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COMPLETENESS PROPERTIES OF HYPERSPACES OF COMPACT FUZZY SETS

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0. Introduction. On an arbitrary uniform space there are two types of "compactlike" fuzzy sets which are widely used in applications: u.s.c. fuzzy sets with compact support (we denote this collection $\Phi_c(X)$) and u.s.c. fuzzy sets with compact levelsets (we denote this collection $\Phi_W(X)$) [**2**], [**12**]. Always $\Phi_c(X) \subset \Phi_W(X)$ but the converse holds only if X itself is compact.

In the first part of our paper we prove that for the global fuzzy hyperspace structure [**8**], [**9**] the completeness of X is equivalent to the completeness of $\Phi_c(X)$ and to either the completeness or the ultracompleteness of $\Phi_W(X)$ [**6**], [**7**].

In the second part we then prove the rather surprising result that the completion of $\Phi_c(X)$ [**7**] is isomorphic to $\Phi_W(\hat{X})$ where $\hat{X}$ denotes the completion of X.

These results not only generalize K. Morita's results on hyperspace of compact subsets [**11**] to the setting of fuzzy hyperspaces of "compact-like" fuzzy subsets but moreover via the isomorphism of the uniform modification of $\Phi_c(X)$ and $\Phi_W(X)$ with hyperspaces of closed subsets of $X \times [0,1]$ [**9**], they also include an extension of K. Morita's classical results to classes of closed subsets of $X \times [0,1]$ which are in general not compact.

1. Preliminaries. In this section we shall recall notations and basic concepts which are used throughout the rest of the paper.

I denotes the unit interval, I_0 stands for $]0,1]$ and I_1 stands for $[0,1[$.
The characteristic function of a subset $Y \subset X$ is denoted 1_Y.

If X is a topological space then contrary to usual notation in hyperspace theory we shall put 2^X for all subsets of X and $\mathcal{F}(X)$ for all closed subsets of X [**9**].

For notations and basic results on prefilters and convergence we refer

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the reader to [3], [6]. We recall however that if $(X, \mathfrak{U})$ is a fuzzy uniform space [4] then a hyper–Cauchy prefilter is a prefilter $\mathfrak{C}$ fulfilling

(HC1) $c(\mathfrak{C}) = 1$;

(HC2) $\forall \varepsilon \in I_0 \; \exists \nu \in \mathfrak{C} : \nu - \varepsilon \leq \mu \Rightarrow \mu \in \mathfrak{C}$;

(HC3) $\forall \nu \in \mathfrak{U} \; \forall \varepsilon \in I_0 \; \exists \mu \in \mathfrak{C} : \mu \times \mu - \varepsilon \leq \nu$ (see [6]).

The set of all minimal hyper–Cauchy prefilters is denoted $\mathcal{M}(X)$. If $\mathfrak{C}$ and $\mathfrak{G}$ are any two prefilters then we put $c(\mathfrak{C}, \mathfrak{G}) = 0$ if $\mathfrak{C} \vee \mathfrak{G}$ does not exist and $c(\mathfrak{C}, G) = c(\mathfrak{C} \vee \mathfrak{G})$ otherwise (for the definition of c and c^- too we refer [3], [5]).

A prefilter $\mathfrak{F}$ is then called a Cauchy prefilter if it fulfills

$$\sup_{\mathfrak{C} \in \mathcal{M}(X)} \inf_{\mathfrak{G} \in \mathcal{P}_m(\mathfrak{F})} c(\mathfrak{C}, \mathfrak{G}) = c^-(\mathfrak{F}).$$

We recall from [6] that hyper Cauchy prefilters and convergent prefilters (i.e., prefilters $\mathfrak{F}$ such that $\sup_{x \in X} \lim \mathfrak{F}(x) = c^-(\mathfrak{F})$) are Cauchy.

A fuzzy uniform space is called complete [6] if every Cauchy prefilter converges; it is called ultracomplete if every hyper–Cauchy prefilter contains a prefilter $\mathfrak{U}(x)$ for some $x \in X$.

Ultracomplete spaces are complete, the converse need however not be true [7].

A fuzzy uniform space is called weakly Hausdorff if it fulfils WT_2 [13], i.e., for any $x, y \in X$, $x \neq y$ there exists $\nu \in \mathfrak{U}$ such that $\nu(x, y) < 1$.

