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SPHERES WITH CONTINUOUS TANGENT PLANES

L.D. LOVELAND

1. Introduction. Burgess [**2**] soved Problem 12 in *The Scottish Book* by exhibiting a wild 2–sphere in E^3 having a continuous family of tangent planes. A 2–sphere Σ in E^3 is said to be *wild* if no space homeomorphism takes Σ onto the sphere S defined by $\{(x,y,z)|x^2+y^2+z^2=1\}$. Spheres that are not wild are called *flat* or *tame.* The definition of a plane being tangent to a surface Σ in Euclidean 3–space E^3 comes from Problem 156 of *The Scottish Book.* A plane $T(q)$ is *tangent* to Σ at a point q of Σ if, for each positive number ϵ, there exists a round ball B centered at q such that the measure of the angle between $T(q)$ and every straight line $L(q,x)$ determined by q and a point x of $\Sigma\cap B\backslash\{q\}$ is less than ϵ. A surface may have infinitely many tangent planes at a single point q as ones sees by examining the surface obtained by rotating the graph of $|x|^{1/2}+|z|^{1/2}=1$ about the z–axis and letting $q=(0,0,1)$, see Figure 1. A 2–sphere Σ is said to have *continuous tangent planes* over a subset K of Σ if, for each q in K, there is a unique tangent plane $T(q)$ to Σ at q such that $\{T(q_i)\}$ converges to $T(q)$ whenever $\{q_i\}$ is a sequence of points of K converging to q. When we say Σ has a continuous family of tangent planes we mean to take K equal to Σ.

The wildness of the spheres described by Burgess [**2**] occurs at points of the 2–sphere Σ that belong to its rim. The *rim* R of Σ is the set of all points q of Σ where the normal to some tangent plane to Σ at q fails to pierce Σ at q. In [**2**] the rim of Σ is a simple closed curve containing the single wild point of Σ. The original motivation for this paper came from a desire to better understand the rim of Σ and its relation to the wild set. A point q of a 2–sphere Σ in E^3 is said to belong to the *wild set* W of Σ if there is no 2–cell K in Σ such that q lies in $\operatorname{Int} K$ and K lies on a tame 2–sphere in E^3. Example 4.2 describes a 2–sphere Σ in E^3 with a continuous family of tangent planes, a 1–dimemsional wild set, and a rim that is the union of a countable sequence of disjoint simple closed curves.

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The close connection between the existence of a tangent plane to Σ at q and the existence of a double cone touching Σ only at its vertex q is explored in the next section where these concepts, when properly stated, are shown to be equivalent. An analysis of the nature of the rim R of Σ when Σ has a family F of tangent planes is given in §3. It turns out that R is independent of the family of planes used to define it (Theorem 3.2), and that a point q of Σ always belongs to R when Σ has more than one plane tangent to it at q (Theorem 3.3). The dimension of R must always be less than two (Theorem 3.5), and when F is continuous, the dimension of the closure $\mathrm{cl}(R)$ is also less than two (Corollary 3.9). The rim need not be closed and need not separate Σ, even when F is continuous. However, Σ is always locally separated by $\mathrm{cl}(R)$ at each point of $\mathrm{cl}(R)$ when F is continuous (Corollary 3.8), so $\mathrm{cl}(R)$ must be one–dimensional. Furthermore, neither R nor its closure can be a point or an arc when F is continuous because Σ–$\mathrm{cl}(R)$ is not connected (Theorem 3.10).

Although the wild set W of a 2–sphere Σ with tangent planes must lie in the closure of R (Theorem 4.1), it need not lie in R. From the results of §2 and §3 it follows that W has dimension less than two when Σ has a continuous family F of tangent planes (Theorem 4.3). Furthermore, the set Q of all points of W that are components of W must be dense in W according to Theorem 4.5. This means W cannot be a connected set when F is continuous and W is nondegenerate

2. The tangent plane–double cone connection. A double cone of height $2h, h > 0$, is any object congruent to the solid double cone C defined by $\{(x,y,z)|x^2+y^2 \leq k^2z^2 \text{ and } |z| \leq h\}$. If f is an isometry defined on E^3, then the *cone angle* 2θ of $f(C)$ is given by $\theta = \arctan|k| (0 < \theta < \pi/2)$, the *cone axis* of $f(C)$ is the image of the z–axis under f, and the *vertex* of $f(C)$ is $f((0,0,0))$. A 2–sphere Σ in E^3 is said to have a *double cone at a point* q if there exists a double cone C with vertex q such that Σ provided Σ has a double cone at each point of K. The equivalence between Σ having a tangent plane at $q \in \Sigma$ and Σ having double cones of arbitrarily large cone angles at q is given in Theorem 2.1 and is exploited throughout the paper. The proof is clear from the two definitions.

THEOREM 2.1. *A 2–sphere Σ in E^3 has a tangent plane at a point q*

of Σ if and only if there exists a plane T through q such that, for each θ between 0 and $\pi/2, \Sigma$ has a double cone with vertex q, cone angle 2θ, and cone axis perpendicular to T.

