ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 17, Number 3, Summer 1987

CONVEX PROGRAMMING WITH SET FUNCTIONS

J.H. CHOU, WEI-SHEN HSIA AND TAN-YU LEE

ABSTRACT. Concepts of subdifferentials and conjugate functions are extended to convex set functions. Optimality conditions for convex programs with set functions are then characterized by subdifferentials of set functions involved. Duality theorems of Wolfe type for convex programs with set functions are also developed.

1. Introduction. Recently, properties of convexity, differentiability and epigraphs of set functions were investigated in several papers (e.g., [**1**,**2**, and **3**]). These results were then used to characterize the optimal solution of mathematical programming problems with set functions. This paper presents some results following the same direction of those papers but with different approaches.

In this paper, we consider the following convex program with set functions:

$$\min_{\Omega\in S} F(\Omega) \quad \text{subject to } G^i(\Omega) \le 0, \quad i = 1, \dots, n,$$

where F and each G^i are convex set functions $\mathcal{A} \to \mathbf{R}\bigcup\{\infty\}$, $\mathcal{S}$ is a nonempty convex subfamily of $\mathcal{A}$, and $(X, \mathcal{A}, \mu)$ is a measure space. Concepts of subgradients, subdifferentials, and conjugate functions of convex functions are extended to convex set functions in §2. Subsequently, their basic properties are also explored. In §3, optimality conditions at Ω^* for convex programs with set functions are then characterized by the sum of subdifferentials of all set functions involved at Ω^*, and the normal cone of the set $\mathcal{D} = \mathcal{S}\bigcap(\operatorname{dom} F)\bigcap_{i=1}^{n}(\operatorname{dom} G^i)$ at Ω^*. This result (see Theorem 3.3) presents general optimality conditions for convex programs with set functions since we include a constrained convex subfamily $\mathcal{S}$ in the setting. Finally, duality theorems of Wolfe type for convex programs with set functions are developed in §4.

Received by the editors on March 20, 1985.
Partially supported by a grant from the Reseach Grants Committee of the Univeristy of Alabama.
AMS (MOS) *Subject classification* (1980): Primary 90C48.
Key words: set functions, subdifferential, conjugate functions, duality.

2. Notations and preliminaries. Let $(X, \mathcal{A}, \mu)$ be a measure space where $\mathcal{A}$ is the σ-algebra of all μ-measureable subsets of X. For $\Omega \in \mathcal{A}$, χ_Ω denotes the characteristic function of Ω. We shall write L_p instead of $L_p(X, \mathcal{A}, \mu)$, and denote by I the unit interval $[0,1]$. Morris showed in [**3**] that if $(X, \mathcal{A}, \mu)$ is atomless and L_1 is separable, then, for any $\Omega, \Lambda \in \mathcal{A}$ and $\lambda \in I$, there exists a sequence $\{\Gamma_n\} \subset \mathcal{A}$ such that $\chi_{\Gamma_n} \xrightarrow{\omega^*} \lambda\chi_\Omega + (1-\lambda)\chi_\Lambda$ where $\xrightarrow{w^*}$ denotes weak* convergence of elements in L_∞. We shall call such a sequence a Morris-sequence associated with $\langle \lambda, \Omega, \Lambda \rangle$. Using Morris-sequence instead of usual convex combinations, a subfamily $S \subset \mathcal{A}$ is said to be convex if, for every $\langle \lambda, \Omega, \Lambda \rangle \in I \times S \times S$ and every Morris-sequence $\{\Gamma_n\}$ associated with $\langle \lambda, \Omega, \Lambda \rangle$ in $\mathcal{A}$, there exists a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ in S. Throughout this paper we will assume that $(\chi, \mathcal{A}, \mu)$ is atomless and L_1 is separable. Therefore, the whole σ-algebra $\mathcal{A}$ is convex and, conventionally, so is the empty subfamily of $\mathcal{A}$.