In this work we shall be occupied mainly with the fuzzy uniform hyperspace of uppersemicontinuous fuzzy sets on a classical uniform space $(X, \mathcal{U})$, i.e., with the space $\Phi_{gl}(X)$ [9]. Since we shall only work with the global structure and not with the horizontal structure of [8] we shall moreover always drop the suffix gl in our notations. Our main interest lies in two particular subspaces of $\Phi(X)$. First we consider the subspace $\Phi_c(X)$ of those fuzzy sets in $\Phi(X)$ which have compact support [2], i.e.,

$$\Phi_c(X) := \{\mu \in \Phi(X) | \overline{\mu^{-1}]0, 1]} \text{ compact}\},$$

and second we consider the subspace $\Phi_W(X)$ of so–called Weiss–compact fuzzy sets in $\Phi(X)$ [12], i.e., which have compact nonzero levelsets,

$$\Phi_W(X) := \{\mu \in \Phi(X) | \forall \alpha \in I_0 : \mu^{-1}[\alpha, 1] \text{ compact}\}.$$

Finally we recall that if $U \in \mathcal{U}$ and $\alpha \in I_0$ then we put

$$B_\alpha := \{(s,t) | |s-t| < \alpha\}$$
$$U \otimes B_\alpha := \{((x,s),(y,t)) | (x,y) \in U, (s,t) \in B_\alpha\}.$$

As in **[9]** if U is an entourage on a basic space then $\tilde{U}$ denotes the induced entourage on the hyperspace and as in **[1]** $\hat{U}$ denotes the induced entourage on the completion.

2. Completeness properties of $\mathbf{\Phi_c(X)}$ and $\mathbf{\Phi_W(X)}$. We begin by stating the theorem which we intend to prove in this section

THEOREM 2.1. *The following are equivalent:*
(1) X *is complete*;
(2) $\Phi_W(X)$ *is ultracomplete*;
(3) $\Phi_W(X)$ *is complete*;
(4) $\Phi_c(X)$ *is complete.*

For clarity we shall scatter the proof of this theorem over a number of propositions.

In order to prove the first of these propositions we have to make some notational conventions and preliminary observations.

We know that the uniform modification of $\Phi(X)$ is isomorphic to a closed subspace of the uniform hyperspace of all closed sets in $X \times I$ (Theorem 5.2 **[9]**) and it will be advantageous to exploit this isomorphism. It is given by the map

$$\mathcal{G} : \Phi(X) \to \mathcal{F}(X \times I)$$
$$\mu \to \mathcal{G}(\mu)$$

where $\mathcal{G}(\mu) := \{(x,t) | t \leq \mu(x)\}$ and where $\mathcal{F}(X \times I)$ is equipped with the Hausdorff–Bourbaki hyperspace structure on closed sets **[10]**. Now if Ψ is a filter on $\mathcal{G}(\Phi_W(X))$ then we shall associate with it a filter on $X \times I$ in the following way.

Put

$$\sum : 2^{\mathcal{G}(\Phi_W(X))} \to 2^{X \times I}$$
$$\mathbf{F} \to \sum(\mathbf{F}) := \bigcup_{F \in \mathbf{F}} F$$

and define

$$\Sigma(\Psi) := \left[\{\Sigma(\mathbf{F}) | \mathbf{F} \in \Psi\}\right].$$

That $\Sigma(\psi)$ is indeed a filter on $X \times I$ is an easy verification which we leave to the reader.

We shall also require a measure of the extent to which ψ contains "vertically" small members. Hereto we define

$$t(\Psi) := \sup\left\{\varepsilon \in I_0 | X \times [0, \varepsilon] \notin \Sigma(\Psi)\right\}.$$

Remark that if $\mathfrak{F}$ is the prefilter on $\Phi_W(X)$ corresponding to Ψ then actually $t(\Psi) = s(\mathfrak{F})$ as in [**9**].

Finally we shall also adhere to the following notational convention. If $A \subset X \times I$ and $\alpha \in I_0$ then

$$A^\alpha := \{x \in X | \exists t \geq \alpha : (x, t) \in A\}.$$

In case A is the endograph of some fuzzy set μ then A^α is nothing else than $\mu^{-1}[\alpha, 1]$.

PROPOSITION 2.2. *If X is complete then $\Phi_W(X)$ is ultracomplete.*

REMARK. The proof of this result is heavily inspired by the paper [**11**] of K. Morita.

PROOF. By Theorem 5.2 [**9**] is suffices to prove that $\mathcal{G}(\Phi_W(X))$ is complete.

Let Ψ be a Cauchy filter on $\mathcal{G}(\Phi_W(X))$.

