x),$$

and

$$\frac{d}{dx}[\psi(x)-\psi(1-x)]=\zeta(2,x)+\zeta(2,1-x). \qquad \square$$

Remark 2. Note that $D_{p,q}(x)$ can be expressed in terms of $E_{u,v}(x)$ and $T_{u,v}(x)$ with the same weight $u+v=p+q$. Besides, just like the differentiation formula (1.18), the higher derivatives of $D_{1,q}(x)$ recursively give the evaluation of $D_{p,q}(x)$ also.

4. The KL system and the proof of Theorem 2. From now on, we assume that $0<x<y<1$. The evaluation of

$$G_{p,q}(x,y)=H_{p,q}(x,y)+\varepsilon H_{p,q}(1-x,1-y),$$

where $\varepsilon=(-1)^{p+q-1}$, is a bit tricky. To this purpose, we introduce two families of Euler sums, where each member consists of four terms and is defined as

$$\begin{aligned}K_{p,q}(x,y):=&\,H_{p,q}(x,y)+\varepsilon H_{p,q}(1-x,1-y)\\&+\varepsilon H_{p,q}(y-x,y)+H_{p,q}(1+x-y,1-y),\end{aligned}\tag{4.1}$$

$$\begin{aligned}L_{p,q}(x,y):=&\,\varepsilon H_{p,q}(y,x)+H_{p,q}(1-y,1-x)\\&+H_{p,q}(y,y-x)+\varepsilon H_{p,q}(1-y,1+x-y)\end{aligned}\tag{4.2}$$

and

$$K^{-}_{p,q}(x,y) := H_{p,q}(x,y) + \varepsilon H_{p,q}(1-x,1-y) - \varepsilon H_{p,q}(y-x,y) - H_{p,q}(1+x-y,1-y), \tag{4.3}$$

$$L^{-}_{p,q}(x,y) := \varepsilon H_{p,q}(y,x) + H_{p,q}(1-y,1-x) - H_{p,q}(y,y-x) - \varepsilon H_{p,q}(1-y,1+x-y). \tag{4.4}$$

Note that the desired quantity $G_{p,q}(x,y)$ is actually equal to

$$\frac{1}{2}(K_{p,q}(x,y) + K^{-}_{p,q}(x,y)) = \frac{1}{2}(L_{p,q}(1-y,1-x) + L^{-}_{p,q}(1-y,1-x)).$$

In view of (1.23) and (1.24), the task of deriving $G_{p,q}(x,y)$ comes down to evaluating $G_{1,q}(x,y)$, and therefore it suffices to evaluate $L_{1,q}(x,y)$ and $L^{-}_{1,q}(x,y)$ instead.

Proposition 6. *For each positive integer q with $q \geq 2$, let $\varepsilon = (-1)^q$. Then*

$$\begin{aligned} L_{1,q}(x,y) = {} & \zeta(q,y-x)\{-\psi(1-y)+\psi(1-x)\} \\ & + \zeta(q,1-x)\{-\psi(y)+\psi(y-x)\} \\ & + \varepsilon\zeta(q,x)\{-\psi(1-y)+\psi(1+x-y)\} \\ & + \varepsilon\zeta(q,1+x-y)\{-\psi(y)+\psi(x)\} \\ & - \sum_{\ell=2}^{q-1}\zeta(\ell,1-x)\zeta(q+1-\ell,y-x) \\ & - \varepsilon\sum_{\ell=2}^{q-1}\zeta(\ell,1+x-y)\zeta(q+1-\ell,x). \end{aligned} \tag{4.5}$$

Proof. This follows from

$$\begin{aligned} L_{1,q}(x,y) = {} & \{H_{1,q}(1-y,1-x) + H_{1,q}(y,y-x)\} \\ & + \varepsilon\{H_{1,q}(1-y,1+x-y) + H_{1,q}(y,x)\} \end{aligned}$$

and Proposition 3. □

Proposition 7. *For each positive integer q with $q \geq 2$, let $\varepsilon = (-1)^q$. Then*