The study here would be facilitated if, instead of double cones, there were double tangent balls to Σ at each point where Σ has tangent planes. Unlike double cones at $q \in \Sigma$, double tangent balls to Σ at q of a given size are unique. Furthermore, embeddings of spheres with double tangent balls have been studied extensively [**1,3,6,7,9,10,11**]. However, the surface obtained by rotating the graph of $z = |x|^{3/2}$ about the z–axis can be shown to have a tangent plane at the origin but not round tangent balls there. This example shows that increasing the cone angle 2θ in Theorem 2.1 may require accepting compensatingly smaller cone heights. On the other hand, it is true that Σ has a tangent plane at every point where it has double tangent balls. In this sense some of the results of §4 are closely related to previous theorems on tangent ball embeddings of 2–spheres in E^3. However, the existence of double tangent balls at each point of a 2–sphere Σ does not insure that Σ has a *continuous* family of tangent planes.

3. The rim and its properties. The *rim* R of a 2–sphere Σ in E^3 that has tangent planes was defined eariler to be all those points of Σ where the normal to a tangent plane at the point of tangency fails to pierce Σ. Theorem 3.2 establishes that the rim is independent of the family of tangent planes. In terms of double cones, the rim consists of those points where the double cones of Theorem 2.1 are always in the closure of one component of E^3–Σ. Theorem 3.3 states that a point q of Σ must belong to R whenever there are two or more distinct tangent planes to Σ at q. Furthermore, the points where Σ has multiple tangent planes are much like the point q in Figure 1 in the sense that the normals to the tangent planes at q sweep out an entire plane (Theorem 3.1). By constructing a convergent sequence of points like q in Figure 1 on 2–sphere, one obtains an example where the rim R fails to be closed, and with more effort one can construct a 2–sphere with a countable dense subset of points like q in FIgure 1. However, the dimension of R is itself always less than two (Theorem 3.5), and when the family of tangent planes is continuous even the closure of R has dimension less than two (Corollary 3.9).

One can construct a 2–sphere Σ having a continuous family F of tangent planes where the closure of R is not connected (see Example 3.11), where R is a continuum that is not locally connected (see Example 3.12), or where R is the union of two disjoint open arcs (Example 3.12). However, when F is continuous $\mathrm{cl}(R)$ cannot be 0–dimensional and $\mathrm{cl}(R)$ cannot be an arc because $\mathrm{cl}(R)$ locally separates Σ (see Corollarly 3.8).

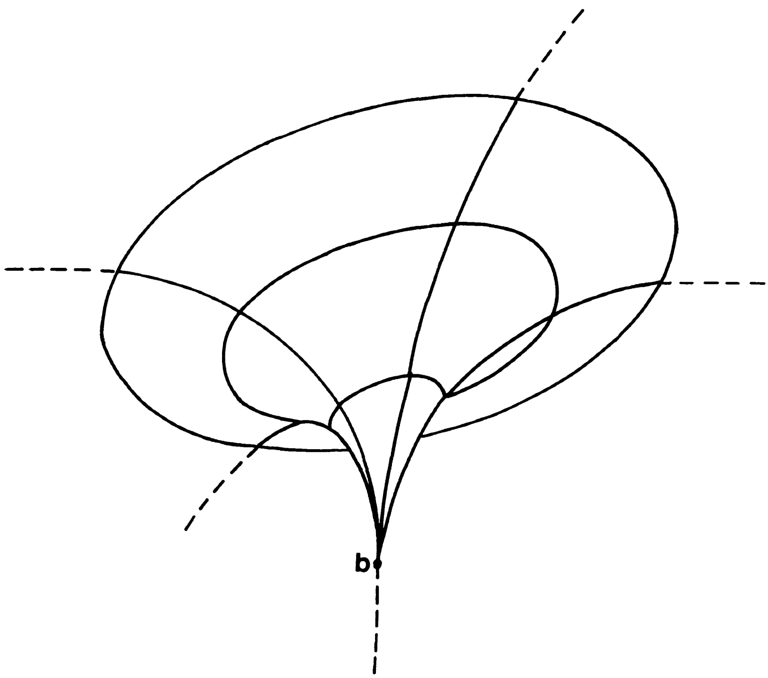

Figure 1.

The rim R of a 2–sphere Σ in E^3 with a family F of tangent planes breaks into two disjoint subsets $U(R)$ and $M(R)$, where a point of R lies in $M(R)$ whenever Σ has more than one tangent plane at p. Of course $M(R)$ is empty when F is a continuous family. The notion that F can fail to be continuous only at $M(R)$ or only at $\mathrm{cl}(R)$ is quickly dispelled by considering a sphere containing a washboard resembling $z = y^2 \sin(1/y)$. It is interesting to note that $M(R)$ is at most a countable set (Theorem 3.4).

If p and q are two points of E^3, then $L(p,q)$ denotes the line containing p and q. The measure of the smaller angle made by the lines L and L' is denoted by $\theta(L,L')$, and the angle between a line L and a plane P has measure $\theta(L,P)$ given by the measure of the angle between L and its orthogonal projection on P.