Case 1. $t(\Psi) = 0$. Fix $U \in {}_S\mathcal{U}$ and $\alpha \in I_0$. Then we can find $\mathbf{F} \in \Psi$ such that

$$\begin{aligned}\Sigma(\mathbf{F}) &\subset X \times [0, \alpha[\\ &\subset (U \otimes B_\alpha)(X \times \{0\})\end{aligned}$$

which implies that for all $F \in \mathbf{F}$

$$(F, X \times \{0\}) \in \widetilde{U \otimes B_\alpha}.$$

By arbitrariness of $U \in {}_s\mathcal{U}$ and $\alpha \in I_0$ this proves that $\Psi \to X \times \{0\}$. Note that $t(\Psi) = 0$ obviously already implies that Ψ is Cauchy.

Case 2. $t(\psi) > 0$.

Assertion 1. For any $\varepsilon < t(\Psi)$, any ultrafilter finer than $\Sigma(\Psi)$ which does not contain $X \times [0, \varepsilon]$ is Cauchy.

From the fact that $X \times [0, \varepsilon] \notin \Sigma(\Psi)$ it is clear that an ultrafilter, say $\mathcal{M}$, fulfilling the suppositions exists. We shall prove $\mathcal{M}$ is Cauchy. Let $U \in {}_s\mathcal{U}$ and $\alpha \in I_0$ be fixed, put

$$4\beta := \alpha \wedge \varepsilon$$

and choose $W, V \in {}_s\mathcal{U}$ such that

$$W \circ W \subset U$$
$$V \circ V \subset W$$

Take $\mathbf{F} \in \Psi$ such that

$$\mathbf{F} \times \mathbf{F} \subset \widetilde{V \otimes B_\beta}$$

and take $F \in \mathbf{F}$. Then

$$\Sigma(\mathbf{F}) \subset V \otimes B_\beta(F) \tag{2.1}$$

and consequently

$$V \otimes B_\beta(F) \in \Sigma(\Psi) \subset \mathcal{M} \tag{2.2}$$

Since $(F^{2\beta} \times [2\beta, 1]) \cap F$ is compact it contains a finite subset S such that

$$(F^{2\beta} \times [2\beta, 1]) \cap F \subset \bigcup_{(x,t) \in S} V \otimes B_\beta(x, t)$$

which implies

$$F \subset \Big(\bigcup_{(x,t) \in S} V \otimes B_\beta(x, t) \Big) \cup (X \times [0, 2\beta]),$$

and consequently

$$V \otimes B_\beta(F) \subset \Big(\bigcup_{(x,t) \in S} W \otimes B_{2\beta}(x, t) \Big) \cup (X \times [0, 3\beta]). \tag{2.3}$$

Since $X \times [0, 3\beta] \notin M$ it follows from (2.2) and (2.3) that there exists $(x, t) \in S$ such that

$$W \otimes B_{2\beta}(x, t) \in \mathcal{M}$$

Finally since

$$W \otimes B_{2\beta}(x, t) \times W \otimes B_{2\beta}(x, t) \subset U \otimes B_\alpha$$

and by the arbitrariness of $U \in {}_s\mathcal{U}$ and $\alpha \in I_0$ this proves $\mathcal{M}$ is indeed Cauchy.

Assertion 2. $K := \bigcap_{L \in \Sigma(\Psi)} \overline{L} \in \mathcal{G}(\Phi_W(X))$.

That K is an endograph is easily seen so that we only need to show it has compact "levelsets."

Let $\delta \in I_0$ be fixed. Then in order to show this it will suffice to prove that $(K^\delta \times [\delta, 1]) \cap K$ is compact which, in turn, by completeness of $X \times I$ and the obvious fact that $(K^\delta \times [\delta, 1]) \cap K$ is closed means it is sufficient to show precompactness.

Let $U \in {}_s\mathcal{U}$ and $\alpha \in I_0$ be fixed, choose $\varepsilon < t(\Psi) \wedge \delta$ and let $\beta \in I_0$, $V, W \in {}_s\mathcal{U}, \mathbf{F} \in \Psi, F \in \mathbf{F}$ and $S \subset (F^{2\beta} \times [2\beta, 1]) \cap F$ be as in the proof of Assertion 1. Then we have from (2.1) and (2.3)

$$\begin{aligned} K &\subset \overline{\Sigma(\mathbf{F})} \subset \overline{V \otimes B_\beta(F)} \\ &\subset \Big(\bigcup_{(x,t) \in S} \overline{W \otimes B_{2\beta}(x, t)} \Big) \cup (X \times [0, 3\beta]). \end{aligned}$$

Since $\varepsilon < \delta$ this implies

$$\begin{aligned} (K^\delta \times [\delta, 1]) \cap K &\subset \bigcup_{(x,t) \in S} \overline{W \otimes B_{2\beta}(x, t)} \\ &\subset \bigcup_{(x,t) \in S} U \otimes B_\alpha(x, t), \end{aligned}$$

which proves the second assertion.