DEFINITION 2.1. Let S be a subfamily of $\mathcal{A}$, and F a set function of S into $\overline{\mathbf{R}} = \mathbf{R} \bigcup \{\infty\}$. F is said to be convex if, for every $\langle \lambda, \Omega, \Lambda \rangle \in I \times S \times S$ and every Morris-sequence $\{\Gamma_n\}$ associated with $\langle \lambda, \Omega, \Lambda \rangle$, there exists a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ in S such that $\limsup_{k \to \infty} F(\Gamma_{n_k}) \leq \lambda F(\Omega) + (1-\lambda)F(\Lambda)$. A set function $G : S \to \mathbf{R} \bigcup \{-\infty\}$ is said to be concave if $-G$ is convex.

For every set function $F : S \to \overline{\mathbf{R}}$, we call the subfamily $\operatorname{dom} F = \{\Omega \in S | F(\Omega) < \infty\}$ the effective domain of F. It is easy to verify that the effective domain of a convex set function is convex. If F is a set function of $S \subset \mathcal{A}$ into $\mathbf{R}$, we can associate with it the function $\tilde{F}$ defined on $\mathcal{A}$ by

$$\begin{aligned} \tilde{F}(\Omega) &= F(\Omega), && \text{if } \Omega \in S \\ \tilde{F}(\Omega) &= \infty, && \text{if } \Omega \notin S. \end{aligned}$$

It is immediate by Definition 2.1 that $\tilde{F}$ is convex if and only if $S \subset \mathcal{A}$ is convex and $F : S \to \mathbf{R}$ is convex. Because of this extension by $+\infty$, we may only consider set functions defined on $\mathcal{A}$.

If $S \subset \mathcal{A}$, the indicator function δ_S of S is defined as

$$\begin{aligned} \delta_S(\Omega), &= 0, && \text{if } \Omega \in S, \\ \delta_S(\Omega) &= \infty, && \Omega \notin S. \end{aligned}$$

Clearly, S is a convex subfamily if and only if δ_S is a convex set function. Thus the study of convex subfamilies is naturally included in the study of convex set functions.

DEFINITION 2.2. The epigraph of a set function $F : \mathcal{A} \to \overline{\mathbf{R}}$ is the set: $\operatorname{epi} F = \{(r, \Omega) \in \mathbf{R} \times \mathcal{A} | F(\Omega) \leq r\}$.

It is the set of points of $\mathbf{R} \times \mathcal{A}$ which lie above the graph of F. The projection of $\operatorname{epi} F$ on $\mathcal{A}$ is $\operatorname{dom} F$.

We shall identify a set $\Omega \in \mathcal{A}$ with χ_Ω, hence $S \subset \mathcal{A}$ is regarded as $\chi_S = \{\chi_\Omega | \Omega \in S\}$ in L_∞. Since, if $F : \mathcal{A} \to \overline{\mathbf{R}}$ is convex then $F(\Omega) = F(\Lambda)$ if $\chi_\Omega \equiv \chi_\Lambda$ a.e., we may regard F as a functional on L_∞ defined on $\chi_{\mathcal{A}} = \{\chi_\Omega : \Omega \in \mathcal{A}\}$. In [**2**], we showed that if $S \subset \mathcal{A}$ is convex, then $\overline{S}$, the w^*-closure of χ_S in L_∞, is the w^*-closed convex hull of χ_S and $\overline{\mathcal{A}} = \{f \in L_\infty : 0 \leq f \leq 1\}$. Furthermore, the convexity of set function can be characterized by their epigraphs [**2**, Theorem 3.3].

PROPOSITION 2.3 [**2**]. *If $F : \mathcal{A} \to \overline{\mathbf{R}}$ is convex, then $\overline{\operatorname{epi} F}$, the w^*-closure of $\operatorname{epi} F$ in $\mathbf{R} \times L_\infty$ is convex.*

DEFINITION 2.4. A set function $F : \mathcal{A} \to \mathbf{R}$ is said to be w^*-continuous on $\operatorname{dom} F$ if, for any $f \in \operatorname{dom} F$, $\{F(\Omega_n)\}$ converges to the same limit for all $\{\Omega_n\}$ in $\operatorname{dom} F$ with $X_{\Omega_n} \xrightarrow{w^*} f$.