THEOREM 3.1. *If P_1 and P_2 are two distinct planes each tangent to a 2–sphere Σ at the point p, then the set of all planes tangent to Σ at p is precisely the set of all planes containing the line $P_1 \cap P_2$.*

PROOF. Let $\{p_i\}$ be a sequence of distinct points of Σ converging to p, and define, for each i, L_i to be the line through p and p_i. From the definition of tangent plane, $\lim_{i\to\infty} \theta(L_i, P_j) = 0$ for $j = 1$ and 2. This means $\{L_i\}$ converges to $P_1 \cap P_2$ and implies every plane through $P_1 \cap P_2$ is tangent to Σ at p. It is also clear now that every plane tangent to Σ at p must contain $P_1 \cap P_2$.

THEOREM 3.2. *The rim of a 2–sphere Σ with a family of tangent planes is independent of the defining family of tangent planes to Σ; that is; there cannot be two planes P_1 and P_2 tangent to Σ at a point p of Σ such that the normal to P_1 pierces Σ at p while the normal to P_2 at p does not.*

PROOF. Suppose there exists two planes P_1 and P_2 each tangent to Σ at p such that the normal N_1 to P_1 at p pierces Σ while the normal N_2 to P_2 at p does not. Using Theorem 2.1, choose, for $i = 1$ and 2, a double cone C_i with vertex p, cone angle larger than $3\pi/4$, and cone axis N_i such that $C_2\backslash\{p\}$ lies in one component U of $E^3\backslash\Sigma$ and C_1 is the union of two single cones C_1^+ and C_1^- whose interiors are on opposite sides of Σ. The contradiction is now evident because the large cone angles force C_1^+ and C_1^- to intersect $\operatorname{Int} C_2$ and, hence, force $C_1\backslash\{p\}$ to also lie in U.

THEOREM 3.3. *If a 2–sphere Σ in E^3 has two distinct tangent planes at a point p of Σ, then p belongs to the rim of Σ.*

PROOF. The idea of the proof of Theorem 3.2 can be applied here to yield a flat, round disk D centered at p such that $D\backslash\{p\}$ intersects the normals to the two hypothesized tangent planes to Σ at p and such that $D\backslash\{p\}$ lies in a single component of $E^3\backslash\Sigma$. Thus, neither normal can pierce Σ at p, and $p \in R$.

THEOREM 3.4. *If a 2–sphere Σ in E^3 has a family of tangent planes, and $M(R)$ is the set of all points of Σ where Σ has more than one tangent plane, then $M(R)$ is a countable set.*

PROOF. From Theorem 2.1, as used in the proofs of Theorems 3.2 and 3.3, it follows that, for each $p \in M(R)$, there exist two double cones whose union contains an entire wheel $W(p)$, where $W(p)$ is obtained by rotating a cone with vertex p about a line through p perpendicular to the cone's axis. To see this, one simply chooses two distinct double cones each with large enough cone angle to cause their interiors to intersect. For each i, define $A_i = \{p \in \Sigma|$ there exists such a wheel $W(p)$ centered at p such that $(\operatorname{Int} W(p)) \cap \Sigma = \emptyset$ and $\operatorname{diam} W(p) \geq \frac{1}{i}\}$. Then A_i is closed for each i. Suppose A_j is infinite, for some j. Then there must exist a sequence $\{W(p_i)\}$ of wheels converging to a wheel $W(p)$ where p_i and p belong to A_j. But $\{p_i\}$ cannot converge to p from either side of $W(p)$ because $W(p) \cup W(p_i)$ eventually closes off a region too small to contain Σ. Thus A_j is finite for every j, and, since $M(R) \subset \cup_1^\infty A_i, M(R)$ is countable.

THEOREM 3.5. *If a 2–sphere Σ in E^3 has a family of tangent planes, then the dimension of its rim R is less than two.*

PROOF. Suppose R contains a 2–cell K. Fix θ between 0 and $\pi/2$, and use Theorem 2.1 to choose, for each $p \in K$, a double cone C_p with vertex p, cone angle 2θ, height h_p, and such that $\operatorname{Int} C_p$ lies in a single component of $E^3\backslash\Sigma$. For each i, let $X_i = \{p \in K|$ there exists a double cone C_p as just described such that $\operatorname{Int} C_p \subset \operatorname{Ext}\Sigma$ and $h_p \geq 1/i\}$, and let $Y_i = \{p \in k|$ such a cone C_p exists with $\operatorname{Int} C_p \subset \operatorname{Int}\Sigma$ and $h_p \geq 1/i\}$. Then, for each i, X_i and Y_i are closed sets, and $K = (\cup X_i) \cup (\cup Y_i)$. A Baire category theorem implies the existence of a 2–cell M and an integer m such that either $M \subset X_m$ or

$M \subset Y_m$. Suppose $M \subset X_m$. There must be a round ball B such that $\operatorname{Int} B \subset \operatorname{Int} \Sigma$ and $\emptyset \neq B \cap \Sigma \subset M$ (see the proof of Theorem 2.1 of [**3**] if necessary); and there must be a double cone C to Σ at a point p of $B \cap \Sigma$ such that $\operatorname{Int} C \subset \operatorname{Ext} \Sigma$. But the geometry of E^3 will not allow the interiors of C and B to be disjoint. Similarly M cannot lie in Y_m.

