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STRUCTURALLY STABLE QUADRATIC FOLIATIONS

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ABSTRACT. We characterize the elements of F_n, the set of polynomial vector fields on the plane of degree at most n without finite singular points, that are structurally stable with respect to perturbations within F_n for $n \leq 2$. We do so with respect to each of the two natural definitions of stability in this setting.

1. Introduction and statement of the main results. Although the characterization of C^r vector fields on compact two-dimensional manifolds that are structurally stable goes back to Peixoto [**11**], the characterization of all structurally stable planar polynomial vector fields of degree n (under perturbation by polynomial vector fields of degree at most n) is still an open problem since it is not known if the "natural" hyperbolicity condition on the limit cycles is needed for stability.

Moreover, even in the case of general families for which we know a *characterization theorem* in terms of singular points, periodic orbits and saddle connections, it is difficult to give an explicit *classification* of all structurally stable phase portraits. An exception to this can be found in [**2**], in which is given the classification of all structurally stable phase portraits for the quadratic family modulo limit cycles. We note, however, that the mathematical object that is structurally stable is not the phase portrait but the specific vector field that realizes it.

One family of polynomial vector fields that is a natural candidate for a complete characterization theorem, and a categorization of structurally stable phase portraits, is what we will refer to as the set of *planar polynomial foliations*, or simply, *foliations*, that is, planar polynomial

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vector fields without singular points (or, equivalently, having all their singularities at infinity), whose phase portrait is a foliation of $\mathbf{R}^2$. For of course within this class of systems there are no cycles in the phase portraits, automatically eliminating the problem of limit cycles. New difficulties, however, particularly for obtaining necessary conditions for structural stability, are introduced by the fact that the set of allowable perturbations is so small. (If we allow general polynomial perturbation of degree at most n, or even smooth perturbation, characterization theorems follow immediately as corollaries of theorems in [**14, 15, 16**].) In an earlier work [**7**] we were able to obtain separate lists of necessary conditions and of sufficient conditions for structural stability (restricting to perturbation within the set of polynomial foliations of degree at most n) when the topological equivalences involved take place on the so-called Poincaré sphere, see Section 2; they are reproduced in Theorem 3.1 below. In this paper we obtain full characterization of structural stability of polynomial foliations of degrees 1 and 2, both on the Poincaré sphere and on the plane, together with a complete catalogue of phase portraits of stable systems.

To describe the results and their context more precisely, we introduce some notation and terminology and give an outline of the two notions of structural stability of polynomial vector fields. A more detailed discussion is given in the next section. Let P_n denote the set of vector fields (P, Q) on $\mathbf{R}^2$ whose components are polynomial functions of x and y of degree at most n, and let F_n denote the set of foliations in P_n. For $X \in \mathrm{P}_n$, we let $\pi(X)$ denote the corresponding Poincaré vector field on $\mathbf{S}^2$. The *tangential eigenvalue* at a singularity p of $\pi(X)$ is the eigenvalue of $d\pi(X)(p)$ whose eigenspace contains the line tangent to the invariant equator $\mathbf{E}$ of $\mathbf{S}^2$.

There are two natural but inequivalent notions of structural stability of an element of F_n with respect to perturbation within F_n. Briefly, $X \in \mathrm{F}_n$ is $\mathbf{S}^2$*-structurally stable* if and only if for any Y in a sufficiently small neighborhood of X in F_n, $\pi(Y)$ is topologically equivalent to $\pi(X)$. $X \in \mathrm{F}_n$ is $\mathbf{R}^2$*-structurally stable* if the topological equivalence is between X and Y themselves. As will become clear, the distinction is already relevant for $X \in \mathrm{F}_1$.

The precise statement of our first main result, whose proof is given near the end of Section 3, is the following. To avoid circumlocutions, throughout this paper we will refer to F_2 as the set of quadratic foliations, even though some of its elements are linear and some constant. Similarly, F_1 will be called the set of linear foliations.

1.1. Theorem. *An element X of F_2 is $\mathbf{S}^2$-structurally stable with respect to perturbation within the set F_2 of all quadratic foliations if and only if it satisfies the following two conditions*:

(1) *The tangential eigenvalue at each singularity of $\pi(X)$ is nonzero.*

(2) *All separatrix connections of $\pi(X)$ lie in the equator $\mathbf{E}$ of the Poincaré sphere $\mathbf{S}^2$.*

The set of $\mathbf{S}^2$-structurally stable elements of F_2 is dense in F_2. Moreover, any requirement that the equivalence homeomorphism be near $\mathrm{id}_{\mathbf{S}^2}$ is redundant.

All possible topological types for $\pi(X)$ when $X \in \mathrm{F}_2$ are shown in [**4**, Figure 1], which we reproduce as our Figure 2 in Appendix A. We will refer to an element X of F_2 as being "of Type k" if $\pi(\pm X)$ is homeomorphic to the phase portrait in Figure 2 (k), which is also Figure 1 (k) of [**4**]. If X is constant, it is Type 8 (which is also the type of some quadratic elements of F_2); if X is linear it is Type 22 or Type 23 (which are not realized by any quadratic element of F_2). By combining Theorem 1.1 with the classification theorem given in [**4**], we obtain the following classification of all structurally stable phase portraits.

1.2. Corollary. *An element X of F_2 is $\mathbf{S}^2$-structurally stable with respect to perturbation within F_2 if and only if X is of Type $1, 3, 4$ or 8* (*and if of Type 8, the unique infinite singular point must have a nonzero tangential eigenvalue*).

Noting that the $\mathbf{S}^2$-structurally stable elements of F_2 have either the maximal or minimal number (counting multiplicity) of singularities at infinity, from the elementary theory of equations we may also express the $\mathbf{S}^2$-structural stability theorem as follows.

1.3. Corollary. *Let X be an element of F_2 and, using the notation of* (2.1), *set $A = a_{0,2}$, $B = a_{1,1} - b_{0,2}$, $C = a_{2,0} - b_{1,1}$, $D = -b_{2,0}$*

and $\Delta = 18ABCD - 4B^3D + B^2C^2 - 4AC^3 - 27A^2D^2$. *Then* X *is* $\mathbf{S}^2$*-structurally stable with respect to perturbation in* F_2 *if and only if either*

(i) $\Delta < 0$ *or*

(ii) $\Delta > 0$ *and there are no saddle connections of* $\pi(X)$ *not lying in* $\mathbf{E}$.

The two corollaries are proved at the end of Section 3.

In [**7**] we conjectured that conditions (1) and (2) in Theorem 1.1, which were proved in [**7**], to be sufficient for $\mathbf{S}^2$-structural stability for *every* degree n, are necessary as well. Theorem 1.1 and Proposition 3.2, which characterize $\mathbf{S}^2$-structural stability in the linear case, confirm the truth of the conjecture for $n \leq 2$.

For $n \geq 2$, the question of $\mathbf{R}^2$-structural stability is exceedingly difficult, because dramatic change in the phase portrait of $\pi(X)$ is compatible with complete lack of change in the phase portrait of X. The only known result for general degree n is Theorem 10 of [**7**], reproduced as Theorem 4.1 below. However, making extensive use of the work done in [**4**], we can give a characterization (and classification) theorem in this setting for low degree. Not surprisingly, all linear foliations are $\mathbf{R}^2$-structurally stable with respect to perturbation within F_1 (Proposition 4.2). The theorem for $n = 2$ is easier to state in terms of instability. The normal forms referred to in the theorem are those of Lemma A.2.

1.4. Theorem. *An element* X *of* F_2 *is* $\mathbf{R}^2$*-structurally unstable with respect to perturbation within the set* F_2 *of all quadratic foliations if and only if it meets one of the following conditions*:

(1) $\deg X < 2$ (*hence is Type* 8 *if constant and Type* 22 *or* 23 *if linear*);

(2) X *is Type* 6 *and has the normal form* $\dot{x} = x^2$, $\dot{y} = d + ax - xy$;

(3) X *is Type* 8 *and the linear part at the unique singularity of* $\pi(X)$ *has two zero eigenvalues but is not identically zero*;

(4) X *is Type* 8, *the linear part at the unique singularity of* $\pi(X)$ *is identically zero, and the normal form of* [**4**] *for* X *is either*

(a) $\dot{x} = x$, $\dot{y} = 1 + x^2$, *or*

(b) $\dot{x} = 1$, $\dot{y} = x^2$, *or*

(c) $\dot{x} = 0$, $\dot{y} = d + ax + lx^2$, *where* $a^2 - 4dl < 0$;

(5) X *is Type* 10, 12, 13, 14, *or* 19.

The set of $\mathbf{R}^2$*-structurally stable elements of* F_2 *is dense in* F_2*. Moreover, any requirement that the equivalence homeomorphism be near* $\mathrm{id}_{\mathbf{R}^2}$ *is redundant.*

The proof, given in Section 4, uses the classification theorem of phase portraits of quadratic foliations, given in [**4**], and the Neumann theorem [**10**] which insures that the relevant orbits in this context are the so-called *inseparable leaves.*

This paper is organized as follows. Section 2 introduces the notation and definitions. Section 3 is devoted to $\mathbf{S}^2$-structural stability. Section 4 deals with $\mathbf{R}^2$-structural stability. Finally, in the Appendix, we review the main parts of [**4**] used in Section 4. In particular, we include the statement and proof of a normal form theorem from [**4**] (Lemma A.2, which is Lemma 2 of [**4**]), which are essential for the proof of Theorem 1.1.

2. Background. Any element $X = (P, Q)$ of P_n is completely determined by the $(n+1)(n+2)$ coefficients of P and Q, hence may be identified with a point of $\mathbf{R}^{(n+1)(n+2)}$. The topology induced on P_n from the usual topology on $\mathbf{R}^{(n+1)(n+2)}$ by this identification is the *coefficient topology.* F_n is given the topology induced from P_n. We place the compact-open topology on the set H of homeomorphisms of $\mathbf{R}^2$ (see [**8**] for a discussion), and the uniform C^0 topology on the set J of homeomorphisms of $\mathbf{S}^2$.

For $X \in \mathrm{F}_n$ the orbits of X form a *foliation* of $\mathbf{R}^2$. An individual orbit of the foliation X is called a *leaf* of the foliation. It escapes to infinity (leaves every compact subset of $\mathbf{R}^2$) in forward and backward time, and divides the plane into two connected unbounded components. The only distinguished orbits of X, that is, the only orbits that a topological equivalence must respect, are the so-called *inseparable leaves.* Two distinct leaves L_1 and L_2 are said to be *inseparable* if for any arcs T_1 and T_2, to which X is nowhere tangent and such that L_i has nonempty intersection with T_i, $i = 1, 2$, there is a third leaf L, distinct from L_1 and L_2, that intersects both T_1 and T_2 (cf. [**6, 9**]). An individual leaf is then said to be an *inseparable leaf* of the foliation if it and some other, distinct leaf (which need not be unique) form a pair of inseparable leaves. For economy of expression we will refer to the inseparable leaves as being possessed by X itself, rather than always referring to them as lying in the phase portrait of X.

For $X \in \mathrm{P}_n$ the associated *Poincaré vector field* $\pi(X)$ is the unique analytic extension to the whole sphere of the vector field induced on $\mathbf{S}^2$ by central projection (after parallel translation of the plane to the north pole) and scaling by the $(n-1)$st power of the height function. The open upper hemisphere of $\mathbf{S}^2$, which in this context will be referred to as the *Poincaré sphere*, will be called the *finite part of the plane*, corresponding to $\mathbf{R}^2$; we will let $\mathbf{E}$ denote the equator of the sphere, which is always invariant for $\pi(X)$, and always contains at least one pair of singularities when $X \in \mathrm{F}_n$. We confine attention to the flow of $\pi(X)$ within the closed upper hemisphere, which is referred to as the *Poincaré disk*. Complete details and simple coordinate expressions using central projection appear for example in [**5**]. Note that $\pi(X)$ depends not only on X but also on the set P_n in which it is regarded as lying. Specifically, if $m := \deg X < n$, then $\mathbf{E}$ is always critical for $\pi(X)$ when X is regarded as an element of P_n, even though generically $\mathbf{E}$ contains but finitely many singularities of $\pi(X)$ when X is viewed as an element of P_m.

We say that $X \in \mathrm{F}_n$ is $\mathbf{R}^2$-*structurally stable* (with respect to perturbation in F_n) if for any neighborhood M of $\mathrm{id}_{\mathbf{R}^2}$ in H there exists a neighborhood N of X in F_n such that every X' in N is topologically equivalent to X by an equivalence homeomorphism h lying in M. We say that $X \in \mathrm{F}_n$ is $\mathbf{S}^2$-*structurally stable* (with respect to perturbation in F_n) if for any neighborhood M' of $\mathrm{id}_{\mathbf{S}^2}$ in J there exists a neighborhood N of X in F_n such that every $\pi(X')$ in N is topologically equivalent to $\pi(X)$ by an equivalence homeomorphism h lying in M'. (The requirement that the equivalence homeomorphism lie in a preassigned neighborhood of the identity is explicitly included in the definitions since it typically is not superfluous when the underlying manifold is noncompact.)