PROPOSITION 2.5 [**2**]. *Let $F : \mathcal{A} \to \overline{\mathbf{R}}$ be w^*-continuous. Then F can be uniquely extended to a w^*-continuous L_∞-functional $\overline{F}$ defined on $\overline{\mathcal{A}}$. Furthermore, F is convex if and only if $\overline{F}$ is convex, and when this is the case we have $\overline{\operatorname{epi} F} = \operatorname{epi} \overline{F}$.*

For basic properties of convex subfamilies and convex set functions, the reader is referred to [**1**].

For a discussion of w^*-semicontinuities and more properties about the epigraphs of set functions, the reference is [**2**].

In the following, we extend the definitions of subgradient and conjugate functions to convex set functions. We denote by $(L_\infty)^*$ the dual space of L_∞ with the norm topology, which can be characterized as the space of finitely additive set functions [**6**].

DEFINITION 2.6. A set function $F : \mathcal{A} \to \overline{\mathbf{R}}$ is said to be subdifferentiable at $\Omega_o \in \operatorname{dom} F$ if there exists an element $x^* \in (L_\infty)^*$ such that

$$F(\Omega) \geq F(\Omega_o) + \langle x^*, \chi_\Omega - \chi_{\Omega_o} \rangle \text{ for all } \Omega \in \mathcal{A}.$$

The set of all subgradients of F at Ω_o is called the subdifferential at Ω_o and is denoted by $\partial F(\Omega_o)$.

DEFINITION 2.7. Let $\mathcal{S}$ be a convex subfamily of $\mathcal{A}$. The normal cone $\mathcal{N}_{\mathcal{S}}(\Omega_o)$ to $\mathcal{S}$ at $\Omega_o \in \mathcal{S}$ is defined by $\mathcal{N}_{\mathcal{S}}(\Omega_o) = \{x^* \in (L_\infty)^* : \langle \chi^*, \chi_\Omega - \chi_{\Omega_o} \rangle \leq 0 \text{ for all } \Omega \in \mathcal{S}\}$.

REMARK 2.8.

(i) It is clear from the definitions that the normal cone of a convex subfamily $\mathcal{S}$ at $\Omega_o \in \mathcal{S}$ is identical to the subdifferential of the indicator function of $\mathcal{S}$ at Ω_o, i.e., $\mathcal{N}_{\mathcal{S}}(\Omega_o) = \partial \delta_{\mathcal{S}}(\Omega_o)$.

(ii) A subdifferential is a closed convex cone in $(L_\infty)^*$.

(iii) If $F : \mathcal{A} \to \overline{\mathbf{R}}$ is defined by $F(\Omega) = \int_\Omega f \, d\mu$ for some $f \in L_1 \subset (L_\infty)^*$, then since $F(\Omega) = F(\Omega_o) + \langle f, \chi_\Omega - \chi_{\Omega_o} \rangle$ for all $\Omega \in \mathcal{A}$; $f \in \partial F(\Omega_o)$. Note that f is the derivative of F at Ω_o [**3**]. If $\mu(X) < \infty$ and $f \equiv$ constant, then $\{g \in L_1 : g \geq 0\} \subset \partial F(X)$.

DEFINITION 2.9. Let $F : \mathcal{A} \to \overline{\mathbf{R}}$ be a convex set function. The function $F^* : (L_\infty)^* \to \overline{\mathbf{R}}$ defined by

$$F^*(x^*) = \sup_{\Omega \in \mathcal{A}} \{\langle x^*, \chi_\Omega \rangle - F(\Omega)\},$$

is called the conjugate function of F. If $G : \mathcal{A} \to \mathbf{R} \bigcup \{-\infty\}$ is concave, then we define $G^*(x^*) = \inf_{\Omega \in \mathcal{A}} \{\langle x^*, \chi_\Omega \rangle - G(\Omega)\}$.