COROLLARY 3.6. *In a 2–sphere Σ in E^3 with tangent planes every point of the rim R is a limit point of Σ–R.*

LEMMA 3.7. *If D is a 2–cell on a 2–sphere Σ in E^3, $\varphi > 0$, and for each point $q \in D$ there exists a double cone $C(q)$ with cone axis $L(q)$ such that:*
(1) $C(q) \cap \Sigma = \{q\}$,
(2) *$C(q)$ has cone angle 2φ,*
(3) *$\{L(q) | q \in D\}$ is a continuous family of lines,*
(4) *$R \cap D = \{q \in D | L(q)$ does not pierce Σ at $q\}$, and*
(5) $p \in \operatorname{cl}(R) \cap \operatorname{Int} D$.
Then there is a positive number ϵ such that K–$\operatorname{cl}(R)$ fails to be connected whenever K is a 2–cell in $N(p,\epsilon) \cap D$ with p in its interior.

PROOF. It is convenient to impose a coordinate system on E^3 with p the origin and $L(p)$ the vertical z–axis. Let π denote the vertical projection of E^3 onto the xy–coordinate plane, and use Conditions (3) and (5) of the hypothesis to choose a positive number ϵ such that
(a) $\theta(L(p), L(q)) < \varphi/2$ whenever $q \in N(p,\epsilon) \cap \Sigma$ and
(b) $(N(p,\epsilon) \cap \Sigma) \subset D$.
For each $q \in N(p,\epsilon) \cap \Sigma$, the double cone $C(q)$ is the union of two congruent single cones $C^+(q)$ and $C^-(q)$, where the "+" denotes the cone whose centroid has the larger z–coordinate. Let K be a 2–cell in Σ such that $p \in \operatorname{Int} K$ and $K \subset N(p,\epsilon)$. Then define sets E and I as follows:

$$E = \{q \in K - \operatorname{cl}(R) | \operatorname{Int} C^+(q) \subset \operatorname{Ext} \Sigma\}$$
$$I = \{q \in K - \operatorname{cl}(R) | \operatorname{Int} C^+(q) \subset \operatorname{Int} \Sigma\}.$$

From the hypothesis it is clear that K–$\operatorname{cl}(R) \subset E \cup I$ and $E \cap I = \emptyset$. If p belongs to R, it is easy to check that neither I nor E can be empty using the fact that there are points arbitrarily close to p that belong to the complementary domain of Σ not containing $\operatorname{Int} C(p)$. For example, suppose $\operatorname{Int} C(p) \subset \operatorname{Ext} \Sigma$. Then there is a point z of $\operatorname{Int} \Sigma$ lying on a vertical segment $I(z)$ whose endpoints lie in $C^+(p)$ and $C^-(p)$, respectively. Order $I(z)$ according to the size of the z–coordinate, and notice that the highest point of $I(z) \cap \Sigma$ must lie in E while the highest point of $I(z) \cap \Sigma$ that lies below z must lie in I. When p is a limit point of R but not in R merely choose a point $p' \in R$ close enough to p that the same argument applies to p'. To complete the proof that $K-\operatorname{cl}(R)$ is not connected it suffices to show that E and I are each open subsets of Σ.

The proofs that E and I are open are similar, so only the proof for E is given here. It follows from Condition (2) of the hypothesis and Condition (a) above that, for each $q \in E$, the cones $C^+(q)$ and $C^-(q)$ contain congruent single cones $B^+(g)$ and $B^-(g)$, respectively, such that $B^+(q) \cup B^-(q)$ is a double cone with a vertical cone axis and with cone angle φ. Choose and fix q in E, and choose $\delta > 0$ small enough that $(N(q,\delta) \cap \Sigma) \subset K-\operatorname{cl}(R)$ and, for each x in $N(q,\delta)$, the line $\pi^{-1}(\pi(x))$ intersects both $B^+(q)$ and $B^-(q)$. Let G^+ and G^- be horizontal 2–cells in $B^+(q)$ and $B^-(q)$, respectively, such that $\pi(G^+) = \pi(G^-)$ and $(G^+ \cup G^-) \subset N(q,\delta)$. For each x in G^+, let $I(x)$ denote the line segment in $\pi^{-1}(\pi(x))$ with endpoints in G^+ and G^-. To construct a function $f : G^+ \to \Sigma$, let $f(x)$ be the point of $I(x) \cap \Sigma$ with largest z–coordinate. For $x \in G^+$ it is clear that $I(x)$ intersects Σ at a point of $K-\operatorname{cl}(R)$ because the endpoints of $I(x)$ are in different components of $E^3 - \Sigma$ and $I(x) \subset N(q,\delta)$. Thus $f : G^+ \to K-\operatorname{cl}(R)$ is defined for each $x \in G^+$, f is clearly injective, and $f(G^+) \subset E$. To see that f is continuous at a point $x \in G^+$, let $\{x_i\}$ be a sequence of points converging to x in G^+, and note that, for i sufficiently large, $f(x_i)$ lies vertically between the two cones $B^+(f(x))$ and $B^-(f(x))$. The reason $f(x_i)$ would lie above $B^-(f(x))$ is because there is no point of $\operatorname{Int}\Sigma$ above $f(x_i)$ on the segment $I(x_i)$ and $\operatorname{Int} B^-(f(x)) \subset \operatorname{Int}\Sigma$; and $f(x_i)$ would lie below $B^+(f(x))$ because otherwise there would be a point of $\Sigma \cap I(x)$ above $f(x)$. Thus $\{f(x_i)\}$ converges to $f(x)$, and f is continuous. Since G^+ is compact, f is a homeomorphism, and it follows that the interior of the 2–cell $f(G^+)$ contains q and lies entirely in E. Then E is open in Σ, and the result follows.