Assertion 3. $\Psi \to K$.

Let again $U \in {}_s\mathcal{U}$ and $\alpha \in I_0$ be fixed and $V \in {}_s\mathcal{U}$ be such that $V \circ V \subset U$. Put $2\beta := \alpha$. Since Ψ is Cauchy we can find $\mathbf{F} \in \Psi$ such that for all $F \in \mathbf{F}$

$$\Sigma(\mathbf{F}) \subset V \otimes B_\beta(F),$$

which in turn implies that for all $(x, s) \in \Sigma(\mathbf{F})$ and for all $F \in \mathbf{F}$ there exists $(y, t) \in F$ such that $((x, s), (y, t)) \in V \otimes B_\beta$.

Now $11x(x, s) \in \Sigma(\mathbf{F})$ then for any $\mathbf{L} \in \Psi$ choosing $F \in \mathbf{F} \cap \mathbf{L}$ we thus have

$$F \cap V \otimes B_\beta(x, s) \neq \emptyset$$

and consequently

$$\Sigma(\mathbf{L}) \cap V \otimes B_\beta(x, s) \neq \emptyset.$$

Together with the obvious fact that for any $\mathbf{L}, \mathbf{K} \in \Psi : \Sigma(\mathbf{L}) \cap \Sigma(\mathbf{K}) = \Sigma(\mathbf{L} \cup \mathbf{K})$, this proves that

$$\mathcal{F}(x, s) := \left[\{\Sigma(\mathbf{L}) \cap V \otimes B_\beta(x, s) | \mathbf{L} \in \Psi\}\right]$$

is a well defined filter on $X \times I$, which by construction is moreover finer than $\Sigma(\Psi)$.

The adherence of this filter is nonempty; indeed if $s \leq \beta$ then obviously

$$(x, 0) \in \bigcap_{\mathbf{L} \in \Psi} \overline{\Sigma(\mathbf{L}) \cap V \otimes B_\beta(x, s)}$$

so that we may now suppose $s > \beta$. In that case since for any $(y, t) \in V \otimes B_\beta(x, s)$ we have $|t - s| < \beta$ it follows that if we put $\varepsilon := s - \beta$ then $X \times [0, \varepsilon] \notin \mathcal{F}(x, s)$.

If $\mathcal{M}$ is an ultrafilter finer than $\mathcal{F}(x, s)$ and which does not contain $X \times [0, \varepsilon]$ then since also $\mathcal{M} \supset \Sigma(\Psi)$ it follows from Assertion 1 that $\mathcal{M}$ is Cauchy and thus by completeness of $X \times I$ it has a nonempty adherence. Since $\mathcal{F}(x, s) \subset \mathcal{M}$ this proves our claim.

Consequently for any $(x, s) \in \Sigma(\mathbf{F})$ we now have

$$\begin{aligned} K \bigcap \overline{V \otimes B_\beta(x, s)} &= \Big(\bigcap_{L \in \Sigma(\Psi)} \overline{L}\Big) \bigcap \overline{V \otimes B_\beta(x, s)} \\ &\supset \bigcap_{L \in \Sigma(\Psi)} \overline{L \cap V \otimes B_\beta(x, s)} \neq \emptyset, \end{aligned}$$

and consequently for any $(x, s) \in \Sigma(\mathbf{F})$:

$$K \cap U \otimes B_\alpha(x, s) \neq \emptyset,$$

which in turn implies that for any $F \in \mathbf{F}$:

$$F \subset U \otimes B_\alpha(K). \tag{2.5}$$

On the other hand by (2.4) and also for any $F \in \mathbf{F}$

$$\begin{aligned} K \subset \overline{\Sigma(\mathbf{F})} &\subset \overline{V \otimes B_\beta(F)} \\ &\subset U \otimes B_\alpha(F). \end{aligned} \tag{2.6}$$

Together (2.5) and (2.6) prove that for all $F \in \mathbf{F}$:

$$(K, F) \in U \widetilde{\otimes} B_\alpha$$

which by the arbitrariness of $U \in {}_s\mathcal{U}$ and $\alpha \in I_0$ shows $\Psi \to K$.

This ends the proof of the proposition.

PROPOSITION 2.3. *If $\Phi_W(X)$ is complete then $\Phi_c(X)$ is complete.*

PROOF. Let $\mathfrak{F}$ be a Cauchy prefilter on $\Phi_c(X)$ and let $\langle\mathfrak{F}\rangle$ denote the prefilter which it generates on $\Phi_W(X)$. We leave it to the reader to verify that this prefilter is again Cauchy (on $\Phi_W(X)$) and that $c^-(<\mathfrak{F}>) = c^-(\mathfrak{F})$. This goes by straightforward verification. If $c^-(\mathfrak{F}) = 0$ there is nothing to prove.