PROPOSITION 2.10. *Let $F : \mathcal{A} \to \overline{\mathbf{R}}$ be a convex set function. Then $F^*(x^*) + F(\Omega) \geq \langle x^*, \chi_\Omega \rangle$ for all $x^* \in (L_\infty)^*$ and $\Omega \in \mathcal{A}$, and $F^*(x^*) + F(\Omega) = \langle x^*, \chi_\Omega \rangle$ if and only if $x^* \in \partial F(\Omega)$ for $\Omega \in \operatorname{dom} F$.*

PROOF. The proof is straightforward from the definitions, thus omitted.

For any given convex set functions $F : \mathcal{A} \to \overline{\mathbf{R}}$ and $G : \mathcal{A} \to \overline{\mathbf{R}}$, it follows directly from the definition that $\partial F(\Omega_o) + \partial G(\Omega_o) \subset \partial(F + G)(\Omega_o)$ for every $\Omega_o \in \operatorname{dom} F \bigcap \operatorname{dom} G$. The following theorem which is parallel to [**5**, Theorem 4] shows that the equality holds in a simple case. Before we state the theorem, we need to set up one more notation.

DEFINITION 2.11. Let $F : \mathcal{A} \to \overline{\mathbf{R}}$ be a convex set function, and $\mathcal{D}$ a convex subfamily of $\operatorname{dom} F$. $F_{\mathcal{D}}$, the restriction of F on $\mathcal{D}$, is defined as: $F_{\mathcal{D}}(\Omega) = F(\Omega)$ for $\Omega \in \mathcal{D}$ and $F_{\mathcal{D}}(\Omega) = \infty$ if $\Omega \notin \mathcal{D}$. Note that $\partial F(\Omega) \subset \partial F_{\mathcal{D}}(\Omega)$ for $\Omega \in \mathcal{D}$.

THEOREM 2.12. *Let $F, G : \mathcal{A} \to \overline{\mathbf{R}}$ be two convex set functions. Let $\mathcal{D} = \operatorname{dom} F \bigcap \operatorname{dom} G$. Suppose that $\overline{\mathcal{D}}$, the w^*-closure of $\mathcal{D}$, contains relative interior points and that either $F_{\mathcal{D}}$ or $G_{\mathcal{D}}$ is w^*-continuous. Then $\partial(F_{\mathcal{D}} + G_{\mathcal{D}})(\Omega) = \partial F_{\mathcal{D}}(\Omega) + \partial G_{\mathcal{D}}(\Omega)$ for every $\Omega \in \mathcal{D}$.*

PROOF. It suffices to show the equality holds for $\Omega_o \in \mathcal{D}$. Given $z_o^* \in \partial(F_{\mathcal{D}} + G_{\mathcal{D}})(\Omega_o)$, we define

$$F_1(\Omega) = G_{\mathcal{D}}(\Omega) \text{ and } F_2(\Omega) = \langle z_o^*, \chi_\Omega \rangle - F_{\mathcal{D}}(\Omega)$$

for $\Omega \in \mathcal{A}$. Then F_1 is convex and F_2 is concave. We have

$$F_1^* x^* = G_{\mathcal{D}}^*(x^*)$$

and

$$\begin{aligned} F_2^*(x^*) &= \inf_{\Omega \in \mathcal{D}} \{ \langle x^*, \chi_\Omega \rangle - \langle z_o^*, \chi_\Omega \rangle + F_{\mathcal{D}}(\Omega) \} \\ &= -\sup_{\Omega \in \mathcal{D}} \{ \langle z_o^* - x^*, \chi_\Omega \rangle - F_{\mathcal{D}}(\Omega) \} \\ &= -F_{\mathcal{D}}^*(z_o^* - x^*), \quad \text{for all } \Omega \in \mathcal{A}. \end{aligned}$$

