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DIVIDED DIFFERENCE OPERATORS AND CLASSICAL ORTHOGONAL POLYNOMIALS

RICHARD ASKEY

ABSTRACT. In an earlier paper J. Wilson and I introduced a divided difference operator that plays the same role for the $_4\varphi_3$ orthogonal polynomials that the derivative does for Jacobi polynomials. Here this operator is used to give a new derivation of the connection coefficient result of L.J. Rogers.

1. Introduction. L.J. Rogers introduced a very attractive set of polynomials in [**10**]. To define them take q fixed with $0 < |q| < 1$. Set

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k) \tag{1.1}$$

and

$$(a;q)_n = (a;q)_\infty/(aq^n;q)_\infty. \tag{1.2}$$

Then, following Rogers (but using a slightly different notation), consider the generating function

$$\frac{(\beta re^{i\theta};q)_\infty(\beta re^{-i\theta};q)_\infty}{(re^{i\theta};q)_\infty(re^{-i\theta};q)_\infty} = \sum_{n=0}^{\infty} C_n(\cos\theta;\beta|q)r^n. \tag{1.3}$$

The q-binomial theorem is

$$\frac{(ar;q)_\infty}{(r;q)_\infty} = \sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}r^n. \tag{1.4}$$

Using this in (1.3) gives

$$\begin{aligned} C_n(\cos\theta;\beta|q) &= \sum_{k=0}^{n}\frac{(\beta;q)_{n-k}(\beta;q)_k}{(q;q)_{n-k}(q;q)_k}\, e^{i(n-2k)\theta} \\ &= \sum_{k=0}^{n}\frac{(\beta;q)_{n-k}(\beta;q)_k}{(q;q)_{n-k}(q;q)_k}\cos(n-2k)\theta. \end{aligned} \tag{1.5}$$

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The second identity in (1.5) follows from the first, because both sides of (1.5) must be real when β and q are real.

It is easy to see that (1.3) and the following recurrence relation imply each other:

$$\begin{aligned}2(1-\beta q^n)xC_n(x;\beta|q) \\ &= (1-q^{n+1})C_{n+1}(x;\beta|q) \\ &\quad + (1-\beta^2q^{n-1})C_{n-1}(x;\beta|q).\end{aligned} \tag{1.6}$$

L.J. Rogers claimed that

$$\begin{aligned}C_n(x;\gamma|q) \\ = \sum_{k=0}^{[n/2]} \beta^k \frac{(\gamma\beta^{-1};q)_k(\gamma;q)_{n-k}}{(q;q)_k(\beta q;q)_{n-k}} \frac{(1-\beta q^{n-2k})}{(1-\beta)} C_{n-2k}(x;\beta|q),\end{aligned} \tag{1.7}$$

and then used (1.6) to prove (1.7) by induction. A second proof of (1.7) was given in [**1**]. Here the orthogonality of $\{C_n(x;\beta|q)\}$ was used, as well as a sum of a very well poised ${}_6\varphi_5$. It would be nice to have a direct derivation of (1.7) that does not assume the form at the beginning, as Rogers did, or use complicated results as in [**1**]. Such a derivation will be given in the next section. It uses a divided difference operator introduced by Wilson and the author in [**2**]. A second very interesting application of this operator is given by Kalnins and Miller [**6**].

2. A divided difference operator and the connection coefficient problem. The operator that plays the role played by the derivative for ultraspherical polynomials can be defined as follows. Take a function of $e^{i\theta}$, say $f(e^{i\theta})$. Define two shift operators by

$$E_q^{\pm} f(e^{i\theta}) = f(q^{\pm 1/2}e^{i\theta}). \tag{2.1}$$

Then define

$$\delta_q f(e^{i\theta}) = (E_q^+ - E_q^-) f(e^{i\theta}) \tag{2.2}$$

and

$$\Delta_q f(x) = \frac{\delta_q f(x)}{\delta_q x}, \tag{2.3}$$

where $x = (e^{i\theta} + e^{-i\theta})/2$. Applying Δ_q to (1.3) gives

$$\begin{aligned}\Delta_q \frac{(\beta re^{i\theta};q)_\infty(\beta re^{-i\theta};q)_\infty}{(re^{i\theta};q)_\infty(re^{-i\theta};q)_\infty} \\ = 2r\frac{(1-\beta)}{(1-q)}\cdot\frac{(\beta rq^{1/2}e^{i\theta};q)_\infty(\beta rq^{1/2}e^{-i\theta};q)_\infty}{(rq^{-1/2}e^{i\theta};q)_\infty(rq^{-1/2}e^{-i\theta};q)_\infty}.\end{aligned} \tag{2.4}$$