By the term *separatrix structure* of $\pi(X)$, $X \in \mathrm{F}_n$, we will mean the union of the separatrices of $\pi(X)$ as a flow on $\mathbf{S}^2$. The absence of finite singularities and polynomial nature of X allow this considerably simplified version of the usual definition, and preserve the truth of the theorems of Markus [**9**] and Neumann [**10**] in our restricted setting: if X, $Y \in \mathrm{F}_n$ and there is a homeomorphism of $\mathbf{S}^2$ carrying the separatrix structure of $\pi(X)$ to that of $\pi(Y)$, then $\pi(X)$ and $\pi(Y)$ are topologically equivalent. The detailed constructions in [**8, 14**] show that, when there are only finitely many singularities and separatrices, the singularities are locally structurally stable, and finite portions of separatrices vary continuously with parameters, which will always be the case in this paper, then the equivalence homeomorphism guaranteed by the theorems of Markus, Neumann, and Peixoto [**9, 10, 12**] can always be chosen to be arbitrarily close to $\mathrm{id}_{\mathbf{S}^2}$, provided the perturbation Y is sufficiently close to X.

Although every inseparable leaf in the foliation of $\mathbf{R}^2$ formed by the orbits of X must be a separatrix of $\pi(X)$, the converse is not true. Thus while $\mathbf{S}^2$-structural stability has the simplifying advantage that the underlying manifold is compact, it has the drawback that it requires that the equivalence homeomorphism respect separatrices of $\pi(X)$, which need not all be distinguished orbits in $\mathbf{R}^2$.

When we need an explicit expression for $X \in \mathrm{P}_n$, we will write it as

(2.1)
$$X(x,y) = (P(x,y), Q(x,y)) = \left(\sum_{0 \le i+j \le n} a_{i,j} x^i y^j, \sum_{0 \le i+j \le n} b_{i,j} x^i y^j \right).$$

Letting U_1 and U_2 be the hemispheres of $\mathbf{S}^2$ corresponding to $x > 0$ and $y > 0$, respectively, the coordinate expressions for $\pi(X)$ restricted to $\mathbf{E}$ are as follows:

In U_1:

(2.2)
$$\begin{aligned} F(s) = b_{n,0} &+ (b_{n-1,1} - a_{n,0})s + (b_{n-2,2} - a_{n-1,1})s^2 + \cdots \\ &+ (b_{0,n} - a_{1,n-1})s^n - a_{0,n}s^{n+1}. \end{aligned}$$

In U_2:

(2.3)
$$\begin{aligned} G(s) = a_{0,n} &+ (a_{1,n-1} - b_{0,n})s + (a_{2,n-2} - b_{1,n-1})s^2 + \cdots \\ &+ (a_{n,0} - b_{n-1,1})s^n - b_{n,0}s^{n+1}. \end{aligned}$$

Also, in U_1:

$$d\pi(X)(0,0) = \begin{pmatrix} b_{n-1,1} - a_{n,0} & b_{n-1,0} \\ 0 & -a_{n,0} \end{pmatrix} \tag{2.4}$$

and in U_2:

$$d\pi(X)(0,0) = \begin{pmatrix} a_{1,n-1} - b_{0,n} & a_{0,n-1} \\ 0 & -b_{0,n} \end{pmatrix}. \tag{2.5}$$

Letting V_1 and V_2 be the hemispheres of $\mathbf{S}^2$ corresponding to $x < 0$ and $y < 0$, respectively, the coordinate expressions are the same as those displayed, but multiplied by $(-1)^{n-1}$.

By the *tangential eigenvalue at a singularity* A of $\pi(X)$, we will mean the eigenvalue of $d\pi(X)(A)$ with eigendirection tangent to the $\pi(X)$-invariant set $\mathbf{E}$, i.e., the upper lefthand entry $b_{1,1} - a_{2,0}$ of the matrix (2.4) when A is located at $(0,0) \in U_1$. The *radial eigenvalue* is the remaining eigenvalue of $d\pi(X)(A)$.

Finally, we recall that if an isolated singularity A of a smooth vector field on a 2-manifold is *elementary*, i.e., at least one eigenvalue of the linear part at A is nonzero, and has a characteristic direction of approach, then A is a node, saddle, or saddle-node (see, for example , [**1**, Section 21, Theorem 65]), and the index (+1, −1, or 0, respectively) distinguishes the topological type of A.

3. Structural stability and genericity theorems on $\mathbf{S}^2$. As mentioned in the introduction, in [**7**] we dealt with the general case of degree n. Although we use a number of the results from that work, in order to reduce the overlap we only state without proof the key result concerning $\mathbf{S}^2$-structural stability in that paper.

3.1. Theorem. *Suppose $X \in \mathrm{F}_n$, the set of polynomial foliations of degree at most n.*

(1) *X is $\mathbf{S}^2$-structurally stable with respect to perturbation within F_n if it satisfies the following two conditions*:

(a) *The tangential eigenvalue at each singularity of $\pi(X)$ is nonzero.*

(b) *All separatrix connections of $\pi(X)$ lie in the equator $\mathbf{E}$ of the Poincaré sphere $\mathbf{S}^2$.*

(2) *X is $\mathbf{S}^2$-structurally stable with respect to perturbation within* F_n *only if it satisfies the following two conditions*:

(a) *$\pi(X)$ has a finite number of singularities, and at any singularity at which the tangential eigenvalue is zero the radial eigenvalue is also zero.*

(b) *All separatrix connections of $\pi(X)$ either lie in the equator $\mathbf{E}$ of the Poincaré sphere $\mathbf{S}^2$, or join two nonelementary singularities.*

The main obstruction to giving necessary and sufficient conditions for structural stability is determining whether an arbitrarily small perturbation of a given planar polynomial foliation is also a planar polynomial foliation. In particular, the only general perturbation of $X = (P, Q) \in \mathrm{F}_n$ which we know (without further conditions on P and Q themselves) to remain in F_n, and which we will use repeatedly, is X' given by

$$X' = \begin{pmatrix} P' \\ Q' \end{pmatrix} = M \begin{pmatrix} P \\ Q \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix} \tag{3.1}$$

for M close to the identity matrix. If a singularity is highly degenerate, it is difficult to be sure that the singularity can be made to bifurcate in $\mathbf{E}$ in such a way that no finite singularity is produced, particularly when there may be other degenerate singularities present. However, this problem can be solved when the degree is low.

It is apparent that a constant foliation is both $\mathbf{S}^2$- and $\mathbf{R}^2$-structurally stable with respect to perturbation within the set of constant foliations. Although the linear case is also quite simple, we will state and prove the relevant result, in part because the corresponding result for $\mathbf{R}^2$-structurally stability is not the same, Proposition 4.2.

3.2. Proposition. *An element X of* F_1 *is $\mathbf{S}^2$-structurally stable with respect to perturbation within* F_1 *if and only if it has exactly two singularity pairs at infinity*; *the set of $\mathbf{S}^2$-structurally stable elements of* F_1 *is dense in* F_1. *Moreover, any requirement that the equivalence homeomorphism be near* $\mathrm{id}_{\mathbf{S}^2}$ *is redundant.*

Proof. If $X \in \mathrm{F}_1$ is constant then by the fact that $\mathbf{E}$ is critical for $\pi(X)$ (since X is being regarded as an element of P_1) and by Lemma 5 in [**7**] (which states that there is an element Y of F_1 arbitrarily close to X but such that $\pi(Y)$ has only finitely many singularities) X is $\mathbf{S}^2$-structurally unstable, even if the equivalence homeomorphism h is allowed to be far from $\mathrm{id}_{\mathbf{S}^2}$. For nonconstant $X \in \mathrm{F}_1$, it is readily verified that, by a series of affine coordinate transformations and a time rescaling, the corresponding system of ordinary differential equations takes either the form

(I) $\dot{x} = y$, $\dot{y} = 1$, or the form

(II) $\dot{x} = 1$, $\dot{y} = y$.

In the former case there is a single singularity pair $\pm A$ at infinity, at which $\pi(X)$ has exactly one elliptic sector and exactly one hyperbolic sector, whose separatrices lie in $\mathbf{E}$; the local type is that of [**1**, Section 22, Figure 239], and globally $\pi(X)$ is like Figure 2 (22) in Appendix A. Both eigenvalues of $d\pi(X)(\pm A)$ are zero. In the latter case $\pm\pi(X)$ has an antipodal pair of hyperbolic sinks and an antipodal pair of saddle-nodes with nonzero tangential eigenvalues; globally $\pi(X)$ is like Figure 2 (23) in Appendix A.

If X is of Form (I), then the arbitrarily close vector field Y corresponding to $\dot{x} = \varepsilon x + y$, $\dot{y} = 1$ is also in F_1 but has two singularity pairs at infinity, so that $\pi(X)$ and $\pi(Y)$ are inequivalent. This shows that the systems of Form (I) are never $\mathbf{S}^2$-structurally stable, even when h may be far from $\mathrm{id}_{\mathbf{S}^2}$, and that the set of systems of Form (II) is dense.

Suppose X is of Form (II), with hyperbolic nodes at $\pm A \in \mathbf{E}$ and saddle-nodes at $\pm B \in \mathbf{E}$. Of course the hyperbolic nodes persist uniquely under perturbation of X. Since F (defined by (2.2)) has a zero of multiplicity one at each saddle-node, under perturbation in F_1 the singularities at $\pm B$ persist as a unique nearby singularity pair $\pm B'$. By (2.4) the tangential eigenvalue at $\pm B'$ is nonzero, so it is a node, saddle, or saddle-node, and since it still has index zero we conclude that it is still a saddle-node. Thus, every system $Y \in \mathrm{F}_1$ near X is of Form (II) so that $\pi(Y)$ has the same separatrix structure as $\pi(X)$, hence is topologically equivalent to $\pi(X)$, and is so by a homeomorphism lying in any pre-assigned neighborhood of $\mathrm{id}_{\mathbf{S}^2}$ in J. □

Note that Proposition 3.2 says precisely that condition (1) of Theorem 3.1 is necessary as well as sufficient for $\mathbf{S}^2$-structural stability when

$n = 1$. Theorem 1.1 says that the same is true for $n = 2$. First we state and prove a lemma and a genericity theorem that will be needed in the proof of Theorem 1.1.

3.1. Lemma. *Let $X \in \mathrm{F}_2$, and suppose $\pi(X)$ has an isolated singularity A at which the tangential eigenvalue is zero. Then for any neighborhood M of X in F_2 and any neighborhood U of A in $\mathbf{S}^2$ there exists $Y \in M$ such that $\pi(Y)$ has a singularity at A for which the tangential eigenvalue is nonzero, and a second, distinct singularity B in U.*

Proof. Suppose X and A are as described in the hypothesis. Then $\deg X > 0$. We give the proof for $\deg X = 2$. The proof for $\deg X = 1$ is similar; simply make perturbations analogous to those made in Case (i) below (Case (ii) does not occur). By a rotation of coordinates place A at $(0, 0) \in U_1$. Then, by (2.2) and (2.4), with $n = 2$, $b_{2,0} = b_{1,1} - a_{2,0} = 0$. We separate the proof into two cases.

Case (i). $(a_{2,0}, a_{1,1}) \neq (0, 0)$. When $a_{2,0} \neq 0$, we choose $M = \begin{pmatrix} 1 & 0 \\ 0 & 1+\varepsilon \end{pmatrix}$ in (3.1) to form Y so that $F_Y(s) = \varepsilon a_{2,0}s + \cdots$; when $a_{2,0} = 0$, we choose $M = \begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix}$ in (3.1) to form Y so that $F_Y(s) = \varepsilon a_{1,1}s + \cdots$. Either way the singularity at A persists and has nonzero tangential eigenvalue. If the first nonzero derivative at $s = 0$ of F_X is of even order, then the fact that $F'_Y(0) \neq 0$ gives existence of a second singularity in U for $|\varepsilon|$ sufficiently small. If it is of odd order, the fact that $F'_Y(0)$ can be chosen of either sign gives existence of a second singularity.