(4.6)
$$\begin{aligned}
L^-_{1,q}(x,y) &= \zeta(q, y-x)\left\{-\psi(x) + \psi(y)\right\} \\
&\quad + \varepsilon\zeta(q, 1+x-y)\left\{-\psi(1-x) + \psi(1-y)\right\} \\
&\quad - \varepsilon\zeta(q, x)\left\{-\psi(y-x) + \psi(y)\right\} \\
&\quad - \zeta(q, 1-x)\left\{-\psi(1+x-y) + \psi(1-y)\right\} \\
&\quad + \sum_{\ell=2}^{q-1}(-1)^{\ell+1}\zeta(q+1-\ell, y-x)\zeta(\ell, x) \\
&\quad + \varepsilon\sum_{\ell=2}^{q-1}(-1)^{\ell+1}\zeta(q+1-\ell, 1+x-y)\zeta(\ell, 1-x).
\end{aligned}$$

Proof. We begin with

$$\begin{aligned}
K^-_{q-1,2}(x,y) &= \left\{H_{q-1,2}(x,y) + \varepsilon H_{q-1,2}(1-x, 1-y)\right\} \\
&\quad - \left\{\varepsilon H_{q-1,2}(y-x, y) + H_{q-1,2}(1+x-y, 1-y)\right\}.
\end{aligned}$$

Employing (2.4) with p and q replaced by $q-1$ and 2, respectively, we get

$$\begin{aligned}
&\frac{1}{(X+\alpha)^2 X^{q-1}} \\
&\qquad = (-1)^q\Bigg(\sum_{\ell=2}^{q-1}(-1)^{\ell+1}(q-\ell)\frac{1}{\alpha^{q+1-\ell}X^\ell} - \frac{1}{\alpha^{q-1}(X+\alpha)^2} \\
&\qquad\qquad + (q-1)\frac{1}{\alpha^q}\left\{\frac{1}{X} - \frac{1}{X+\alpha}\right\}\Bigg).
\end{aligned}$$

It follows that, with $\varepsilon = (-1)^q$,

$$\begin{aligned}
H_{q-1,2}(x,y) = \varepsilon\Bigg(&\sum_{\ell=2}^{q-1}(-1)^{\ell+1}(q-\ell)\zeta(q+1-\ell, y-x)\zeta(\ell, x) \\
&- H_{q-1,2}(y-x, y) + H_{1,q}(y, y-x) \\
&+ (q-1)(q-1)\zeta(q, y-x)\left\{-\psi(x) + \psi(y)\right\}\Bigg).
\end{aligned}$$

We then apply the same process to other terms defining $K^{-}_{q-1,2}(x,y)$ and sum together to get

(4.7)

$$\begin{aligned}K^{-}_{q-1,2}(x,y) = {} & K^{-}_{q-1,2}(x,y) - \varepsilon(q-1)L^{-}_{1,q}(x,y)\\
& + \varepsilon\sum_{\ell=2}^{q-1}(-1)^{\ell+1}(q-\ell)\zeta(q+1-\ell,y-x)\zeta(\ell,x)\\
& + \sum_{\ell=2}^{q-1}(-1)^{\ell+1}(q-\ell)\zeta(q+1-\ell,1+x-y)\zeta(\ell,1-x)\\
& - \sum_{\ell=2}^{q-1}(-1)^{\ell+1}(q-\ell)\zeta(q+1-\ell,x)\zeta(\ell,y-x)\\
& - \varepsilon\sum_{\ell=2}^{q-1}(-1)^{\ell+1}(q-\ell)\zeta(q+1-\ell,1-x)\zeta(\ell,1+x-y)\\
& + \varepsilon(q-1)\zeta(q,y-x)\left\{-\psi(x)+\psi(y)\right\}\\
& + (q-1)\zeta(q,1+x-y)\left\{-\psi(1-x)+\psi(1-y)\right\}\\
& - (q-1)\zeta(q,x)\left\{-\psi(y-x)+\psi(y)\right\}\\
& - \varepsilon(q-1)\zeta(q,1-x)\left\{-\psi(1+x-y)+\psi(1-y)\right\}.\end{aligned}$$

Thus, our assertion then follows from an elementary calculation. □

Proof of Theorem 2. Adding (4.5) and (4.6) together and changing the variables $(x,y)\to(1-y,1-x)$, we immediately derive Theorem 2. □

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