A subset X of a 2–sphere Σ in E^2 is said to *locally separate* Σ if, for each point $x \in X$, there exists a positive number ϵ such that whenever K is a 2–cell of diameter less than ϵ on Σ with x in its interior it follows that $K–X$ is not connected. Corollary 3.8 follows immediately from Lemma 3.7 and an obvious global generalization of Theorem 2.1.

COROLLARY 3.8. *If a 2–sphere Σ in E^3 has a continuous family of tangent planes and the rim R of Σ is not empty, then* $\mathrm{cl}\,(R)$ *must locally separate Σ.*

COROLLARY 3.9. *If R is the rim of a 2–sphere Σ in E^3, and Σ has a continuous family of tangent planes, then*
(1) $\mathrm{cl}\,(R)$ *is one–dimensional,*
(2) $\mathrm{cl}\,(R)$ *cannot be an arc or a singleton set, and*
(3) *every point of R is a limit point of R.*

THEOREM 3.10. *If a 2–sphere Σ in E^3 has a continuous family of tangent planes and the rim R of Σ is not empty, then* $\Sigma–\mathrm{cl}\,(R)$ *is not connected.*

PROOF. Let φ be a number such that $0 < \varphi < \pi/2$. Now convert the continuous family of tangent planes to a family $\{C(q)|q \in \Sigma\}$ of double cones to Σ (the equivalence is stated in a local form as Theorem 2.1) and a corresponding continuous family $\{L(q)|q \in \Sigma\}$ of lines, where $L(q)$ is the cone exis of $C(q)$ and all of Conditions (1), (2), (3), and (4) of Lemma 3.7 are satisfied with $D = \Sigma$. Let S denote a round 2–sphere centered at the origin, and think of the projective plane P^2 as the space obtained by identifying antipodal points of S. There is a natural map $f : \Sigma \to P^2$, where $f(q)$ is obtained by first translating $L(q)$ to a parallel line $L'(q)$ which passes through the origin and then by letting $f(q) = \sigma(L'(q) \cap S)$, where σ is the projection map from S to P^2. Then, by standard techniques, f can be lifted to a continuous map $\hat{f} : \Sigma \to S$ such that $\sigma\hat{f} = f$. For each $q \in \Sigma$, let $R(q)$ be the ray on $L(q)$ with endpoint q such that the direction of $R(q)$ is the same as that of the ray from the origin through $\hat{f}(q)$. Then $\{R(q)|q \in \Sigma\}$ is a continuous family of rays.

Divide each double cone $C(q)$ into two congruent cones $C^+(q)$ and

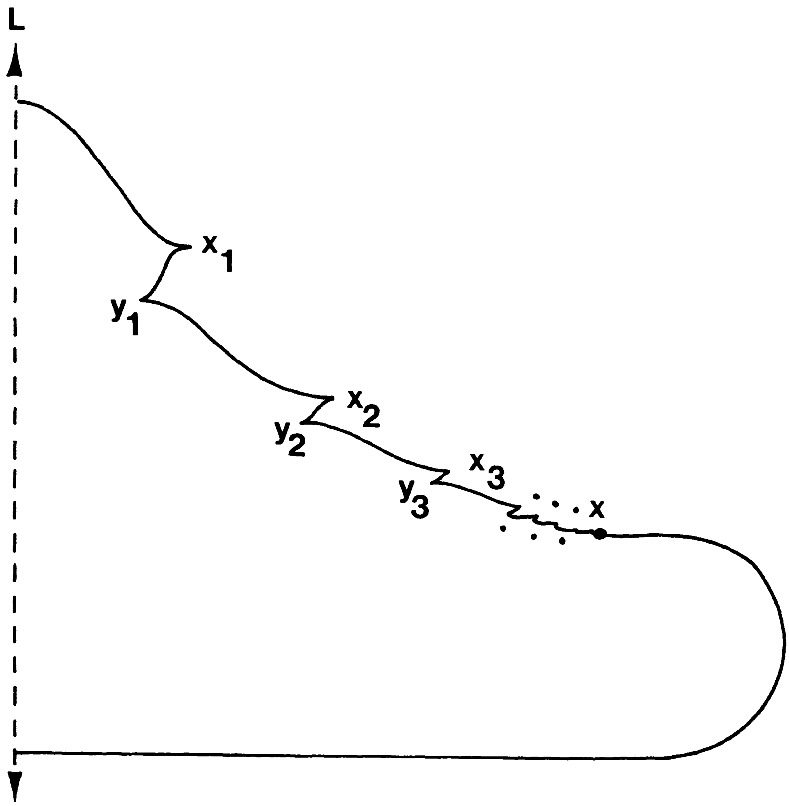

Figure 2.