Case (ii). $(a_{2,0}, a_{1,1}) = (0, 0)$. Then by (2.4), $b_{1,1} = 0$ (in addition to $b_{2,0} = 0$). First suppose $b_{0,2} = 0$, so that $a_{0,2} \neq 0$, else X is not quadratic. As a preliminary perturbation, choose $M = \begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix}$ in (3.1). None of the quadratic coefficients change except $b_{0,2}$, which becomes $\varepsilon a_{0,2}$. Thus, $\pi(X)'$ still has a singularity $A = (0, 0) \in U_1$ at which the linear part still has the form $M = \begin{pmatrix} 0 & * \\ 0 & 1 \end{pmatrix}$ that it had originally, but now $b_{0,2} \neq 0$. Relabel the new system X. Thus, we have reduced the problem to a consideration of $X \in \mathrm{F}_2$ satisfying

$a_{2,0} = a_{1,1} = b_{2,0} = b_{1,1} = 0$ but $b_{0,2} \neq 0$. Either $a_{0,2} = 0$ already, or there is a second singularity B which by a shear transformation of coordinates in $\mathbf{R}^2$, namely, $\binom{u}{v} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \binom{x}{y}$, $b = -a_{0,2}/b_{0,2}$, we can move to $(0,0) \in U_2$, without moving A from $(0,0) \in U_1$. Thus, we may assume that all quadratic coefficients are zero except $b_{0,2}$, which by a time rescaling we can make equal to 1. If $(a_{1,0}, a_{0,1}) \neq (0,0)$ a translation of coordinates in $\mathbf{R}^2$ if necessary makes $a_{0,0} = 0$, so that we have reduced to

$$\begin{aligned} P(x,y) &= a_{0,0} \\ Q(x,y) &= b_{0,0} + b_{1,0}x + b_{0,1}y + y^2 \end{aligned} \tag{3.2a}$$

or to

$$\begin{aligned} P(x,y) &= a_{1,0}x + a_{0,1}y \\ Q(x,y) &= b_{0,0} + b_{1,0}x + b_{0,1}y + y^2. \end{aligned} \tag{3.2b}$$

In either case $F(s) = s^2$, so that $A = (0,0) \in U_1$ is sure to be locally unstable if we can make $F_Y'(0) \neq 0$, i.e., $\operatorname{sgn}(F_Y'(0))$ is no longer important to the argument. In case (3.2a), if $a_{0,0} \neq 0$ we may change $b_{1,1}$ to any $\varepsilon \neq 0$, while if $a_{0,0} = 0$ we may change $a_{2,0}$ to any $-\varepsilon \neq 0$, either way obtaining $F_Y(s) = \varepsilon s + s^2$. In case (3.2b), if $a_{0,1} = 0$, then there is always a choice of $\operatorname{sgn}(\varepsilon)$ so that changing $a_{2,0}$ to $-\varepsilon$ is admissible, again yielding $F_Y(s) = \varepsilon s + s^2$. If $a_{0,1} \neq 0$ in (3.2b), the choice $Y = (\widehat{P}, \widehat{Q}) = (P, Q + \varepsilon xy)$ once more yields $F_Y(s) = \varepsilon s + s^2$, so the proof is complete once we show that Y is indeed a foliation. To see that it is, we regard $\widehat{P}(x,y)$ and $\widehat{Q}(x,y)$ as polynomial functions of y, parametrized by $x \in \mathbf{R}$, and for any choice of ε compute, see for example [**3**],

$$\begin{aligned} \mathcal{R}_\varepsilon(x) &= \text{Resultant}\,(\widehat{P}[x], \widehat{Q}[x]) \\ &= a_{0,1}^2 b_{0,0} + a_{0,1}(a_{0,1}b_{1,0} - a_{1,0}b_{0,1})x + a_{1,0}(a_{1,0} - \varepsilon a_{0,1})x^2. \end{aligned}$$

It is apparent that for all choices of x and ε the resultant is a meaningful expression and $\deg \widehat{P}$ and $\deg \widehat{Q}$ are unchanged. By hypothesis either $\mathcal{R}_0(x)$ has no zeros at all, or at any of its zeros the corresponding common zero of $P[x]$ and $Q[x]$ is complex. Since the degree of $\mathcal{R}_\varepsilon(x)$ does not increase as ε is changed from 0, the same result continues to hold for $\widehat{P}[x]$ and $\widehat{Q}[x]$. □

3.4. Theorem. *Let* $\Gamma \subset \mathrm{F}_2$ *denote those* X *such that every singularity of* $\pi(X)$ *has a nonzero tangential eigenvalue, and there are no separatrix connections outside the equator* $\mathbf{E}$ *of the Poincaré sphere. Then* Γ *is a dense open subset of* F_2.

Proof. Lemma 5 in [**7**] states in this context that the set Γ'' consisting of those X in F_2 for which $\pi(X)$ has finitely many singularities is dense in F_2; it is clearly open. Note that if $X \in \mathrm{F}_2$ has $\deg X < 2$, then $\mathbf{E}$ is critical for $\pi(X)$ when X is treated as an element of F_2, so that $X \notin \Gamma''$.

Let $X \in \Gamma''$ be such that $\pi(X)$ has a singularity A at which the tangential eigenvalue is zero. Since F is a cubic polynomial function whose derivative at any singularity is the tangential eigenvalue there, $\pi(X)$ has exactly one such singularity A and additionally a single remote singularity, which is simple. (If $\pi(X)$ had a singularity at $(0,0) \in U_2$, then a shear transformation of coordinates in $\mathbf{R}^2$ exists to move it into U_1, making the statements about F true.) Placing A at $(0,0) \in U_1$ and perturbing X to the Y given by Lemma 3.3, F_Y has three simple zeros. Thus, the set Γ' of elements X in Γ'' such that every singularity of $\pi(X)$ has a nonzero tangential eigenvalue is dense in Γ'', hence in F_2; it is clearly open in Γ'', hence in F_2. Lemma 8 of [**7**] states that if $X \in \mathrm{F}_n$ is such that $d\pi(X)$ has a nonzero tangential eigenvalue at every singularity of $\pi(X)$, then X can be approximated arbitrarily closely by an element Y in F_n such that $\pi(Y)$ has no separatrix connections not contained in the equator $\mathbf{E}$ of $\mathbf{S}^2$. Thus, Γ is dense in Γ', hence in F_2. It is certainly open in Γ', hence is open in F_2, and the theorem follows. □

Proof of Theorem 1.1. Sufficiency follows from part (1) of Theorem 3.1, which was proved under the condition that the equivalence homeomorphism be close to $\mathrm{id}_{\mathbf{S}^2}$ in J.

We will prove the necessity of conditions (1) and (2) for $\mathbf{S}^2$-structural stability even when the equivalence homeomorphism between $\pi(X)$ and $\pi(Y)$ for nearby $Y \in \mathrm{F}_2$ is not required to be close to $\mathrm{id}_{\mathbf{S}^2}$; this will automatically prove the necessity of these conditions for $\mathbf{S}^2$-structural stability as we have defined it (restricted h). Hence, suppose $X \in \mathrm{F}_2$ is $\mathbf{S}^2$-structurally stable (h unrestricted) with respect to perturbation within F_2. Then, by the same reasoning as at the beginning of the proof of Proposition 3.2, $\deg X = 2$. Moreover, $\pi(X)$ possesses every topological property possessed by the Poincaré vector fields corresponding to the elements of the dense subset Γ described in Theorem 3.4. Thus, $\pi(X)$ has only finitely many singularities, at each of which the direction of flow in $\mathbf{E}$ changes, and there are no separatrix connections not wholly contained in $\mathbf{E}$. We must show that the tangential eigenvalue at every singularity of $\pi(X)$ is nonzero.

Suppose to the contrary that $\pi(X)$ has a singularity A at which the tangential eigenvalue is zero. The reversal of the flow of $\pi(X)|\mathbf{E}$ at A implies that the multiplicity of the zero at A of the function F of (2.2) is three, so that A and $-A$ are the only singularities of $\pi(X)$. But, applying Lemma 3.3, we obtain $Y \in F_2$ that is arbitrarily close to X but such that $\pi(Y)$ has at least one more singularity pair than $\pi(X)$. Thus, $\pi(Y)$ is not topologically equivalent to $\pi(X)$ by any homeomorphism of $\mathbf{S}^2$, whether close to $\mathrm{id}_{\mathbf{S}^2}$ or not, contradicting the $\mathbf{S}^2$-structural stability of X. The structurally stable elements of F_2 must therefore satisfy condition (1).

The density statement is now a direct corollary of Theorem 3.4. □

Proof of Corollary 1.2. From an examination of Figure 2, the following conclusions follow immediately from the geometry of certain phase portraits:

(i) if X is a system of Type 2, then $\pi(X)$ has a saddle connection, hence X is $\mathbf{S}^2$-structurally unstable;

(ii) if X is a system of Type 5, 6, 7, or 9, then $\pi(X)$ has a singularity with tangential eigenvalue zero, hence X is $\mathbf{S}^2$-structurally unstable;

(iii) if X is a system of Type k, $10 \leq k \leq 21$, then $\pi(X)$ has a nonelementary singularity, hence X is $\mathbf{S}^2$-structurally unstable;

(iv) if X is a system of Type 22 or 23, then X is linear, hence when X is regarded as an element of F_2, the set in which perturbation takes place, $\pi(X)$ has a singularity at every point of $\mathbf{E}$, hence X is $\mathbf{S}^2$-structurally unstable;

(v) if X is a system of Type 1, 3, or 4, then $\pi(X)$ has no separatrix connections, and since there are three singularity pairs, each zero of $F(s)$ and $G(s)$ in (2.2) and (2.3) is simple, hence the tangential eigenvalue at each singularity is nonzero, so X is $\mathbf{S}^2$-structurally stable.

Systems of Type 8 remain to be considered. If X is constant, then it is $\mathbf{S}^2$-structurally unstable for the same reason that linear systems in F_2 are. Otherwise, X is $\mathbf{S}^2$-structurally stable if and only if the tangential eigenvalue at the unique singularity is nonzero. □

Proof of Corollary 1.3. By (2.2) and (2.3), $-F(s) = As^3 + Bs^2 + Cs + D$ and $G(s) = Ds^3 + Cs^2 + Bs + A$, and Δ is the discriminant of F and G scaled by $1/A^4$ or $1/D^4$, respectively. It follows easily from the elementary theory of equations, for example, as in [**3**, subsection 8.4], that $\pi(X)$ has a unique, simple singularity pair on $\mathbf{E}$ if $\Delta < 0$, a multiple singularity pair if $\Delta = 0$, and three simple singularity pairs on $\mathbf{E}$ if $\Delta > 0$. Simplicity of the unique singularity pair when $\Delta < 0$ eliminates Types 11, 12, 13, and 22, and separates out from all Type 8 systems those that are stable. $\Delta > 0$ picks out systems of Type 1, 2, 3, or 4. Moreover, in the unstable situation, $\deg X < 2$ and Δ is also zero. Corollary 1.3 follows from these facts and Corollary 1.2. □

Using the tables in the appendix it is easy to identify all $\mathbf{S}^2$-structurally stable elements of F_2 in normal form, except for the question of the existence of a separatrix connection, which distinguishes Types 1 and 2.

Types 1 through 4 are distinguished by the existence of three singularity pairs in $\mathbf{E}$: forms I.1, I.4, and so on through form IX.1. As to Type 8, we first identify all normal forms having a single singularity pair in $\mathbf{E}$, then automatically eliminate the types other than Type 8 that have a single singularity pair in $\mathbf{E}$ (Types 11, 12, 13, and 22) when we impose the condition that the tangential eigenvalue be nonzero; for example, in Table A.1 we reduce from I.2, I.7, and I.13 to I.2 and I.7 when we impose $n \neq 1$, the condition for a nonzero tangential eigenvalue at $(0, 0) \in U_2$ for normal form family I.

Thus, the $\mathbf{S}^2$-structurally stable families are: Types 1, 3, and 4: I.1, I.4, I.6, I.9; II.1, II.4, II.6, II.9; (III–IX).1 (and the nonalgebraic condition on the coefficients that there be no separatrix connection not lying in $\mathbf{E}$).

Type 8: I.2, I.7; II.2, II.7; (III–IX).2; X.1.

4. Structural stability and genericity theorems on $\mathbf{R}^2$. As we pointed out above, there is no analogue of Theorem 3.1 (arbitrary degree) for $\mathbf{R}^2$-stability. The one result in this direction, proved in [**7**], is the following.

TABLE 1. $\mathbf{R}^2$-unstable elements of F_2 and their perturbations, $\deg X < 2$.