Let $c^-(\mathfrak{F}) > 0$ and let $\varepsilon \in]0, c^-(\mathfrak{F})[$. By completeness of $\Phi_W(X)$ we can find $\mu \in \Phi_W(X)$ such that

$$\lim <\mathfrak{F}> (\mu) \geq c^-(\mathfrak{F}) - \varepsilon.$$

Now in case $\sup \mu \leq \varepsilon$ put $\mu_\varepsilon := 0$ and in case $\sup \mu > \varepsilon$ put $\mu_\varepsilon := \mu \wedge 1_{\mu^{-1}[\varepsilon,1]}$.

In both cases $\mu_\varepsilon \in \Phi_c(X)$ and $e(\mu, \mu_\varepsilon) \leq \varepsilon$. Consequently from Lemma 4.9 [**9**] and Remark (2) following the proof of Theorem 4.3 (c) [**9**], it follows that first

$$\lim \omega(\iota(\langle\mathfrak{F}\rangle))(\mu) \geq c^-(\mathfrak{F}) - \varepsilon$$

and consequently that second

$$\begin{aligned} \lim\langle\mathfrak{F}\rangle(\mu_\varepsilon) &= c^-(\mathfrak{F}) \wedge \lim \omega(\iota(\langle\mathfrak{F}\rangle))(\mu_\varepsilon) \\ &\geq c^-(\mathfrak{F}) \wedge \lim \omega(\iota(\langle\mathfrak{F}\rangle))(\mu) \wedge (1 - e(\mu, \mu_\varepsilon)) \\ &\geq c^-(\mathfrak{F}) \wedge (c^-(\mathfrak{F}) - \varepsilon) \wedge (1 - \varepsilon) = c^-(\mathfrak{F}) - \varepsilon. \end{aligned}$$

Since obviously $\lim \mathfrak{F}(\mu_\varepsilon) = \lim\langle\mathfrak{F}\rangle(\mu_\varepsilon)$ this proves that

$$\sup_{\xi \in \Phi_c(X)} \lim \mathfrak{F}(\xi) = c^-(\mathfrak{F})$$

and thus $\Phi_c(X)$ is complete.

In the next proposition we shall denote φ the canonical injection of X into $\Phi_c(X)$, i.e., $\varphi(x) := 1_{\{x\}}$ (see also [**9**]).

PROPOSITION 2.4. *If $\Phi_c(X)$ is complete then X is complete.*

PROOF. If $\mathcal{F}$ is a Cauchy filter on X then a straightforward verification shows that $\omega(\widetilde{\varphi(\mathcal{F})})$ is hyper Cauchy and consequently Cauchy on $\Phi_c(X)$. Thus we have

$$\sup_{\xi \in \Phi_c(X)} \lim \omega(\widetilde{\varphi(\mathcal{F})})(\xi) = 1.$$

By Theorem 6.1 [**9**] this implies there exists $\xi \in \Phi(X)$ such that

$$\lim \omega(\widetilde{\varphi(\mathcal{F})})(\xi) = 1.$$

Again staightforward verification shows this is equivalent to

$$\lim \omega(\varphi(\mathcal{F}))(\xi) = 1$$

(actually this equivalence holds in any fuzzy neighborhood space) and consequently by Theorem 4.2 [**9**] we have

$$\varphi(\mathcal{F}) \to \xi \text{ in } \iota_u(\Phi(U)).$$

But since $1_{\varphi(X)} \in \varphi(\mathcal{F})$ and $\varphi(X)$ is $\iota_u(\Phi(U))$–closed it follows that $\xi \in \varphi(X)$, i.e., there exists $x \in X$ such that $\xi = 1_{\{x\}}$. Clearly $\mathcal{F} \to x$.

3. Completion of $\mathbf{\Phi_C(X)}$. In the previous section we have seen that $\Phi_c(X)$ is complete if and only if X is complete. However since a complete space need not be ultracomplete and since the completion constructed in [**7**] is automatically ultracomplete, the question–even for a complete X–poses itself whether we can describe the completion of $\Phi_c(X)$ in a concise concrete way. That this is indeed the case shall be shown in this section

Let X be arbitrary, i.e., not necessarily complete and let $\hat{X}$ be its completion.