$C^-(q)$ where $C^+(q)$ is the one whose axis is $R(q)$. Then define

$$E = \{q \in \Sigma - \mathrm{cl}\,(R) | \mathrm{Int}\, C^+(q) \subset \mathrm{Ext}\, \Sigma\}$$

and

$$I = \{q \in \Sigma - \mathrm{cl}\,(R) | \mathrm{Int}\, C^+(q) \subset \mathrm{Int}\, \Sigma\}.$$

It is clear that $\Sigma - \mathrm{cl}\,(R) = I \cup E, I \cap E = \emptyset$, and, as in the proof of Lemma 3.7, I and E are proven to be open. The fact that $R \neq \emptyset$ is used to prove that neither I nor E can be empty; a brief outline is given at the end of the first paragraph of the proof of Lemma 3.7. These facts show that $\Sigma - \mathrm{cl}\,(R)$ is not connected.

EXAMPLE 3.11. There exists a 2–sphere Σ in E^3 having a continuous family of tangent planes such that the rim of Σ is an infinite union of disjoint circles.

Such a 2–sphere Σ can be obtained by rotating the curve pictured in Figure 2 about the line L. Both sequences $\{x_i\}$ and $\{y_i\}$ converge to the same point x and, for each i, x_i and y_i sweep out circles M_i and K_i, respectively, under the rotation. Then $R = (\cup M_i) \cup (\cup K_i)$, and $\mathrm{cl}\,(R) = R \cup J$ where J is the circle generated by x under the rotation.

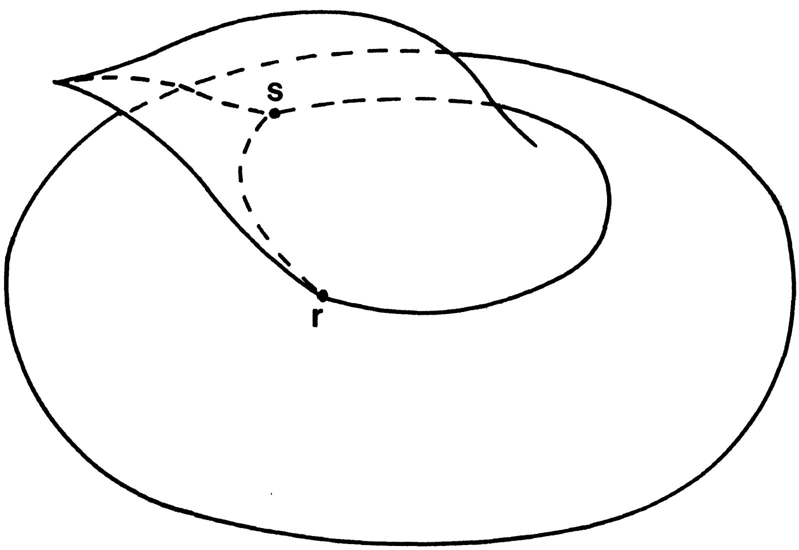

Figure 3.

EXAMPLE 3.12. There are examples of 2–spheres in E^3 having a continuous family of tangent planes such that either:

(1) R is the union of two disjoint open arcs with $\mathrm{cl}\,(R)$ a simple closed curve,

(2) $\mathrm{cl}\,(R)$ has a point or an arc as a component,

(3) there are points of R that are limit points of $\mathrm{cl}\,(R) - R$, or

(4) R is a continuum that is not locally connected.

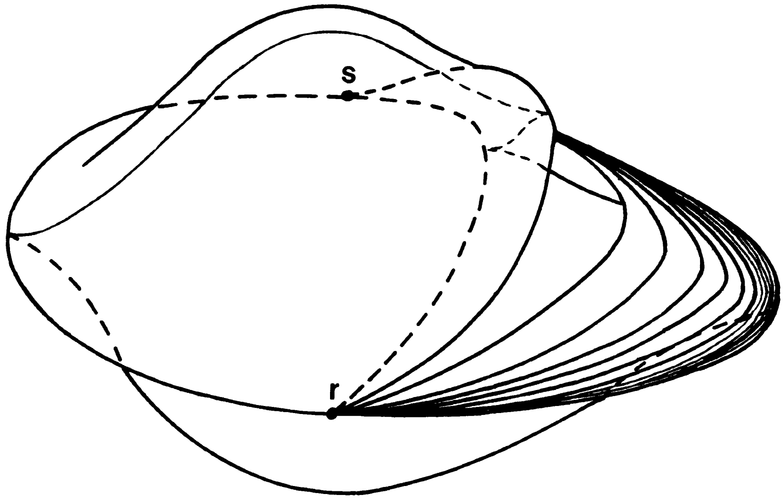

Figure 4.