Now since $z_o^* \in \partial(F_{\mathcal{D}} + G_{\mathcal{D}})(\Omega_o)$ implies that

$$\langle z_o^*, \chi_\Omega - \chi_{\Omega_o} \rangle \leq (F_{\mathcal{D}} + G_{\mathcal{D}})(\Omega) - (F_{\mathcal{D}} + G_{\mathcal{D}})(\Omega_o) \quad \forall \Omega \in A,$$

we have

$$F_{\mathcal{D}}(\Omega_o) + G_{\mathcal{D}}(\Omega_o) - \langle z_o^*, \chi_{\Omega_o} \rangle \leq F_{\mathcal{D}}(\Omega) + G_{\mathcal{D}}(\Omega)$$

and

$$-\langle z_o^*, \chi_\Omega \rangle = F_1(\Omega) - F_2(\Omega).$$

It follows from the Fenchel Duality Theorem [**2**] that there exists an $x_o^* \in (L_\infty)^*$ such that

$$\begin{aligned} &F_{\mathcal{D}}(\Omega_o) + G_{\mathcal{D}}(\Omega_o) - \langle z_o^*, \chi_{\Omega_o} \rangle \\ &\quad = F_2^*(x_o^*) - F_1^*(x_o^*) = -F_{\mathcal{D}}^*(z_o^* - x_o^*) - G_{\mathcal{D}}^*(x_o^*), \end{aligned}$$

i.e.,

$$F_{\mathcal{D}}^*(z_o^* - x_o^*) + F_{\mathcal{D}}(\Omega_o) + G_{\mathcal{D}}^*(x_o^*) + G_{\mathcal{D}}(\Omega_o)$$
$$= \langle z_o^* - x_o^*, \chi_{\Omega_o}\rangle + \langle x_o^*, \chi_{\Omega_o}\rangle.$$

It follows from Proposition 2.10 that we must have $z_o^* - x_o^* \in \partial F_{\mathcal{D}}(\Omega_o)$ and $x_o^* \in \partial G_{\mathcal{D}}(\Omega_o)$, and hence $z_o^* \in \partial F_{\mathcal{D}}(\Omega_o) + \partial G_{\mathcal{D}}(\Omega_o)$. This shows that

$$\partial(F_{\mathcal{D}} + G_{\mathcal{D}})(\Omega) = \partial F_{\mathcal{D}}(\Omega) + \partial G_{\mathcal{D}}(\Omega), \quad \text{for all } \Omega \in \mathcal{A}.$$

COROLLARY 2.13. *Let everything be as in Theorem* 2.12. *Assume* $\alpha > 0, \beta > 0$. *Then* $\partial(\alpha F_{\mathcal{D}} + \beta G_{\mathcal{D}})(\Omega) = \alpha\partial F_{\mathcal{D}}(\Omega) + \beta\alpha G_{\mathcal{D}}\Omega)$ *for all* $\Omega \in \mathcal{A}$.

3. Optimality conditions. In this section, we first consider the following general constrained minimization problem of a set function over a convex subfamily $\mathcal{S}$ of $\mathcal{A}$:

$$\text{(A)} \qquad \min F(\Omega) \text{ subject to } \Omega \in \mathcal{S},$$

where $F : \mathcal{A} \to \overline{\mathbf{R}}$ is a convex set function.

For the unconstrained case, we have a direct consequence from the definition of subdifferentials:

LEMMA 3.1. $F(\Omega^*) = \min_{\Omega \in \mathcal{A}} F(\Omega) = \min_{\Omega \in \operatorname{dom} F} F(\Omega)$ if and only if $0 \in \partial F(\Omega^*) = \partial F_{\operatorname{dom} F}(\Omega^*)$.

THEOREM 3.2. *Suppose* $\overline{\mathcal{D}}$ *has relative interior points where* $\mathcal{D} = (\operatorname{dom} F) \bigcap \mathcal{S}$. *Then* Ω^* *solves problem* (A) *if and only if* $0 \in \partial F_{\mathcal{D}}(\Omega^*) + \mathcal{N}_{\mathcal{D}}(\Omega^*)$.