From the elementary fact that compactness is absolute, i.e., independent of the superspace, the following map is well defined

$$i : \Phi_c(X) \to \Phi_W(\hat{X}),$$

where $\iota(\mu)(x) = \mu(x)$ if $x \in X$ and $i(\mu)(x) = 0$ if $x \notin X$. Remark moreover that actually $i(\Phi_c(X)) \subset \Phi_c(\hat{X})$ and that i is an embedding; in particular for any $U \in {}_s\mathcal{U}$ and $\alpha \in I_0$ we have

$$(i \times i)^{-1}(\omega_{(\hat{U},\alpha)}) = \omega_{(U,\alpha)}.$$

THEOREM 3.1. *In the category of fuzzy uniform spaces and maps the pair* $(i, \Phi_W(\hat{X})$ *is universal for* $\Phi_c(X)$ *with respect to the full subcategory of weakly Hausdorff ultracomplete spaces, i.e.,*

$$\widehat{\Phi_c(X)} \approx \Phi_W(\hat{X}).$$

REMARK. That the first claim of the theorem implies the isomorphism between $\widehat{\Phi_c(X)}$ and $\Phi_W(\hat{X})$ is a purely categorical result which follows immediately from the results of [**7**].

PROOF. In order to verify this first claim let $(Y, \mathfrak{U})$ be a weakly Hausdorff ultracomplete fuzzy uniform space and let

$$\Phi_c(X) \xrightarrow{f} Y$$

be a uniformly continuous map.

We shall prove that there exists a unique uniformly continuous factorization $\hat{f}$ over $\Phi_W(\hat{X})$, i.e., such that $\hat{f} \circ i = f$.

Step 1. *Construction of* $\hat{f}$.

Let $\mu \in \Phi_W(\hat{X})$ be fixed.

Assertion 1. For any $U \in {}_s\mathcal{U}$ and $\alpha \in I_0$ there exists $\mu_{(U,\alpha)} \in \Phi_c(X)$ such that $\omega_{(\hat{U},\alpha)}(i(\mu_{(U,\alpha)}),\mu) = 1$. Indeed in case $\mu^{-1}[\alpha,1] = \emptyset$ put $\mu_{(U,\alpha)} := 0$. Then

$$\omega_{(\hat{U},\alpha)}(i(\mu_{(U,\alpha)}),\mu) \geq (1 - e(i(\mu_{(U,\alpha)}),\mu) + \alpha) \wedge 1 = 1.$$

In case $\mu^{-1}[\alpha,1] \neq \emptyset$, for any $k \in \mathbf{N}_0$ put

$$C_k := \mu^{-1}[k\alpha,(k+1)\alpha[$$

and let

$$K := \{k \in \mathbf{N}_0 | C_k \neq 0\}.$$

Choose $V \in {}_s\mathcal{U}$ such that $V \circ V \subset U$. Then by the precompactness of C_k we can find a finite set of distinct points

$$x_1^k,\ldots,x_{n(k)}^k \in C_k$$

such that

$$C_k \subset \bigcup_{j=1}^{n(k)} \hat{V}(x_j^k).$$

Since $\hat{X}$ is Hausdorff we can find $W \in {}_s\mathcal{U}$, $W \subset V$ such that

$$\hat{W}(s) \cap \hat{W}(t) = \emptyset$$

for all $s,t \in \bigcup_{k\in K}\{x_1^k,\ldots,x_{n(k)}^k\}, s \neq t$. By denseness of X we can then pick $y_j^k \in \hat{W}(x_j^k)$ for each $j \in \{1,\ldots,n(k)\}$ and $k \in K$.

It follows that all points y_j^k are distinct and that for each $k \in K$

$$C_k \subset \bigcup_{j=1}^{n(k)} \hat{U}(y_j^k).$$

Consequently the following fuzzy set

$$\mu_{(U,\alpha)}(x) := \begin{cases} \mu(x_j^k) & x = y_j^k \\ 0 & \text{elsewhere on } X \end{cases}$$

is well defined and clearly belongs to $\Phi_c(X)$. Now if $x \notin \bigcup_{k\in K}\{y_j^k | j = 1, \dots, n(k)\}$ then

$$i(\mu_{(U,\alpha)}) - \alpha \leq 0 \leq 1_{\hat{U}} < \mu > (x),$$

whereas if $x = y_j^k$ for some $k \in K$ and $j \in \{1, \dots, n(k)\}$ then

$$\begin{aligned} i(\mu_{(U,\alpha)})(y_j^k) - \alpha &\leq \mu(x_j^k) \wedge 1_{\hat{U}}(x_j^k, y_j^k) \\ &\leq 1_{\hat{U}}\langle\mu\rangle(y_j^k), \end{aligned}$$

which shows that $i(\mu_{(U,\alpha)}) - \alpha \leq 1_{\hat{U}}\langle\mu\rangle$. On the other hand if $x \notin \mu^{-1}[\alpha, 1]$ then