Normal Form of X	Type of X	Perturbation Y	Type of Y
$\dot{x} = 1$ $\dot{y} = 0$	8	$\dot{x} = 1$ $\dot{y} = \varepsilon x^2$	8
$\dot{x} = y$ $\dot{y} = 1$	22	$\dot{x} = y$ $\dot{y} = 1 + \varepsilon xy$	10
$\dot{x} = 1$ $\dot{y} = y$	23	$\dot{x} = 1$ $\dot{y} = y + \varepsilon x^2$	13

4.1. Theorem. *Suppose $X \in \mathrm{F}_n$ has no inseparable leaves and that there exists a neighborhood U of X in F_n such that if Y is in U, then Y has no inseparable leaves. Then X is $\mathbf{R}^2$-structurally stable in F_n.*

This result allows us to quickly dispose of linear foliations. Note the contrast with the situation for $\mathbf{S}^2$-structural stability, Proposition 3.2.

4.2. Proposition. *Every element of F_1 is $\mathbf{R}^2$-structurally stable with respect to perturbation within F_1, whether the equivalence homeomorphism is restricted to be near $\mathrm{id}_{\mathbf{R}^2}$ or not.*

Proof. No element of F_1 has an inseparable leaf, so $\mathbf{R}^2$-structural stability follows immediately from Theorem 4.1. □

The rest of this section is devoted to proving Theorem 1.4 using the results of [**4**]. Essential to our proof is the fact that [**4**] shows not only what phase portraits are possible, but how they are realized (for example, it is implicit in [**4**] that Type 9 occurs if and only if one singularity pair at infinity is hyperbolic and the other has nonzero linear part with both eigenvalues zero). Remember that we will refer to an element X of F_2 as being "of Type k" if $\pi(\pm X)$ is homeomorphic to the phase portrait in Figure 2 (k).

Proof of Theorem 1.4. The first column of Table 1 lists the normal form for each system X described by part (1) of the theorem. In each case the perturbation Y listed in the third column is equivalent to X, in $\mathbf{R}^2$, but appears as an element of Table 2, hence is unstable. (The transformation $x = x_1$, $y = \varepsilon y_1$ places the system Y in line one (respectively, three) in the form of the X in line four (respectively, eight) of Table 2.) If X is as described in part (5) of the theorem, that is, if it is of Type 10, 12, 13, 14, or 19, then it has one specific normal form (from Lemma A.2), and each system with the corresponding normal form is of that same type. Table 2 lists the normal form for each of these five types, together with the nearby element Y of F_2 that either has distinct type, or that appears elsewhere in the table along with a small perturbation of distinct type. Thus, these five types need not be considered further. By contrast, Types 6 and 8 each occur for more than one normal form. The first five lines of Table 2 show in order the normal form that corresponds to each of the unstable situations listed in points (2), (3), (4a), (4b), and (4c) of the theorem. Verification that the perturbations in either table actually lie in F_2 and have the types indicated are left to the reader. Identification of type is often facilitated by following the scheme for reduction to normal form outlined in the proof of Lemma A.2 in Appendix A. In every case the inequivalence arises from a change in the number of inseparable leaves.

Because for every system X listed in Table 1 or Table 2 the arbitrarily close perturbation Y is either $\mathbf{R}^2$-structurally stable, or is approached arbitrarily closely by an $\mathbf{R}^2$-structurally stable system Y', the $\mathbf{R}^2$-structurally stable systems are dense in F_2. Note further that, because every unstable system is approached arbitrarily closely by a system with a different number of inseparable leaves, and the number of inseparable leaves is a topological invariant, the systems in Tables 1 and 2 are $\mathbf{R}^2$-structurally unstable even if the equivalence homeomorphism is not required to lie in a pre-assigned neighborhood of $\mathrm{id}_{\mathbf{R}^2}$.

We now show that all remaining elements of F_2 are $\mathbf{R}^2$-structurally stable. To do so, we first note that if for some neighborhood U of $X \in \mathrm{F}_2$, every element Y of U has the same number of inseparable leaves as does X, then X is $\mathbf{R}^2$-structurally stable. For given such

X and U, if X has no inseparable leaves, then it is $\mathbf{R}^2$-structurally stable by Theorem 4.1. If X has exactly two inseparable leaves, then it and any $Y \in U$ are of Type 3 or 18. In this case we consider the vector fields $\sigma(X)$ and $\sigma(Y)$ induced on $\mathbf{S}^2$ minus one point by *stereographic* projection, as outlined, say, in [**13**]; there is a topological equivalence between $\sigma(X)$ and $\sigma(Y)$ which for Y sufficiently near X can be chosen so as to induce a near-identity equivalence between X and Y. Finally, if the phase portrait of X contains three inseparable leaves (the maximum possible), then it and $Y \in U$ are both Type 19. Thus, for Y sufficiently close to X, the equivalence homeomorphism that therefore exists between $\pi(X)$ and $\pi(Y)$ on $\mathbf{S}^2$ can be chosen sufficiently near $\mathrm{id}_{\mathbf{S}^2}$ so as to induce a near-identity equivalence between X and Y on $\mathbf{R}^2$. All systems not appearing in Tables 1 and 2 will thus be shown to be $\mathbf{R}^2$-structurally stable when the equivalence homeomorphism h is required to lie in a pre-assigned neighborhood of $\mathrm{id}_{\mathbf{R}^2}$, hence they are $\mathbf{R}^2$-structurally stable when h is not so restricted. This fact, together with the last sentence in the previous paragraph, shows that any requirement that the equivalence homeomorphism be required to lie in a pre-assigned neighborhood of $\mathrm{id}_{\mathbf{R}^2}$ is redundant.

Henceforth we will let X denote the original element of F_2 and X' a nearby element of F_2. By the discussion in the previous paragraph, we need only show that if X' is sufficiently near X in F_2, then X' has the same number of inseparable leaves as X.

With [**4**] we will say that an isolated singularity of $\pi(X)$ is of type:

(i) E if the singularity is hyperbolic;

(ii) S if exactly one eigenvalue of the linear part is zero;

(iii) H if both eigenvalues are zero but the linear part is not zero; and

(iv) T if the linear part is zero.

An expression like (E, E, S) will mean that $\pi(X)$ has three singularity pairs, of the types indicated.

Types 1–4. $\pi(X)$ has three singularity pairs, at each of which the tangential eigenvalue is nonzero. Lemma 7 of [**7**] asserts that in such a situation, under sufficiently small perturbation of X within F_2, each singularity persists uniquely as a singularity of the same topological type. Type 2 can change to Type 1, while in the remaining cases the separatrix structure on $\mathbf{S}^2$ is fixed under sufficiently small perturbations of X within F_2; the number of inseparable leaves is unchanged under small perturbation (2 for Type 2, 0 for Types 1, 3, and 4).

TABLE 2. $\mathbf{R}^2$-unstable elements of F_2 and their perturbations, $\deg X = 2$.

Normal Form of X	Type of X	Perturbation Y	Type of Y	Note
$\dot{x} = x^2$ $\dot{y} = d + ax - xy, \quad d \neq 0$	6	$\dot{x} = x^2$ $\dot{y} = d + ax - (1+\varepsilon)xy$ $\varepsilon > 0$	19	
$\dot{x} = y$ $\dot{y} = d + ax + by + lx^2$ $a^2 - 4dl < 0$	8	$\dot{x} = y$ $\dot{y} = d + ax + by + lx^2 - \varepsilon^2 ly^2$	3	
$\dot{x} = x$ $\dot{y} = 1 + x^2$	8	$\dot{x} = \varepsilon + x + \varepsilon x^2 + \varepsilon^2 xy$ $\dot{y} = 1 + x^2 + \varepsilon xy$	3	
$\dot{x} = 1$ $\dot{y} = x^2$	8	$\dot{x} = 1 + (9/4)\varepsilon y$ $\dot{y} = -2\varepsilon y + x^2 + \varepsilon xy - 2\varepsilon^2 y^2$	3	
$\dot{x} = 0$ $\dot{y} = d + ax + lx^2$ $a^2 - 4dl < 0$	8	$\dot{x} = \varepsilon(d/l) + \varepsilon x^2 + (\varepsilon^2/l)xy$ $\dot{y} = d + ax + lx^2 + \varepsilon xy$	3	
$\dot{x} = -lx + y$, $\dot{y} = d + ax$ $+by + mxy$, $lm \neq 0$, $(a+bl)^2 - 4dlm < 0$ or $m \neq 0$, $l = a = 0$, $d \neq 0$	10	$\dot{x} = -lx + y$ $\dot{y} = d + ax + by + mxy + \varepsilon y^2$ $\mathrm{sgn}(\varepsilon) = -\mathrm{sgn}(ml)$ if $l \neq 0$ $\mathrm{sgn}(\varepsilon) = -\mathrm{sgn}(d)$ if $l = 0$	3	(2)
$\dot{x} = x$ $\dot{y} = -1 + x^2$	12	$\dot{x} = x$ $\dot{y} = -1 + x^2 + \varepsilon xy$	14	(1)
$\dot{x} = 1$ $\dot{y} = y + x^2$	13	$\dot{x} = 1 + \eta y$ $\dot{y} = y + x^2 + \varepsilon xy - (5/16)\varepsilon^2 y^2$ $\eta < -(9/16)\varepsilon^2$, $\varepsilon > 0$ $4\eta + 5\varepsilon - 5\varepsilon^2 < 0$	3	
$\dot{x} = x$ $\dot{y} = 1 + xy$	14	$\dot{x} = x + \varepsilon y + \eta xy$ $\dot{y} = 1 + xy$ $\eta < 0$, $\varepsilon < 0$ $\eta^2 + 4\varepsilon < 0$	3	
$\dot{x} = 1 + xy$ $\dot{y} = (1/m)y^2 \ \ m < -1$	19	$\dot{x} = 1 + xy$ $\dot{y} = \varepsilon xy + (1/m)y^2 \ \ \varepsilon > 0$	2	

(1) Y is $\mathbf{R}^2$-equivalent to X but is $\mathbf{R}^2$-unstable.

(2) Normal form of X derived from the normal form of [**4**] $\dot{x} = y$, $\dot{y} = d + ax + by + lx^2 + mxy$ by the shear transformation $x_1 = x$, $y_1 = (l/m)x + y$.

Type 5. Since there are three singularity pairs on $\mathbf{E}$, counting multiplicity, the nodes are type E or S, hence persist uniquely as nodes of the same stability under perturbation. The saddle-nodes are of type S, since they cannot be of type E and according to [**4**] configurations (E,T), (E,H), (S,T), and (S,H) yield types other than Type 5. If the saddle-nodes disappear completely, then the new system is Type 8. If the saddle-nodes split into two singularity pairs, the new singularities are saddles and nodes, so X' has Type 1. If the saddle-nodes persist uniquely, since only one eigenvalue was zero originally, the same is true of the new singularities, so by an index argument they remain saddle-nodes; continuous dependence of the separatrices on parameters implies that X' is not of Type 6 or 7, hence is of Type 5. Thus, in all cases any X' sufficiently near X in F_2 has, like X, no inseparable leaves.

Type 6. By [**4**] the configuration of singularity pairs is (E,S), (S,S), or (E,T). The first two situations correspond to structurally stable X, exactly as for Type 5; the type can change to Types 1, 5, 7, or 8 under small perturbation, none with any inseparable leaves. Suppose then that the original situation was (E,T). When the T-singularity pair disappears or splits, the same analysis applies, but when it persists uniquely, the new nearby singularities need not be saddle-nodes. (Indeed, this is the mechanism for instability in the first line of Table 2.) *A priori* we know only that $\pi(X)p$ has a sink/source pair plus one additional pair of singularities, hence is of Type 5, 6, 7, 9, 15, 18, or 19. In the first five situations, if they occur, X' still has no inseparable leaves. Type 18 is in fact impossible: erect at the saddle-node of the original Type 6 system X transverse sections Σ_1 and Σ_2 to each separatrix that lies outside $\mathbf{E}$, and a transverse section Σ to the separatrix that lies within $\mathbf{E}$. For any point $A \in \Sigma_i$, $i = 1,2$, such that $o^+(A)$ or $o^-(A)$ intersects Σ before leaving a small neighborhood of the singularity, under sufficiently small perturbation $o^+_{X'}(A)$ or $o^-_{X'}(A)$ has the same behavior, so the new singularity has a hyperbolic sector on each side of $\mathbf{E}$, ruling out Type 18.