PROOF. Since $F(\Omega^*) = \min_{\Omega \in \mathcal{S}} F(\Omega)$ if and only if $F(\Omega^*) = \min_{\Omega \in \mathcal{D}}(F + \delta_{\mathcal{D}})(\Omega)$,Ω^* solves problem (A) if and only if $0 \in \partial(F_{\mathcal{D}} + \delta_{\mathcal{D}})(\Omega^*)$. Observe that $\delta_{\mathcal{D}}$ is w^*-continuous, by Theorem 2.12 and Remark 2.8(i), $0 \in \partial F_{\mathcal{D}}(\Omega^*) + \mathcal{N}_{\mathcal{D}}(\Omega^*)$ if and only if $0 \in \partial(F_{\mathcal{D}} + \delta_{\mathcal{D}})(\Omega^*)$.

Next, we consider the primal problem

$$\text{(P)} \qquad \min_{\Omega \in \mathcal{S}} F(\Omega) \text{ subject to } G^i(\Omega) \leq 0, \quad i = 1, \ldots, n$$

where F and each G^i are convex set functions $\mathcal{A} \to \overline{\mathbf{R}}$, and $\mathcal{S}$ is a nonempty convex subfamily of $\mathcal{A}$.

THEOREM 3.3. *Suppose that* $\mathcal{D}$ *has nonempty relative interior, where* $\mathcal{D} = \mathcal{S} \bigcap (\operatorname{dom} F) \bigcap_{i=1}^{n} (\operatorname{dom} G^i)$. *Suppose that at least n functions of* F *and* $G^i, i = 1, \ldots, n$ *are* w^**-continuous on* $\mathcal{D}$, *and* $\bigcap_{i=1}^{n} \{\Omega \in \mathcal{S} : G^i(\Omega) < 0\} \neq \phi$. *Then* Ω^* *solves problem* (P) *if and only if*

(i) $G^i(\Omega^*) \leq 0$, *for* $i = 1, \ldots, n$,

(ii) *there exist* $\lambda_i^* \geq 0$, *for* $i = 1, \ldots, n$ *such that* $\lambda_i^* G^i(\Omega^*) = 0, i = 1, \ldots, n$, *and*

(iii) $0 \in \partial F_{\mathcal{D}}(\Omega^*) + \sum_{i=1}^{n} \lambda_i^* \partial G_{\mathcal{D}}^i(\Omega^*) + \mathcal{N}_{\mathcal{D}}(\Omega^*)$.

PROOF. If Ω^* solves problem (P), then, by [**1**, Theorem 3.2], there exist $\lambda_1^* \geq 0, i = 1, \ldots, n$, such that $\lambda_i^* G^i(\Omega^*) = 0$ and $F(\Omega^*) + \sum_{i=1}^{n} \lambda_i^* G^i(\Omega^*) = \min_{\Omega \in \mathcal{S}} F(\Omega) + \sum_{i=1}^{n} \lambda_i^* G^i(\Omega)$.

By Theorem 3.2, $0 \in \partial(F_{\mathcal{D}} + \sum_{i=1}^{n} \lambda_i^* G_{\mathcal{D}}^i)(\Omega^*) + \mathcal{N}_{\mathcal{D}}(\Omega^*)$, and Corollary 2.13 asserts that $0 \in \partial F_{\mathcal{D}}(\Omega^*) + \sum_{i=1}^{n} \lambda_i^* \partial G_{\mathcal{D}}^i(\Omega^*) + \mathcal{N}_{\mathcal{D}}(\Omega^*)$. Conversely, if $G^i(\Omega^*) \leq 0$ for $i = 1, \ldots, n$, and there exist $\lambda_i^* \geq 0$ for $i = 1, \ldots, n$, such that (ii) and (iii) are satisfied, then $0 \in \partial(F_{\mathcal{D}} + \sum_{i=1}^{n} \lambda_i^* G_{\mathcal{D}}^i)(\Omega^*) + \mathcal{N}_{\mathcal{D}}(\Omega^*)$. Hence, by Theorem 3.2., Ω^* minimizes $(F + \sum_{i=1}^{n} \lambda_i^* G^i)(\Omega)$ subject to $\Omega \in \mathcal{D}$, i.e., $F(\Omega^*) + \sum_{i=1}^{n} \lambda_i^* G^i(\Omega^*) = F(\Omega^*) = \min_{\Omega \in \mathcal{D}} F(\Omega) + \sum_{i=1}^{n} \lambda_i^* G^i(\Omega) = \min_{\Omega \in \mathcal{S}} F(\Omega)$ subject to $G^i(\Omega) \leq 0, i = 1, \ldots, n$. This completes the proof.