Expanding both sides of (2.4) in a power series in r gives

$$\Delta_q C_n(x;\beta|q) = \frac{2(1-\beta)}{(1-q)} q^{(1-n)/2} C_{n-1}(x;\beta q|q). \tag{2.5}$$

This is an extension of

$$\frac{d}{dx}C_n^\lambda(x) = 2\lambda\, C_{n-1}^{\lambda+1}(x), \tag{2.6}$$

where

$$(1-2xr+r^2)^{-\lambda} = \sum_{n=0}^\infty C_n^\lambda(x) r^n. \tag{2.7}$$

Apply Δ_q to both sides of the second form of (1.5). The result is (1.7) when γ is $q\gamma, n$ is replaced by $n-1$, and β in (1.7) is now q. Then, to derive (1.7), it is sufficient to apply Δ_q successively. The result is equivalent to (1.7) when $\beta = q^k$, $k = 1,2,\ldots.$ That is sufficient to prove (1.7), for both sides are analytic in β for $|\beta| < 1$ and they agree for infinitely many values of $\beta = q^k$, $k = 1,2,\ldots.$

For applications of (1.7) see [**3, 9, 10**].

3. Comments and open problems. The operator Δ_q was introduced in [**2**], but was not used there except to derive some formulas which extend known formulas for the Jacobi, Laguerre and Hermite polynomials. These polynomials satisfy many properties, and some of these properties only hold for these polynomials, either as polynomials, or as orthogonal polynomials. Here are three such facts.

(3A) If a set of polynomials $\{p_n(x)\}$ is orthogonal with respect to a positive measure on the real line, if $q_n(x) = \frac{d}{dx}p_{n+1}(x)$, and if

$\{q_n(x)\}$ is also orthogonal, then $\{p_n(x)\}$ is orthogonal with respect to the beta, gamma or normal distributions, and the polynomials are Jacobi, Laguerre or Hermite polynomials.

(3B) If $\{p_n(x)\}$ satisfies the differential equations

$$a(x)y'' + b(x)y' + \lambda_n y = 0, \quad y = p_n(x),$$

where $a(x)$ and $b(x)$ are independent of n and λ_n is independent of x, and if $\{p_n(x)\}$ are orthogonal polynomials, then $\{p_n(x)\}$ are as in (3A).

(3C) If there are functions $w(x)$ and $l(x)$ so that

$$w(x)p_n(x) = \frac{d^n}{dx^n}\left(w(x)\left(l(x)\right)^n\right),$$

where $p_n(x)$ is a polynomial of degree n, then $\{p_n(x)\}$ are as in (3A) or else are a class of polynomials known as Bessel polynomials. Bessel polynomials are not orthogonal with respect to a positive measure on the real line, but are orthogonal with respect to some signed measures.

See [**4**, Chapter 10] for statements of these theorems and for references. Hahn [**5**] found similar theorems for the operator

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}.$$

One can ask if there are similar results for Δ_q. There is a set of polynomials which satisfies identities like those in (3A,B,C), see [**2**]. Leonard [**7**] has proven a theorem that essentially shows that, if an identity like (3B) holds for Δ_q, then the polynomials are those in [**2**] or limiting cases of them. It is natural to see if something similar is true for (3A) and (3C) with the derivative being replaced by a divided difference operator. The first question to ask is to find all such operators that have the property that a polynomial of degree n is taken to a polynomial of degree $(n-1)$ for $n = 1, 2, \ldots$. Magnus [**8**] has solved this problem, and the operators are divided difference operators like (2.3). They have a bit more freedom, but that is almost illusory. Now one has an inherent structural reason for considering (2.3), so it is natural to look for the analogues of the uniqueness part of (3A) and (3C) for divided difference operators like Δ_q.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WI 53706