The normal form for a Type 6 element of F_2 with an (E,T) singularity configuration is [**4**, pages 777 ff.]

$$\dot{x} = x^2, \qquad \dot{y} = d + ax + mxy$$
$$d \neq 0, \quad -1 < m < 1.$$

(Type 6 also occurs when $m = -1$ but is the unstable system in line 1 of Table 2.) To show that no element of F_2 near X is of Type 19, we start with this normal form, perturb, and show that in reduction to normal form to determine the type of X' (following the sequence of coordinate changes in the proof of Lemma A.2) we cannot arrive at Type 19. There is no loss of generality in maintaining the location of the singularity at $(0,0) \in U_2$, which means maintaining $a_{0,2} = 0$. The perturbation is of Type 19 only if it has configuration (E,T), hence only if the singularity at $(0,0) \in U_2$ is still of Type T, hence by (2.5) only if for X' we have

$$a_{0,1} = a_{1,1} = b_{0,2} = 0, \tag{4.1}$$

so that X' is

$$\begin{aligned}\dot{x} &= \alpha_{0,0} + \alpha_{1,0}x + (1+\alpha_{2,0})x^2\\ \dot{y} &= (d+\beta_{0,0}) + (a+\beta_{1,0})x + \beta_{0,1}y + \beta_{2,0}x^2 + (m+\beta_{1,1})xy.\end{aligned}$$

Now apply the transformation that places the foliation satisfying these conditions into normal form. In the proof of Lemma A.2 we have $m_1 = b_1 = 0$ and $l_1 \neq 0$, so we form $k := a_1^2 - 4d_1l_1 = \alpha_{1,0}^2 - 4(1+\alpha_{2,0})\alpha_{0,0}$. If k is nonzero, we will obtain normal form V or VI, neither of which yields Type 19, hence we must also choose $\alpha_{0,0} = \alpha_{1,0}^2/[4(1+\alpha_{0,0})]$. Applying the final change of coordinates that the yields normal form VII of Lemma A.2, the coefficient of x_1y_1 in $\dot{y}_1$ is $\overline{m} = (m+\beta_{1,1})/(1+\alpha_{2,0})$. Since $-1 < m < 1$ and $a_{2,0}$ and $b_{1,1}$ are close to zero, $-1 < \overline{m} < 1$, so X' is still of Type 6, not Type 19.

Type 7. Under small perturbation within F_2 the source/sink pair persist uniquely as a source/sink pair, while the saddle-nodes either disappear, split, or persist uniquely as nearby saddle-nodes. As in the case of Type 6, just discussed, the only difficulty is in ruling out the possibility that X' be Type 19. We follow exactly the same procedure as in Type 6: in a perturbation of X we may again choose $\alpha_{0,2} = 0$, and must choose $a_{0,1}$, $a_{1,1}$, and $b_{0,2}$ satisfying (4.1), so that any nearby X' that can be of Type 19 must have the form

$$\begin{aligned}\dot{x} &= (-1+\alpha_{0,0}) + \alpha_{1,0}x + (1+\alpha_{2,0})x^2\\ \dot{y} &= (d+\beta_{0,0}) + (a+\beta_{1,0})x + \beta_{0,1}y + (l+\beta_{2,0})x^2 + \beta_{1,1}xy.\end{aligned}$$

We must again compute k and find that it has value $k := a_1^2 - 4d_1l_1 = \alpha_{1,0}^2 - 4(1+\alpha_{2,0})(-1+\alpha_{0,0}) \approx 4 > 0$, so the normal form is form V, hence X' is not of Type 19.

Type 8. The single singularity pair can be of type (E), (S), (H), or (T). Proof of stability in the first two cases is straightforward, although normal hyperbolicity must be used for situation (S) when the tangential eigenvalue is the one that is zero. Situation (H) is always unstable: this is the second line in Table 2. Situation (T) occurs for the standard forms (VI.5), (VII.5), (VIII.6), (IX.6), and (X.3) in the tables in Appendix A. The last three are unstable and correspond to lines 3–5 of Table 2. To establish stability in the remaining two cases, we make a general perturbation of the standard form (without loss of generality maintaining $\alpha_{0,2} = 0$), and follow the series of coordinate changes described in the proof of Lemma A.2 to reduce the perturbation to normal form. Tedious but direct computation on a symbolic manipulator such as *Maple* shows that one can never obtain Types 3, 18, or 19 (the types having inseparable leaves).

Type 9. The configuration of singularity pairs is (E, H), so the source/sink pair persist uniquely as a nearby source/sink pair under small perturbation. If the saddle-node pair disappear, then X' is Type 8. If the saddle-nodes split, then each new singularity has a nonzero tangential eigenvalue, hence is a node, saddle, or a saddle-node. The index implies that we have either

(i) a pair of nodes and a pair of saddles, so that X' is Type 1 or (less likely) Type 2, or

(ii) two pairs of saddle-nodes, so that X' is Type 4; in any case, X' is equivalent to X. If the saddle-node pair persist uniquely, then each singularity is of index zero and has nonzero linear part; it is Type S or H. If Type S, then X' has configuration (E, S) with a source/sink pair and a saddle-node pair, hence is of Type 5, 6, or 7, hence equivalent to X. If Type H, then Theorem 11 of [**4**] indicates that it is of Type 9, again equivalent to X.

Type 11. When X is placed in standard form the singularity, located at $(0, 0) \in U_2$, is of Type (T); quadrants II and III are hyperbolic sectors, quadrants I and IV are elliptic sectors, and the separatrices lie in the coordinate axes. For X' sufficiently near X, $\pi(X')$ has one, two or three singularities near $(0,0)$ in U_2. By erecting transverse sections to the separatrices as was done in the discussion of X of Type 6, we find that the left-most singularity has a pair of hyperbolic sectors that lie on opposite sides of $\mathbf{E}$. This implies immediately that if $\pi(X')$ has exactly one singularity near $(0,0) \in U_2$, then X' is of Type 11, since no other phase portrait in the catalogue in Figure 2 in Appendix A has a unique singularity pair with hyperbolic sectors so situated.

If $\pi(X')$ has three singularities near $(0,0)$, each is elementary, and by an index argument and the discussion immediately above $\pi(X')$ has a saddle and two nodes in U_2, hence X' is of Type 1 or (less likely) Type 2.

If $\pi(X')$ has exactly two singularities near $(0,0)$, then $G(s)$ in (2.3) changes sign at exactly one of them. If the sign change occurs at the left singularity, which is thus elementary and has the hyperbolic sectors on each side of $\mathbf{E}$, then that singularity must be a saddle. The only compatible phase portraits in the catalogue are Types 16 and 17, which have no inseparable leaves. If the sign change occurs at the right singularity, then the phase portrait of $\pi(X')$

(i) has two singularity pairs,

(ii) has one singularity pair at which there are two or more hyperbolic sectors, at least one on each side of $\mathbf{E}$, and

(iii) has a node, saddle, or saddle-node at the remaining singularity pair (which are elementary). Of the phase portraits meeting these three conditions (Types 5, 6, 7, 9, 15, and 19), all but Type 19 have no inseparable leaves. To show that X' cannot be of Type 19, we begin with the standard form for X of Type 11,

$$\begin{aligned}\dot{x} &= x^2\\ \dot{y} &= -1 + x^2 + xy\end{aligned}$$

and make a general perturbation, without loss of generality maintaining $\alpha_{0,2} = 0$. As in previous cases, X' can be of Type 19 only if we require conditions (4.1), so that X' has the form

$$\begin{aligned}\dot{x} &= \alpha_{0,0} + \alpha_{1,0}x + (1 + \alpha_{2,0})x^2\\ \dot{y} &= (-1 + \beta_{0,0}) + \beta_{1,0}x + \beta_{0,1}y + (1 + \beta_{2,0})x^2 + (1 + \beta_{1,1})xy.\end{aligned}$$

Since in the terminology of the proof of Lemma A.2 $m_1 = b_1 = 0$ but $l_1 \neq 0$ we compute $k = a_1^2 - 4d_1l_1 = \alpha_{1,0}^2 - 4\alpha_{0,0}(1+\alpha_{2,0})$. X' is of Type 19 only if $k = 0$. With this requirement we make the coordinate change $x_1 = x + \alpha_{1,0}/(2(1+\alpha_{2,0}))$, $y_1 = y$, $t_1 = t$ and find that the coefficient m of x_1y_1 in $\dot{y}_1$ is $m = 1 + \beta_{1,1}$, which is not strictly less than -1, which is the condition that X' be of Type 19, cf. [**4**, pages 777–778].

Type 15. The configuration of singularity pairs is (E,T) or (S,T), so the source/sink pair persist uniquely as a nearby source/sink pair under small perturbation. If, under small perturbation the nonelementary singularity disappears, then X' is of Type 8, hence equivalent to X. If it splits into two nearby singularities, then they are either a pair of saddle-nodes or a saddle and a node; in either case it is easy to see that X' has no inseparable leaves. If the nonelementary singularity persists uniquely, then as in previous cases the only possible inequivalent phase portrait is Type 19. As before, we begin with the normal form for X, make an arbitrary small perturbation, and show that Type 19 cannot occur. X can be reduced to three standard forms in the tables in Appendix A: VI.4, which reduces to the case $m < 1$; IX.5, which reduces, cf. [**4**], to $\dot{x} = 1$, $\dot{y} = d' \mp xy$; and X.2. The treatment of the first two situations is exactly as in the corresponding discussion of Type 11 immediately above. The X.2 normal form is

$$\begin{aligned}\dot{x} &= 0\\ \dot{y} &= d + ax + by + lx^2 + mxy + ny^2\end{aligned}$$

with several side conditions, including $n \neq 0$. The configuration of singularity pairs is (E,T); the hyperbolic singularity is located at $(0,0)$ and the degenerate singularity at $((-m/2n),0)$ in U_2. Thus, as a preliminary step, we interchange x and y, then apply the shear transformation $x_1 = 2nx + my$, $y_1 = y$ to move the degenerate singularity to $(0,0)$ in U_2. Reverting to the names x and y for the coordinates, X is now

$$\begin{aligned}\dot{x} &= 2dn + bx + \frac{1}{2}x^2\\ \dot{y} &= 0.\end{aligned}$$

We can now proceed exactly as in the previous cases, and as before find that no nearby system is of Type 19.

Type 16. The configuration of singularity pairs is (E, T) or (S, T), so the saddle pair persist uniquely as a nearby saddle pair under small perturbation. The nonelementary singularity pair have index $+2$, hence either split or persist as a unique nearby pair. In the former case, the nearby singularities are elementary and have indices summing to $+2$, hence are each nodes; X' is of Type 1 or Type 2. In the latter case, the only phase portraits having a pair of saddles and a pair of index $+2$ singularities are Types 16 and 17.

Type 17. The configuration of singularity pairs is (E, T) or (S, T), and the same discussion as given for Type 16 applies, with the only difference that when the degenerate singularity persists uniquely, no nearby system can be of Type 16, since the separatrices in the finite part of the plane would have to make a discontinuous jump to connect.

Type 18. The configuration of singularity pairs is (E, T). Under small perturbation the hyperbolic source/sink pair persists uniquely as a source/sink pair. Near the degenerate singularity, erect a transverse section Σ to the stable separatrix that lies in the finite part of the plane, and other transverse sections to the remaining separatrices. Near the degenerate singularity erect a transverse section Σ to the stable separatrix that lies in the finite part of the plane, transverse sections Σ_1 and Σ_2 at the unstable separatrices, and transverse section Σ_3 at the remaining stable separatrix. Let A denote the unique point of intersection of Σ with the local stable set, and let K denote a closed proper subinterval of Σ containing A. For X' close enough to X, the sets $K_i := \{B \in K \mid o^+_{X'}(B) \cap \Sigma \neq \varnothing\}$, $i = 1, 2$, are relatively open in K, hence there is a point A' in $K \setminus (K_1 \cup K_2)$. Since $o^+_{X'}(A') \cap \Sigma_3 = \varnothing$, the degenerate singularity does not disappear under perturbation. If it persists uniquely, then a study of the flow from section to section shows that it has three hyperbolic sectors on the same side of $\mathbf{E}$, which implies that X' is also Type 18.

If the degenerate singularity splits into two singularities, they are elementary, and each has a hyperbolic sector. Since their indices sum to 0, they are both saddle-nodes, and it is clear that X' is Type 3.