The basis for parts (1), (2),and (3) is the example pictured in Figure 3 where the rim is the union of two open arcs from r to s. The example in Figure 3 satisfies (1) above, and examples for (2) and (3) are obtained from it by adding a null sequence of such "hats" converging to a point or to an arc. For (3) one must be sure the point or arc converged to lies in R, but this can be done much as in Figure 4 where the example for (4) is pictured. In Figure 4 the rim is the union of an infinite family of simple closed curves each two of which contain the horizontal semicircle with endpoints r and s.

4. The wildness of spheres with tangent planes. It is clear from Cannon's $*$–taming theory [**5**] that the wild set W of a 2–sphere Σ in E^3 having tangent planes must lie in the closure of the rim R of Σ because it follows from Theorem 2.1 that Σ can be touched with the tip of a cone from both sides of Σ at each point of $\Sigma - \mathrm{cl}\,(R)$.

THEOREM 4.1. *If U is an open subset of a 2–sphere Σ in E^3 such that Σ has tangent planes over U, then the wild set of Σ in U must lie in the closure of the rim of Σ.*

If Σ has continuous tangent planes, then from Theorem 4.1 and Corollary 3.9 it follows that the dimemsion of W is less than two. Example 4.2 shows a 2–sphere with continuous tangent planes such that dimension $W = 1$, and Burgess' original example [**2**] shows such a 2–sphere with dimension $W = 0$.

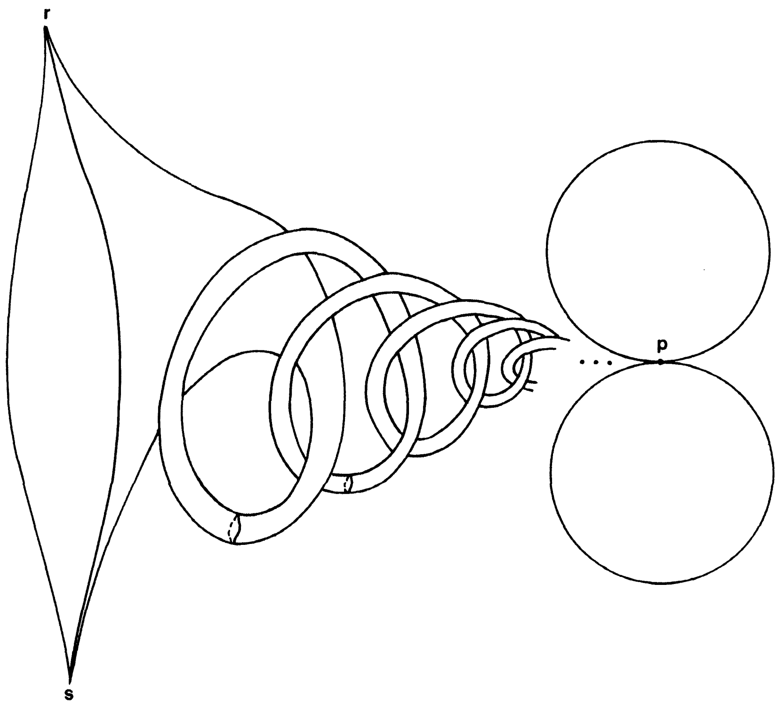

Figure 5.

EXAMPLE 4.2. There exists a 2–sphere Σ in E^3 having a continuous family of tangent planes such that the wild set of Σ is 1–dimensional and $\mathrm{cl}\,(R)$ is an infinite union of disjoint simple closed curves.

Depicted in Figure 6, this example is constructed by modifying Example 3.11 using a null sequence of attached wild disks as pictured in Figure 5. The wild Fox–Artin disk pictured in Figure 5 is a slight modification of the example given by Burgess [**2**]. One should visualize the two balls that are tangent at p as being perpendicular to the page so that the plane of the paper is tangent to the disk at p. The rim of this disk is an arc through p with endpoints r and s. The desired example is obtained by attaching 2^i of these disks to Σ along small neighborhoods of 2^i disjoint arcs on each of the circles M_i of Example 3.11. The locations of these attached disks are pictured as small bumps in Figure 6. In this modification the circle M_i becomes a nearly horizontal simple closed curve J_i which contains exactly 2^i wild points like p. The construction puts every point of J in the limiting set of the wild points so that J itself belongs to W. In this example, $R = (\cup J_i) \cup (\cup K_i)$ and $\mathrm{cl}\,(R) = R \cup J$.