$$\mu(x) - \alpha \leq 0 \leq 1_{\hat{U}}\langle\mu_{(U,\alpha)}\rangle,$$

whereas if $x \in \mu^{-1}[\alpha, 1]$ then, taking $k \in K$ such that $x \in C_k$, we can find $j \in \{1, \dots, n(k)\}$ such that also $x \in \hat{U}(y_j^k)$ and it follows that

$$\begin{aligned} \mu(x) - \alpha &\leq k\alpha \leq \mu(x_j^k) \\ &= i(\mu_{(U,\alpha)})(y_j^k) \wedge 1_{\hat{U}}(y_j^k, x) \leq 1_{\hat{U}}\langle i(\mu_{(U,\alpha)})\rangle(x), \end{aligned}$$

which shows that also $\mu - \alpha \leq 1_{\hat{U}}\langle i(\mu_{(U,\alpha)})\rangle$. This proves Assertion 1.

Now for any $W \in {}_s\mathcal{U}$ and $\theta \in I_0$ let

$$F^{\mu}_{(W,\theta)} := \{\mu_{(U,\alpha)} | U \subset W, \alpha \leq \theta\}$$

and put

$$\mathcal{F}(\mu) := [\{F^{\mu}_{(W,\theta)} | W \in {}_s\mathcal{U}, \theta \in I_0\}].$$

Assertion 2. $\omega(\widetilde{\mathcal{F}(\mu)})$ is a hyper Cauchy prefilter on $\Phi_c(X)$.

Indeed, that $\omega(\widetilde{\mathcal{F}(\mu)})$ is a prefilter and that both (HC1) and (HC2) are fulfilled is clear by construction.

To prove (HC3) let $U \in {}_s\mathcal{U}$ and $\alpha \in I_0$ and take $V \in {}_s\mathcal{U}, V \circ V \subset U$ and $2\beta := \alpha$.

If $\mu_{(V',\beta')}, \mu_{(V'',\beta'')} \in F^{\mu}_{(V,\beta)}$ then if follows from Assertion 1 and the remarks following the definition if i that

$$\begin{aligned} &\omega_{(U,\alpha)}(\mu_{(V',\beta')}, \mu_{(V'',\beta'')}) \\ &\quad = \omega_{(\hat{U},\alpha)}(i(\mu_{(V',\beta')}), i(\mu_{(V'',\beta'')})) \\ &\quad \geq \omega_{(\hat{V},\beta)}(i(\mu_{(V',\beta')}), \mu) \wedge \omega_{(\hat{V},\beta)}(\mu, i(\mu_{(V'',\beta'')})) = 1 \end{aligned}$$

and thus

$$1_{F^{\mu}_{(V,\beta)}} \times 1_{F^{\mu}_{(V,\beta)}} \leq \omega_{(U,\alpha)},$$

which proves Assertion 2.

Now by the uniform continuity of f and the fact that $(Y,\mathfrak{U})$ is a weakly Hausdorff ultracomplete space it follows from Proposition 5.4 [7] that

$$f(\omega(\widetilde{\mathcal{F}(\mu)}))$$

is hyper Cauchy on Y that there exists a unique point $y_\mu \in Y$ such that

$$\mathfrak{U}(y_\mu) \subset f(\omega(\widetilde{\mathcal{F}(\mu)}))$$

or by Lemma 8.1 [7] equivalently, such that

$$\operatorname{adh} f(\omega(\widetilde{\mathcal{F}(\mu)}))(y_\mu) = 1.$$

Define

$$\hat{f}(\mu) := y_\mu.$$

Step 2. $\hat{f} \circ i = f$.

Let $\mu \in \Phi_c(X)$ then based on the remarks following the definition of i and applying Assertion 1 once again we find that

$$\begin{aligned}
&\operatorname{adh} \omega(\widetilde{\mathcal{F}(i(\mu))})(\mu) \\
&= \inf_{\substack{W \in_s \mathcal{U} \\ \theta \in I_0}} \inf_{\substack{U \in_s \mathcal{U} \\ \alpha \in I_0}} \sup_{\xi \in \Phi_C(X)} 1_{F^{i(\mu)}_{(W,\theta)}}(\xi) \wedge \omega_{(U,\alpha)}(\xi, \mu) \\
&\geq \inf_{\substack{W \in_s \mathcal{U} \\ \theta \in I_0}} \inf_{\substack{U \in_s \mathcal{U} \\ \alpha \in I_0}} \sup_{\substack{V \subset W \cap U \\ \gamma \leq \theta \wedge \alpha}} \omega_{(U,\alpha)}(i(\mu)_{(V,\gamma)}, \mu) \\
&\geq \inf_{\substack{W \in_s \mathcal{U} \\ \theta \in I_0}} \inf_{\substack{U \in_s \mathcal{U} \\ \alpha \in I_0}} \sup_{\substack{V \subset W \cap U \\ \gamma \leq \theta \wedge \alpha}} \omega_{(\hat{V},\alpha)}(i(i(\mu)_{(V,\gamma)}), i(\mu)) \\
&= 1.
\end{aligned}$$