Type 20. By Lemma A.3, X can be placed in the form

$$\begin{aligned}\dot{x} &= 1 + xy\\ \dot{y} &= y^2.\end{aligned} \tag{4.2}$$

Let X' be a small perturbation of X. If $\mathbf{E}$ is critical for $\pi(X')$, then X' is equivalent to X. Otherwise, there is a suitable rotation of coordinates R_θ in $\mathbf{R}^2$ (not necessarily small) which places any singularity of $\pi(R_\theta X')$ at $(0,0) \in U_2$. X is unstable only if there is a system of Type 3, 18, or 19 lying in any neighborhood of X, which holds if and only if for every θ there is a system $R_\theta X'$ of like type lying in any neighborhood of $R_\theta X$. To investigate the situation, conceptually we perturb X to X' and then rotate to place the singularity of $\pi(R_\theta X')$ at $(0,0) \in U_2$, but it is equivalent, and simpler computationally, to rotate X with arbitrary θ to $R_\theta X$, then make small changes in the coefficients of $R_\theta X$ to create $R_\theta X'$ (specifying X' only implicitly). From (4.2), we obtain

$$\begin{aligned}\dot{x} &= C - Sx^2 + Cxy\\ \dot{y} &= S - Sxy + Cy^2\end{aligned} \tag{4.3}$$

for $R_\theta X$, where $S = \sin(\theta)$ and $C = \cos(\theta)$. A general perturbation of $R_\theta X$ in $\mathbf{R}^{12}$ is given by

(4.4)

$$\begin{aligned}\dot{x} &= (C + \alpha_{0,0}) + \alpha_{1,0}x + \alpha_{0,1}y + (-S + \alpha_{2,0})x^2 + (C + \alpha_{1,1})xy\\ \dot{y} &= (S + \beta_{0,0}) + \beta_{1,0}x + \beta_{0,1}y + \beta_{2,0}x^2 + (-S + \beta_{1,1})xy + (C + \beta_{0,2})y^2\end{aligned}$$

where S and C have been chosen so that $\alpha_{0,2} = 0$.

To show that the system $R_\theta X'$ given by (4.4) cannot be of Type 3, we recall that any system of Type 3 has three distinct pairs of infinite singular points; one of them is a hyperbolic node, and the other two are saddle-nodes, hence have radial eigenvalue zero. We may choose θ so that the hyperbolic node is located at $(0,0) \in U_2$.

When $\beta_{2,0} \neq 0$, the saddle-nodes, call them s_+ and s_-, have coordinates in U_2 of the form $(s,t) = (u+v, 0)$ and $(s,t) = (u-v, 0)$. The radial eigenvalues are

$$\lambda_\pm = (-S + \alpha_{2,0})(u \pm v) + (C + \alpha_{1,1}).$$

Since $v \neq 0$, else s_+ and s_- are not distinct, λ_+ and λ_- are both zero only if $S = \alpha_{2,0}$ and $C = -\alpha_{1,1}$, which is impossible since $|\alpha_{2,0}| + |\alpha_{1,1}| \ll 1$.

When $\beta_{2,0} = 0$, then one of the saddle-nodes lies at $(0,0) \in U_1$, and similar considerations lead to the same contradiction. Thus, there is no foliation X' of Type 3 near X.

To show that the system $R_\theta X'$ of (4.4) cannot be of Type 18 or 19, we recall that such system has a T-singularity, which by proper choice of θ can be placed at $(0,0) \in U_2$. The linear part being identically zero there gives the following conditions:

$$\begin{aligned} \alpha_{1,1} &= \beta_{0,2} = -C \\ \alpha_{0,1} &= 0. \end{aligned}$$

Notice that the first condition means that $\theta \approx \pi/2$, hence $S \approx 1$ (one can check that the case $S \approx -1$ can be reduced to this case). System (4.4) becomes

(4.5)
$$\begin{aligned} \dot{x} &= (C + \alpha_{0,0}) + \alpha_{1,0}x + (-S + \alpha_{2,0})x^2, \\ \dot{y} &= (S + \beta_{0,0}) + \beta_{1,0}x + \beta_{0,1}y + \beta_{2,0}x^2 + (-S + \beta_{1,1})xy, \end{aligned}$$

which yields the coordinate expression for $\pi(R_\theta X')$ in U_2 of

(4.6)
$$\begin{aligned} \dot{s} &= (\alpha_{2,0} - \beta_{1,1})s^2 + (\alpha_{1,0} - \beta_{0,1})st + (C + \alpha_{0,0})t^2 \\ &\quad - \beta_{2,0}s^3 - \beta_{1,0}s^2t - (S + \beta_{0,0})st^2 \\ \dot{t} &= -(-S + \beta_{1,1})st - \beta_{0,1}t^2 - \beta_{2,0}s^2t - \beta_{1,0}st^2 - (S + \beta_{0,0})t^3. \end{aligned}$$

We may assume that $\beta_{2,0}$ is nonzero. For when $\beta_{2,0} = 0$, if (i) $\alpha_{2,0} - \beta_{1,1} = 0$ then the line at infinity is critical for $\pi(R_\theta X')$, so X' has the same topological type as X, while if (ii) $\alpha_{2,0} - \beta_{1,1} \neq 0$, then the second singularity of $\pi(R_\theta X')$ lies at $(0,0) \in U_1$, and a similar argument to the case we consider applies. Assuming that $\beta_{2,0} \neq 0$, we must have $\alpha_{2,0} - \beta_{1,1} \neq 0$ as well, else $\pi(R_\theta X')$ has only one singularity pair at infinity, and cannot be Type 18 or 19.

Directions of approach to the singularity of $\pi(R_\theta X')$ at $(0,0) \in U_2$ are given by the zeros of the *characteristic polynomial*

$$H(s,t) = t\left[(-S + \alpha_{2,0})s^2 + \alpha_{1,0}st + (C + \alpha_{0,0})t^2\right],$$

where $-S+\alpha_{2,0} < 0$ since $S \approx 1$. We distinguish three cases, depending on the zeros of the polynomial H.

If the equation $H = 0$ has no other solutions than $t = 0$ there will be only one *simple* characteristic direction to the origin of U_2 and consequently, by going back through the blow-up, its local phase portrait cannot be as in Type 18 or 19 ($t = 0$ cannot be a double or triple characteristic direction because $-S + \alpha_{2,0} \neq 0$). We note that a *simple* characteristic direction implies that after the corresponding blow-up the singular point is at worst an S-singularity.

Now we assume that $H = 0$ has two distinct solutions (each different from $t = 0$), i.e., that the origin of U_2 has three simple characteristic directions (one of them is $t = 0$). In this case, since $H(s,t)/t$ is the homogenization of $\dot{x}$ in (4.5), $\dot{x}$ has two real zeros denoted by $\lambda_\pm$. If we now compute $\dot{y}$ along the lines $x = \lambda_\pm$, the condition that $R_\theta X'$ have no finite singular points is that

$$(S + \beta_{0,0}) + \beta_{1,0}\lambda_\pm + \beta_{2,0}\lambda_\pm^2 + (\beta_{0,1} + \lambda_\pm(-S + \beta_{1,1}))y \neq 0$$

for any y. If either of λ_+ and λ_- is zero, then $\beta_{0,1} = 0$, which in turn forces the other of λ_+ and λ_- to be zero (since $S \approx 1$), contradicting $\lambda_+ \neq \lambda_-$. Thus $\lambda_+\lambda_-(-S + \beta_{1,1}) \neq 0$, which implies that both $\beta_{0,1} = \lambda_+ + (-S + \beta_{1,1})$ and $\beta_{0,1} = \lambda_- + (-S + \beta_{1,1})$, which again is impossible if $\lambda_+ \neq \lambda_-$.

The remaining case is that $H = 0$ have a double solution away from $t = 0$, that is, that the origin of U_2 have one simple characteristic direction $t = 0$ and one double characteristic direction. We may assume that $\alpha_{1,0}(c + \alpha_{0,0}) \neq 0$. If not, the double characteristic direction is $s = 0$ and we have to introduce a preliminary linear transformation (as is done below) in order to control both characteristic directions $s = 0$ and $t = 0$ at the same time, but the end result is identical.

Since $H = 0$ has a solution different from $t = 0$, we know that in (4.5) $\dot{x}$ has one (double) zero along $x = -\alpha_{1,0}/(2(-S + \alpha_{2,0}))$. Therefore, substituting this expression into $\dot{y}$ in (4.5) we get two further conditions in order for $R_\theta X'$ to be a foliation:

$$\begin{aligned} &\beta_{0,1} = \frac{\alpha_{1,0}(-S + \beta_{1,1})}{2(-S + \alpha_{2,0})} \\ &\nu := (S + \beta_{0,0}) - \frac{\alpha_{1,0}\beta_{1,0}}{2(-S + \alpha_{2,0})} + \frac{\alpha_{1,0}^2\beta_{2,0}}{4(-S + \alpha_{2,0})^2} \neq 0. \end{aligned} \tag{4.7}$$

To resolve the singularity at $(0,0) \in U_2$ of the system $Z := \pi(R_\theta X')$ we perform a series of directional blow-ups. We start with $z = s$, $\mu = t/s$, followed by a rescaling by $1/z$. The resulting system $\overline{Z}$ has a hyperbolic singularity at $(0,0)$, with linear part given by the matrix

$$\begin{bmatrix} \alpha_{2,0} - \beta_{1,1} & 0 \\ 0 & -(-s + \alpha_{2,0}) \end{bmatrix},$$

and a T-singularity at $(z, \mu) = (0, \mu_0)$, $\mu_0 = -2(-s + \alpha_{2,0})/\alpha_{1,0}$. Translate the latter singularity to the origin of a new coordinate system by means of the transformation $w = z$, $\eta = \mu - \mu_0$. Directions of approach are given by zeros of the characteristic polynomial

$$H(w,\eta) = \frac{1}{\alpha_{1,0}^2} w\eta \left[4(\alpha_{2,0} - S)^2 w\nu + \frac{\alpha_{1,0}^3}{2(\alpha_{2,0} - S)} \left[\alpha_{2,0} + \beta_{1,1} - 2S\right] \eta \right],$$

where we have used the first condition in (4.7). In order to handle all characteristic directions simultaneously, we make the linear transformation $u = w - \eta$, $v = \eta$, followed by the single blow-up $U = u$, $V = v/u$, in turn followed by a rescaling by $1/U$. The singular points on the line $U = 0$ for the resulting system are $(0,0)$, $(0,-1)$, and $(0,\lambda)$, where

$$\lambda = -\frac{8(\alpha_{2,0} - S)^3}{\nu_1} \nu$$

and

$$\nu_1 := (\alpha_{2,0} + \beta_{1,1} - 2S)\alpha_{1,0}^3 + 2(\alpha_{2,0} - S)\beta_{2,0}\alpha_{1,0}^2 - 4(\alpha_{2,0} - S)^2 \beta_{1,0}\alpha_{1,0} + 8(\alpha_{2,0} - S)^3(\beta_{0,0} + S).$$

The linear parts at these singular points are given by the matrices

$$\begin{bmatrix} -(4(-s + \alpha_{2,0})^2 \nu/\alpha_{1,0}^2) & 0 \\ 0 & (4(-s + \alpha_{2,0})^2 \nu/\alpha_{1,0}^2) \end{bmatrix},$$

$$\begin{bmatrix} -(\alpha_{1,0}/2) & 0 \\ 0 & (\alpha_{1,0}(-2s + \alpha_{2,0} + \beta_{1,1})/2(-s + \alpha_{2,0})) \end{bmatrix},$$

and

$$\begin{bmatrix} -(\alpha_{1,0}(-s + \alpha_{2,0})^3 \nu/\nu_1) & 0 \\ & -(\alpha_{1,0}(-2s + \alpha_{2,0} + \beta_{1,1})(-s + \alpha_{2,0})^2 \nu/\nu_1) \end{bmatrix},$$

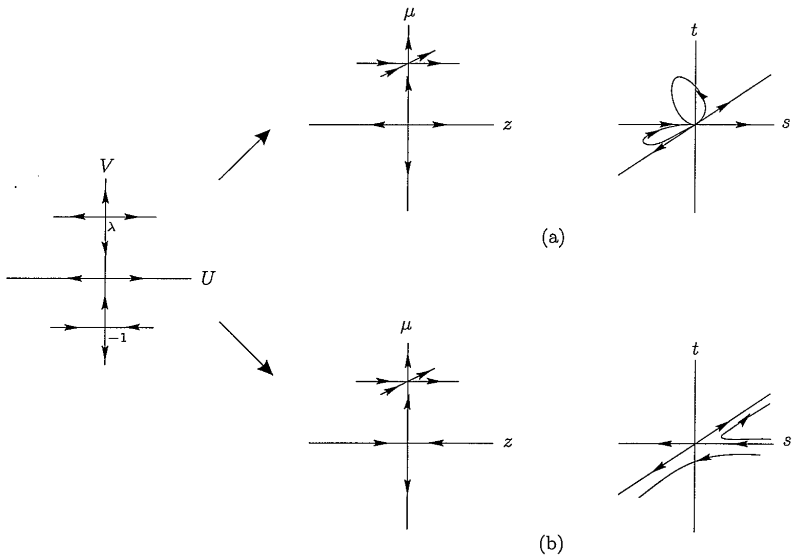

FIGURE 1. Going back through the blow-ups when the origin of U_2 has a double characteristic direction $\lambda > 0$. Case (a), respectively (b), corresponds to $\alpha_{2,0} - \beta_{1,1} > 0$, respectively $\alpha_{2,0} - \beta_{1,1} > 0$. If $\lambda < 0$ the picture is similar.

respectively. Thus, the singular points $(0,0)$ and $(0,-1)$ are saddle points while $(0,\lambda)$ is a node. In Figure 1 we show that, going back through the blow-ups, we never get the local phase portrait at the origin of U_2 corresponding to Type 18 or 19.