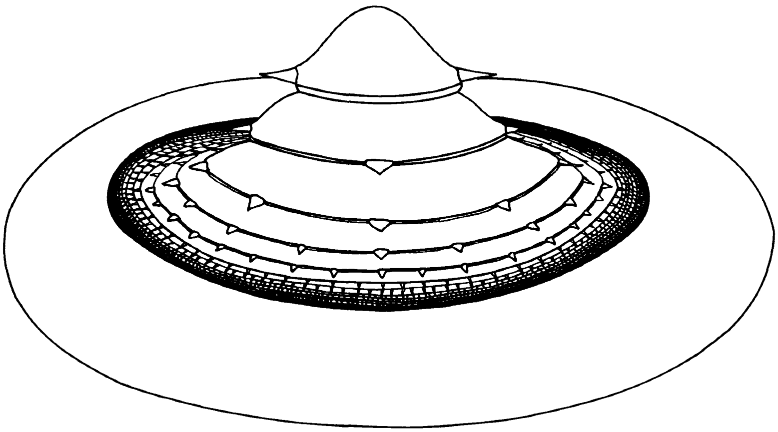

Figure 6.

THEOREM 4.3. *If U is an open subset of a 2–sphere Σ in E^3 such that Σ has continuous tangent planes over U, and W is the set of wild points of Σ, then the dimension of $W \cap U$ is less than two.*

The "continuity" of the family of tangent planes in the hypothesis is necessary because there are 2–spheres in E^3 that are wild at every point and that have tangent planes everywhere. Such an example can be constructed inductively by attaching a dense set of points like p in Figure 3 to a 2–sphere; the details are left to the reader.

In contrast to the double cones stressed in most of this paper, the next theorem deals with just single cones. A (single) *cone* of height h is a set isometric with $\{(x, y, z)|x^2 + y^2 \leq kz^2$ where $k > 0$ and $0 \leq z \leq h\}$, and a subset X of E^3 is said to be *touched by the tip of such a cone* C at a point x of X if $X \cap (\operatorname{Int} C) = \emptyset$ and x is the vertex of C. Wright [**12**] proved that a 0– dimensional compact subset X of E^3 is tamely embedded in E^3 if it can be touched at each of its points by the tip of a cone. Of course this weak cone condition will not tame a 2–sphere; a reasonable embedding of the Fox–Artin [**8**] sphere can be touched by the tip of a cone at each of its points but is wild. However, the following theorem demonstrates that a stronger cone condition is enough to imply the tameness of subsets of 2–spheres in E^3, and it leads to a taming theorem using continuous tangent planes.

THEOREM 4.4. *If a 2–sphere Σ can be touched by the tip of a cone from a continuous family F of congruent cones at each point of a subset X of Σ, then each point x of X lies in a neighborhood N of x such that $N \cap X$ lies on a tame 2–sphere in E^3.*

PROOF. Let $x \in X$, let $C(x)$ be a cone from F with vertex x, let p be the centroid of $C(x)$, and let B be a round ball centered at p that lies in $\operatorname{Int} C(x)$. Because F is continuous, the union G of all cones from F containing B contains a neighborhood N of x in X. The radial map from p is a homeomorphism from $Bd\ B$ to $Bd\ G$, so G is a starlike 3–cell whose boundary contains N.

THEOREM 4.5. *If a 2–sphere Σ is wildly embedded in E^3, Q is the set of all points of the wild set W of Σ that are components of W, and Σ has a family of continuous tangent planes over W, then Q is dense in W.*

PROOF. Suppose there is a point w of W that does not belong to the closure $\operatorname{cl}(Q)$ of Q, and let G be a 2–cell in Σ such that $w \in \operatorname{Int} G$ and $G \cap \operatorname{cl}(Q) = \emptyset$. Let 2θ be fixed. From Theorem 2.1 it follows that for each $x \in W$ there is double cone C_x with cone angle 2θ and vertex x whose axis is orthogonal to the plane tangent to Σ at x such that $(\operatorname{Int} C_x) \cap \Sigma = \emptyset$. Let $X_i = \{p \in W \cap G|$ there exists a double cone C_p, with cone angle 2θ and vertex p, whose axis is orthogonal to the plane tangent to Σ at p such that C_p has height at least $1/i$ and $(\operatorname{Int} C_p) \cap \Sigma = \emptyset\}$. Then X_i is closed, for each i, and $(G \cap W) \subset \cup_{i=1}^{\infty} X_i$. A Baire category theorem applies to yield a 2–cell K in G and an integer m such that $\emptyset \neq (K \cap W) \subset X_m$. Now $W \cap K$ satisfies the conditions on X specified in the hypothesis of Theorem 4.4, so there exists a 2–cell H in K such that $W \cap \operatorname{Int} H \neq \emptyset$ and $H \cap W$ lies on a flat 2–sphere. But Theorem 1.1 of [**4**] contradicts the fact that the wild set of Σ can intersect the interior of H.

COROLLARY 4.6. *If W is the wild set of a 2–sphere Σ in E^3, Σ has a continuous family of tangent planes over W, and W contains at least two points, then W is not connected.*

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DEPARTMENT OF MATHEMATICS, UTAH STATE UNIVERSITY, LOGAN, UT 84322