By continuity of f it then follows that also

$$\operatorname{adh} f(\omega(\widetilde{\mathcal{F}(i(\mu))}))(f(\mu)) = 1,$$

which by the construction of $\hat{f}$, the fact that Y is weak Hausdorff and upon applying Corollary 8.2 [7] implies that $f(\mu) = \hat{f}(i(\mu))$.

Step 3. $\hat{f}$ *is uniformly continuous*

Let $\nu \in \mathfrak{U}, \varepsilon \in I_0$ and choose $\xi \in \mathfrak{U}$ such that

$$\xi^3 - \frac{\varepsilon}{2} \leq \nu.$$

Then choose $V \in {}_s\mathcal{U}, \beta \in I_0$ such that

$$\omega_{(V,\beta)}|_{\Phi_c(X)} \leq (f \times f)^{-1}(\xi)$$

and finally take $U \in {}_S\mathcal{U}, U^3 \subset V$ and $3\alpha := \beta$. Fix $\mu, \varsigma \in \Phi_W(\hat{X})$.

Assertion 3. There exist $W \in {}_s\mathcal{U}, W \subset U, \gamma \in I_0, \gamma \leq \alpha$ such that

$$\xi(\hat{f}(\mu), f(\mu_{(W,\gamma)})) \geq 1 - \varepsilon/2,$$
$$\xi(\hat{f}(\xi), f(\varsigma_{(W,\gamma)})) \geq 1 - \varepsilon/2.$$

Indeed, from the construction of $\hat{f}$ it follows that we can find $D \in {}_s\mathcal{U}$ and $\delta \in I_0$ such that

$$\xi\langle\hat{f}(\mu)\rangle \geq f(1_{F^{\mu}_{(D,\delta)}}) - \varepsilon/2,$$
$$\xi\langle\hat{f}(\xi)\rangle \geq f(1_{F^{\varsigma}_{(D,\delta)}}) - \varepsilon/2.$$

The reader can easily verify that $W := D \cap U$ and $\gamma := \delta \wedge \alpha$ fulfill the claim of the assertion.

Applying once again the remarks following the definition of i, Assertions 1 and 3 we then have

$$\begin{aligned}
&\nu(\hat{f}(\mu), \hat{f}(\varsigma)) \\
&\geq \xi(\hat{f}(\mu), f(\mu_{(W,\delta)})) \wedge \xi(f(\mu_{(W,\delta)}), f(\varsigma_{(W,\delta)})) \\
&\quad \wedge \xi(f(\varsigma_{(W,\delta)}), \hat{f}(\varsigma)) - \varepsilon/2 \\
&\geq (1 - \varepsilon/2) \wedge (f \times f)^{-1}(\xi)(\mu_{(W,\gamma)}, \varsigma_{(W,\gamma)}) - \varepsilon/2 \\
&\geq \omega_{(V,\beta)}(\mu_{(W,\gamma)}, \varsigma_{(W,\gamma)}) - \varepsilon \\
&\geq \omega_{(\hat{U},\alpha)}(i(\mu_{(W,\gamma)}), \mu) \wedge \omega_{(\hat{U},\alpha)}(\mu, \varsigma) \\
&\quad \wedge \omega_{(\hat{U},\alpha)}(\varsigma, i(\varsigma_{(W,\gamma)})) - \varepsilon \\
&= \omega_{(\hat{U},\alpha)}(\mu, \varsigma) - \varepsilon.
\end{aligned}$$

By Corollary 2.6 [**4**] this proves $\hat{f}$ is indeed uniformly continuous.

Step 4. *$\hat{f}$ is unique.*

Let $h : \Phi_W(\hat{X}) \to Y$ be a continuous extension of f and let $\mu \in \Phi_W(\hat{X})$ be fixed. Then by a calculation similar to the one of Step 2 we find

$$\operatorname{adh} i(\omega(\widetilde{\mathcal{F}(\mu)}))(\mu)) = 1.$$

Since h is continuous and extends f this implies

$$\operatorname{adh} f(\omega(\widetilde{\mathcal{F}(\mu)}))(h(\mu)) = 1,$$

which by the construction of $\hat{f}$ finally yields $h(\mu) = \hat{f}(\mu)$.

This ends the proof of the theorem.

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