Type 21. By Lemma A.3, X can be placed in the form

$$\begin{aligned} \dot{x} &= xy \\ \dot{y} &= 1 + by + y^2, \end{aligned} \tag{4.8}$$

where $|b| < 2$.

We repeat the discussion in the first paragraph of the previous case, Type 20, this time obtaining

$$\begin{aligned} \dot{x} &= -S + bS^2x - bCSy - Sx^2 + Cxy \\ \dot{y} &= C - bCSx + bC^2y - Sxy + Cy^2 \end{aligned} \tag{4.9}$$

for $R_\theta X$ and

$$\begin{aligned}\dot{x} &= (-S+\alpha_{0,0}) + (bS^2+\alpha_{1,0})x + (-bCS+\alpha_{0,1})y \\ &\quad + (-S+\alpha_{2,0})x^2 + (C+\alpha_{1,1})xy \\ \dot{y} &= (C+\beta_{0,0}) + (-bCS+\beta_{1,0})x + (bC^2+\beta_{0,1})y \\ &\quad + \beta_{2,0}x^2 + (-S+\beta_{1,1})xy + (C+\beta_{0,2})y^2\end{aligned} \tag{4.10}$$

for $R_\theta X'$, where $|b|<2$, $S=\sin(\theta)$, $C=\cos(\theta)$, and θ has been chosen so that there is no term y^2 in the horizontal component of $R_\theta X'$.

The proof that $R_\theta X'$ cannot be of Type 3, 18, or 19 is similar to the corresponding proof in the previous case (X of Type 20), so we will only outline the argument. That $R_\theta X'$ cannot have Type 3 is so much like the previous case that it will not be discussed further.

The condition that $R_\theta X'$ have a T-singularity at $(0,0)\in U_2$ (the analogue of (4.7)) is now

$$\begin{aligned}\alpha_{1,1} &= -C \\ \beta_{0,2} &= -C \\ \alpha_{0,1} &= bCS.\end{aligned} \tag{4.11}$$

Thus again $\theta\approx\pi/2$ and consequently $S\approx 1$. System (4.10) becomes

$$\begin{aligned}\dot{x} &= (-S+\alpha_{0,0}) + (bS^2+\alpha_{1,0})x + (-S+\alpha_{2,0})x^2 \\ \dot{y} &= (C+\beta_{0,0}) + (-bCS+\beta_{1,0})x + (bC^2+\beta_{0,1})y \\ &\quad + \beta_{2,0}x^2 + (-S+\beta_{1,1})xy,\end{aligned} \tag{4.12}$$

which yields the coordinate expression for $\pi(R_\theta X')$ in U_2 of

(4.13)

$$\begin{aligned}\dot{s} &= (\alpha_{2,0}-\beta_{1,1})s^2 + (b(S^2-C^2)+\alpha_{1,0}-\beta_{0,1})st + (-S+\alpha_{0,0})t^2 \\ &\quad - \beta_{2,0}s^3 + (bCS-\beta_{1,0})s^2t - (C+\beta_{0,0})st^2 \\ \dot{t} &= -(-S+\beta_{1,1})st - (bC^2+\beta_{0,1})t^2 \\ &\quad - \beta_{2,0}s^2t + (bCS-\beta_{1,0})st^2 - (C+\beta_{0,0})t^3.\end{aligned}$$

The discussion in the paragraph following (4.6) applies verbatim, so that we may assume that $\beta_{2,0}(\alpha_{2,0}-\beta_{1,1})\neq 0$.

The origin in the local chart U_2 is a linearly zero singular point for which directions of approach are determined by zeros of the characteristic polynomial

$$H(s,t) = t\left[(-S+\alpha_{2,0})s^2 + (bS^2+\alpha_{1,0})st + (-S+\alpha_{0,0})t^2\right].$$

If $H = 0$ admits only $t = 0$ as a solution then the same argument as before (for Type 20) shows that it is not possible to have the local phase portraits corresponding to Figures 18 and 19 at the origin of U_2.

If $H = 0$ has two distinct solutions each different from $t = 0$, then an argument analogous to the one given before (for Type 20) shows that the condition that $R_\theta X'$ be a foliation leads to a contradiction.

The only remaining case is that $H = 0$ have a unique double zero in addition to and distinct from $t = 0$. This is the case that the discriminant of $H(s,t)/t$ be zero and that $\dot{x}$ vanish along $x = -(bS^2+\alpha_{1,0})/(2(-S+\alpha_{2,0}))$. Evaluating $\dot{y}$ along this line, the condition that $R_\theta X'$ be a foliation leads to two further conditions

$$\beta_{0,1} = \frac{(bS^2+\alpha_{1,0})(-S+\beta_{1,1})}{2(-S+\alpha_{2,0})} - C^2 b$$

$$\nu := (C+\beta_{0,0}) - \frac{(bS^2+\alpha_{1,0})(-bCS+\beta_{1,0})}{2(-S+\alpha_{2,0})} + \frac{(-S+\alpha_{0,0})\beta_{2,0}}{(-S+\alpha_{2,0})} \neq 0.$$

To resolve the singularity at $(0,0) \in U_2$ of the system $Z := \pi(R_\theta X')$ we perform a series of directional blow-ups, beginning with $z = s$, $\mu = t/s$, followed by a rescaling by $1/z$. The resulting system $\overline{Z}$ has a hyperbolic singularity at $(0,0)$, with linear part given by the matrix

$$\begin{bmatrix} \alpha_{2,0} - \beta_{1,1} & 0 \\ 0 & -(-s+\alpha_{2,0}) \end{bmatrix},$$

and a T-singularity at $(z,\mu) = (0,\mu_0)$, $\mu_0 = \sqrt{S-\alpha_{2,0}}/\sqrt{S-\alpha_{0,0}}$. Directions of approach to the second singularity are determined by the characteristic polynomial

$$H(s,t) = st\left[-\frac{(\alpha_{2,0}-S)}{(\alpha_{0,0}-S)}\nu s - \frac{(\alpha_{2,0}-\beta_{1,1}-2S)\sqrt{(\alpha_{0,0}-S)(\alpha_{2,0}-S)}}{(\alpha_{2,0}-S)}t\right],$$

which has three simple zeros. We proceed as in the previous case (Type 20): translate the degenerate singularity to the origin, apply a suitable linear transformation so that all directions of approach can be handled in one blow-up, and perform a directional blow-up followed by a rescaling, to obtain a system with three simple singular points.

Two of the singular points are saddles and one is a node. Figure 1 finishes the proof. □

We conclude this section by stating a corollary of Theorems 1.1, 1.4 and A.1 on the maximum number of inseparable leaves in structurally stable quadratic foliations. A general result for $\mathbf{S}^2$-structurally stable degree n foliations with respect to perturbation within P_n can be found in [**7**].

4.3. Corollary. *The maximum number of inseparable leaves for a quadratic $\mathbf{S}^2$- or R^2-structurally stable foliation X within P_n is 2.*

APPENDIX

A. **Chordal quadratic systems: A review.** We state here a few key results from [**4**] which are crucial to the proof of Theorem 1.4, and reproduce an important figure and tables from that paper.

A.1. Theorem [**4**]. *For any X in F_2, the phase portrait of $\pi(\pm X)$ is homeomorphic to one of the separatrix configurations shown in Figure 2. Furthermore, every separatrix configuration in Figure 2 is realized by $\pi(X)$ for some X in F_2.*

The proof of Theorem A.1 relies in part on the following normal form classification from [**4**] which reduces the study of quadratic foliations to ten families. We include the proof, to which reference is made in our proof of Theorem 1.4.

A.2. Lemma [**4**]. *For any X in F_2 there is an affine change of variables and change of scale placing X into one of the following ten normal forms:*

$$\text{(I)}\begin{cases}\dot{x} = 1 + xy,\\ \dot{y} = Q(x,y)\end{cases} \qquad \text{(II)}\begin{cases}\dot{x} = xy,\\ \dot{y} = Q(x,y)\end{cases}$$

$$\text{(III)}\begin{cases}\dot{x} = y + x^2,\\ \dot{y} = Q(x,y)\end{cases} \qquad \text{(IV)}\begin{cases}\dot{x} = y,\\ \dot{y} = Q(x,y)\end{cases}$$

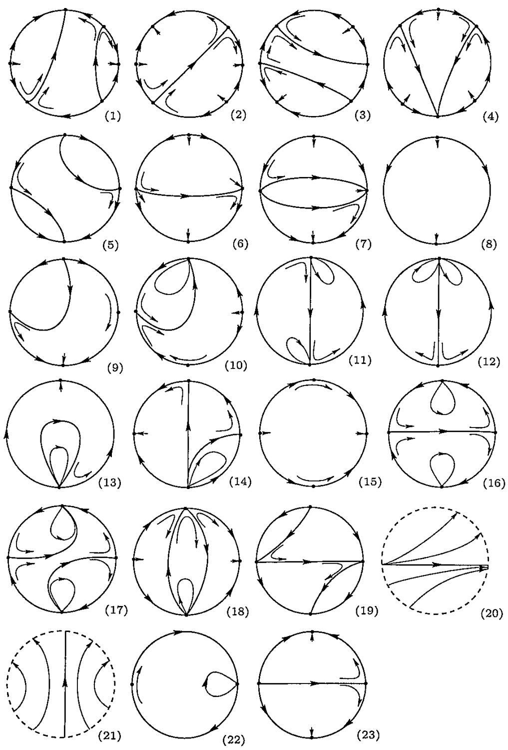

FIGURE 2. The separatrix configurations on the Poincaré disk of elements of F_2, up to the direction of flow; **E** is critical in Types 20 and 21.

$$(\mathrm{V})\begin{cases}\dot{x}=-1+x^2,\\ \dot{y}=Q(x,y)\end{cases}\qquad(\mathrm{VI})\begin{cases}\dot{x}=1+x^2,\\ \dot{y}=Q(x,y)\end{cases}$$

$$(\mathrm{VII})\begin{cases}\dot{x}=x^2,\\ \dot{y}=Q(x,y)\end{cases}\qquad(\mathrm{VIII})\begin{cases}\dot{x}=x,\\ \dot{y}=Q(x,y)\end{cases}$$

$$(\mathrm{IX})\begin{cases}\dot{x}=1,\\ \dot{y}=Q(x,y)\end{cases}\qquad(\mathrm{X})\begin{cases}\dot{x}=0,\\ \dot{y}=Q(x,y)\end{cases}$$

where $Q(x,y)=d+ax+by+lx^2+mxy+ny^2$.

Proof. We write X as

$$\begin{aligned} \dot{x} &= d_1 + a_1x + b_1y + l_1x^2 + m_1xy \\ \dot{y} &= Q(x, y), \end{aligned} \tag{A.1}$$

where without loss of generality we have assumed that a singularity of $\pi(X)$ is located at the origin of U_2.

If $m_1 \neq 0$, we apply the translation $x_1 = x + b_1/m_1$, $y_1 = y$ and system (A.1) becomes (always reverting to the labels x and y)

$$\begin{aligned} \dot{x} &= d_1' + a_1'x + l_1'x^2 + m_1xy \\ \dot{y} &= Q(x, y), \end{aligned}$$

which under the change of variables $x_1 = x$, $y_1 = a_1' + l_1'x + m_1y$ becomes

$$\begin{aligned} \dot{x} &= d_1' + xy \\ \dot{y} &= Q(x, y). \end{aligned}$$

If $d_1' = 0$, this is (II) while if $d_1' \neq 0$ we make the change of variables $x_1 = x/d_1'$, $y_1 = y$ to obtain (I).

If $m_1 = 0$ and $b_1 \neq 0$, we make the change of variables $x_1 = x$, $y_1 = d_1 + a_1x + b_1y$, which transforms system (A.1) into

$$\begin{aligned} \dot{x} &= y + l_1x^2 \\ \dot{y} &= Q(x, y). \end{aligned} \tag{A.2}$$

If $l_1 = 0$ this is (IV), while if $l_1 \neq 0$ the change of variables $x_1 = x$, $y_1 = y/l_1$ and time rescaling $t_1 = l_1t$ transforms system (A.1) into (III).

If $m_1 = b_1 = 0$ and $l_1 \neq 0$, we consider $k := a_1^2 - 4l_1d_1$. If $k \neq 0$, the change of variables $x_1 = 2l_1(x + a_1/(2l_1))/\sqrt{|k|}$, $y_1 = y$ and time rescaling $t_1 = \sqrt{|k|}t/2$ converts system (A.1) into (V) or (VI) according to whether k is positive or negative. If $k = 0$, then the change of variables $x_1 = x + a_1/(2l_1)$, $y_1 = y$ and time rescaling $t_1 = l_1t$ converts system (A.1) into (VII).

If $m_1 = b_1 = l_1 = 0$ and $a_1 \neq 0$, then the change of variables $x_1 = x + d_1/a_1$, $y_1 = y$ and time rescaling $t_1 = a_1t$ converts system (A.1) into (VIII).

Finally, suppose $m_1 = b_1 = l_1 = a_1 = 0$. If $d_1 = 0$ we already have (X), while if $d_1 \neq 0$ the time rescaling $t_1 = d_1 t$ converts system (A.1) into (IX). $\square$

Next we reproduce from [**4**] the tables showing the number and types of the singularities of $\pi(X)$ for each of the ten families. An isolated singularity is of type E if hyperbolic, type S if exactly one eigenvalue of the linear part is zero, type H if the linear part is not zero but both its eigenvalues are zero, and type T if the linear part is zero. A notation like (E, T) in the second column means that there are two distinct singularity pairs on $\mathbf{E}$, one of type E, one of type T. The notation "(degenerate)" in the second column means that the equator $\mathbf{E}$ is composed entirely of singularities of $\pi(X)$.

In Table A.1, the notations (A), (B) and (C) mean:

(A) The polynomial $lx^4 + ax^3 + (d-m)x^2 - bx + n$ has no real roots different from 0.

(B) Either $a = 0$, $d - m \neq 0$ and $b^2 - 4n(d-m) < 0$,

(1) or $a = b = n = 0$ and $d - m \neq 0$,

(2) or $a = d - m = n = 0$ and $b \neq 0$,

(3) or $b = n = d - m = 0$ and $a \neq 0$,

(4) or $a = b = d - m = 0$ and $n \neq 0$,

(5) or $n = 0$, $a \neq 0$ and $(d-m)^2 + 4ab < 0$.

(C) Either $b = 0$ and $a^2 - 4l(d-m) < 0$, or $b = a = d - m = 0$.

In Table A.2, the notations (A), (B), (C) and (D) mean:

(A) $b^2 - 4nd < 0$ and $a^2 - 4ld < 0$.

(B) $a = 0$ and $b^2 - 4nd < 0$.

(C) $b = 0$ and $a^2 - 4ld < 0$.

(D) $a = b = 0$ and $d \neq 0$.

In Table A.3, the notations (A) and (B) mean:

(A) The polynomial $nx^4 - mx^3 + (l-b)x^2 + ax + d$ has no real roots.

(B) Either $m = 0$, $l - b \neq 0$ and $a^2 - 4d(l-b) < 0$, or $a = m = l - b = 0$ and $d \neq 0$.

TABLE A.1.

Infinite Singularities for System (I)					
(E,E,E)			$m^2-4(n-1)l>0$	(I.1)	
(E)		$l\neq 0$	$m^2-4(n-1)l<0$	(I.2)	(A)
(E,S)	$n\neq 0,1$		$m^2-4(n-1)l=0$	(I.3)	
(E,E,S)		$l=0$	$m\neq 0$	(I.4)	(B)
(E,T)			$m=0$	(I.5)	
(E,E,S)			$m^2+4l>0$	(I.6)	
(S)		$l\neq 0$	$m^2+4l<0$	(I.7)	(C)
(S,S)			$m^2+4l=0$	(I.8)	
(E,S,S)	$n=0$		$m\neq 0$	(I.9)	
(S,H)		$l=0$	$m=0$ $a\neq 0$	(I.10)	(B)
(S,T)			$m=a=0$	(I.11)	
(E,S)		$l\neq 0$	$m\neq 0$	(I.12)	(A)
(S)			$m=0$	(I.13)	
(S,S)	$n=1$	$l=0$	$m\neq 0$	(I.14)	(B)
(degenerate)			$m=0$	(I.15)	

TABLE A.2.

Infinite Singularities for System (II)					
(E,E,E)			$m^2-4(n-1)l>0$	(II.1)	
(E)		$l\neq 0$	$m^2-4(n-1)l<0$	(II.2)	(A)
(E,S)	$n\neq 0,1$		$m^2-4(n-1)l=0$	(II.3)	
(E,E,S)		$l=0$	$m\neq 0$	(II.4)	(B)
(E,T)			$m=0$	(II.5)	
(E,E,S)			$m^2+4l>0$	(II.6)	
(S)		$l\neq 0$	$m^2+4l<0$	(II.7)	(C)
(S,S)	$n=0$		$m^2+4l=0$	(II.8)	
(E,S,S)		$l=0$	$m\neq 0$	(II.9)	(D)
(S,T)			$m=0$	(II.10)	
(E,S)		$l\neq 0$	$m\neq 0$	(II.11)	(A)
(S)	$n=1$		$m=0$	(II.12)	
(S,S)		$l=0$	$m\neq 0$	(II.13)	(B)
(degenerate)			$m=0$	(II.14)	

In Table A.4, $\Delta = 4an^2 - 2bmn - m^2$, and it is assumed that either $l\neq 0$ and $a^2-4ld<0$, or $a=l=0$ and $d\neq 0$.

TABLE A.3.

Infinite Singularities for System (I)				
(E,E,E)	$n \neq 0$	$(m-1)^2 - 4nl > 0$	(III.1)	(A)
(E)		$(m-1)^2 - 4nl < 0$	(III.2)	
(E,S)		$(m-1)^2 - 4nl = 0$	(III.3)	
(E,H)	$n = 0$		(III.4)	(B)

TABLE A.4

Infinite Singularities for System (IV)			
(E,S,S)	$n \neq 0$	$m^2 - 4nl > 0$	(IV.1)
(E)		$m^2 - 4nl < 0$	(IV.2)
(E,H)		$m^2 - 4nl = 0,\ \Delta \neq 0$	(IV.3)
(E,T)		$m^2 - 4nl = \Delta = 0$	(IV.4)
(S,H)	$n = 0$	$m \neq 0$	(IV.5)
(H)		$m = 0,\ l \neq 0$	(IV.6)
System (IV) is linear when		$n = m = l = 0$	(IV.7)

TABLE A.5.

Infinite Singularities for System (V)				
(E,E,E)		$(m-1)^2 - 4nl > 0$	(V.1)	(A)
(E)	$n \neq 0$	$(m-1)^2 - 4nl < 0$	(V.2)	
(E)	(E,S)	$(m-1)^2 - 4nl = 0$	(V.3)	
(E,T)	$n = 0$		(V.4)	(B)

TABLE A.6.

Infinite Singularities for System (VI)			
(E,E,E)	$n \neq 0$	$(m-1)^2 - 4nl > 0$	(VI.1)
(E)		$(m-1)^2 - 4nl < 0$	(VI.2)
(E,S)		$(m-1)^2 - 4nl = 0$	(VI.3)
(E,T)	$n = 0$	$m \neq 1$	(VI.4)
(T)		$m = 1,\ l \neq 0$	(VI.5)
(degenerate)		$m = 1,\ l = 0$	(VI.6)

In Table A.5, the notations (A) and (B) mean:

(A) $(m+b)^2 - 4n(d+a+l) < 0$ and $(m-b)^2 - 4n(d-a+l) < 0$.

(B) $m = b = 0$ and $d + l \neq \pm a$.

In Table A.8, $\Delta = 2an - (b-1)m$.

In Table A.9, $\Delta = 2an - bm$.

In Table A.10 the equation $d + ax + by + lx^2 + mxy + ny^2 = 0$ has no real solutions. Moreover,

$$D = \det \begin{pmatrix} l & m/2 & a/2 \\ m/2 & n & b/2 \\ a/2 & b/2 & d \end{pmatrix}.$$

TABLE A.7.

Infinite Singularities for System (VII)				
(E,E,E)	$n \neq 0$	$(m-1)^2 - 4nl > 0$	(VII.1)	$b^2 - 4nd < 0$
(E)		$(m-1)^2 - 4nl < 0$	(VII.2)	
(E,S)		$(m-1)^2 - 4nl = 0$	(VII.3)	
(E,T)	$n = 0$	$m \neq 1$	(VII.4)	$b = 0,\ d \neq 0$
(T)		$m = 1,\ l \neq 0$	(VII.5)	
(degenerate)		$m = 1,\ l = 0$	(VII.6)	

TABLE A.8.

Infinite Singularities for System (VIII)				
(E,S,S)	$n \neq 0$	$m^2 - 4nl > 0$	(VIII.1)	$b^2 - 4nd < 0$
(E)		$m^2 - 4nl < 0$	(VIII.2)	
(E,H)		$m^2 - 4nl = 0,\ \Delta \neq 0$	(VIII.3)	
(E,T)		$m^2 - 4nl = \Delta = 0$	(VIII.4)	
(S,T)	$n = 0$	$m \neq 0$	(VIII.5)	$b = 0,\ d \neq 0$
(T)		$m = 0,\ l \neq 0$	(VIII.6)	
System (VIII) is linear when	$n = m = l = 0$		(VIII.7)	

TABLE A.9.

Infinite Singularities for System (IX)			
(E,S,S)	$n \neq 0$	$m^2 - 4nl > 0$	(IX.1)
(E)		$m^2 - 4nl < 0$	(IX.2)
(E,H)		$m^2 - 4nl = 0,\ \Delta \neq 0$	(IX.3)
(E,T)		$m^2 - 4nl = \Delta = 0$	(IX.4)
(S,T)	$n = 0$	$m \neq 0$	(IX.5)
(T)		$m = 0,\ l \neq 0$	(IX.6)
System (IX) is linear or of degree zero when	$n = m = l = 0$		(IX.7)

TABLE A.10.

Infinite Singularities for System (X)			
(E)	$n \neq 0$	$m^2 - 4nl < 0$ and either $lD > 0$ or $D = 0$	(X.1)
(E,T)		$m^2 - 4nl = 2an - bm = D = 0,\ b^2 - 4nd < 0$	(X.2)
(T)	$n = 0$	$b = m = 0,\ l \neq 0$ and $a^2 - 4ld < 0$	(X.3)
System (X) is of degree zero when	$a = b = l = m = n = 0,\quad d \neq 0$		(X.4)

We close with a lemma on (properly) degenerate quadratic foliations that simplifies some computations in the final part of the proof of Theorem 1.4. A *properly degenerate system* X *in* F_2 is one that is neither constant nor linear, but such that the equator **E** is composed entirely of singularities of $\pi(X)$.

A.3. Lemma. *For a (properly) degenerate system $X \in \mathrm{F}_2$, there exists an affine transformation and a scaling of the time variable which reduces X to one of the systems*:

$$\text{(D.1)} \quad \begin{cases} \dot{x} = 1 + xy \\ \dot{y} = y^2 \end{cases} \qquad \text{(D.2)} \quad \begin{cases} \dot{x} = xy \\ \dot{y} = 1 + by + y^2, \end{cases}$$

where $|b| < 2$.

Proof. From the tables above the only (properly) degenerate quadratic systems correspond to (I.15), (II.14), (VI.6) and (VII.6).

System (I.15) with $d = 0$ is already (D.1). If $d \neq 0$ we have $d > 0$, since we are assuming $b^2 - 4d < 0$. Therefore, the change of variables

$x_1 = \sqrt{d}x$, $y_1 = 1/\sqrt{d}y$ and time rescaling $t_1 = \sqrt{d}\,t$ converts system (I.15) to

$$\text{(A.3)} \qquad \begin{aligned} \dot{x} &= 1 + xy \\ \dot{y} &= 1 + b'y + y^2, \end{aligned}$$

which, after the new change $x_1 = x - b' - y$, $y_1 = y$, becomes system (D.2). Similarly, system (II.14) becomes (D.2).

Any system (VI.6) converts to system

$$\text{(A.4)} \qquad \begin{aligned} \dot{x} &= x^2 - 2bx + b^2 + 1 \\ \dot{y} &= d' + xy \end{aligned}$$

after applying the translation $x_1 = x + b$, $y_1 = y + a$. If $d' \neq 0$, we apply the change of variables $x_1 = \sqrt{b^2+1}/d'y$, $y_1 = 1/\sqrt{b^2+1}x$ and time rescaling $t_1 = \sqrt{b^2+1}\,t$ to system (A.4) to get system (A.3), and consequently system (D.2). If $d' = 0$, the change of variables $x_1 = y$, $y_1 = 1/\sqrt{b^2+1}x$ and time rescaling $t_1 = \sqrt{b^2+1}\,t$ converts system (VII.6) into system (D.2).

Finally the change of variables $x_1 = (y+a)/d$, $y_1 = x$ converts system (VII.6) to system (D.1). □

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