4. Duality theorems of Wolfe type for set functions. A Wolfe type dual problem [**4**] to (P) can be formulated as

$$\text{(D)} \qquad \max F(\Omega) + \sum_{i=1}^{n} \lambda_i G^i(\Omega)$$

subject to

$$\begin{cases} \lambda_i \geq 0, i = 1, \ldots, n \\ 0 \in \partial F_{\mathcal{D}}(\Omega) + \sum_{i=1}^{m} \lambda_i \partial G_{\mathcal{D}}^i(\Omega) + \mathcal{N}_{\mathcal{D}}(\Omega) \end{cases}$$

where $\mathcal{D} = \mathcal{S} \bigcap \operatorname{dom} F \bigcap_{i=1}^{n} \operatorname{dom} G^i$. We shall write

$$L(\Omega, \lambda) = F(\Omega) + \sum_{i=1}^{n} \lambda_i G^i(\Omega) \text{ where } \lambda = (\lambda_1, \ldots, \lambda_m).$$

THEOREM 4.1. (WEAK DULAITY). *Suppose that* $\hat{\Omega}$ *and* $(\overline{\Omega}, \overline{\lambda})$ *are feasible solutions to* (P) *and* (D) *respectively. Then* $F(\hat{\Omega}) \geq L(\overline{\Omega}, \overline{\lambda})$.

PROOF. Assume that $(\overline{\Omega}, \overline{\lambda})$ is feasible to (D); then $\overline{\lambda}_i \geq 0, i = 1, \ldots, n$, and there exist $u^* \in \partial F_{\mathcal{D}}(\overline{\Omega})$, $v_i^* \in \partial G^i_{\mathcal{D}}(\overline{\Omega})$ and $w^* \in \mathcal{N}_{\mathcal{D}}(\overline{\Omega})$ such that $0 = u^* + \sum_{i=1}^n \overline{\lambda}_i v_i^* + w^*$, and hence

$$0 = \langle u^*, \chi_\Omega - \chi_{\overline{\Omega}} \rangle + \langle \sum_{i=1}^n \overline{\lambda}_i v_i^*, \chi_\Omega - \chi_{\overline{\Omega}} \rangle \\ + \langle w^*, \chi_\Omega - \chi_{\overline{\Omega}} \rangle \quad \text{for all } \Omega \in \mathcal{D}.$$

Now since $\langle w^*, \chi_\Omega - \chi_{\overline{\Omega}} \rangle \leq 0$ for $\Omega \in \mathcal{D}$, $\langle u^* + \sum_{i=1}^n \overline{\lambda}_i v_i^*, \chi_\Omega - \chi_{\overline{\Omega}} \rangle \geq 0$ for $\Omega \in \mathcal{D}$. Therefore,

$$\begin{aligned} F(\hat{\Omega}) - L(\overline{\Omega}, \overline{\lambda}) &= F(\hat{\Omega}) - \left(F(\overline{\Omega}) + \sum_{i=1}^n \overline{\lambda}_i G^i(\overline{\Omega}) \right) \\ &= (F(\hat{\Omega}) - F(\overline{\Omega})) - \sum_{i=1}^n \overline{\lambda}_i G^i(\overline{\Omega}) \\ &\geq \langle u^*, \chi_{\hat{\Omega}} - \chi_{\overline{\Omega}} \rangle - \sum_{i=1}^n \overline{\lambda}_i G^i(\overline{\Omega}) \quad (\text{since } u^* \in \partial F_{\mathcal{D}}(\Omega)) \\ &\geq -\langle \sum_{i=1}^n \overline{\lambda}_i v_i^*, \chi_{\hat{\Omega}} - \chi_{\overline{\Omega}} \rangle - \sum_{i=1}^n \overline{\lambda}_i G^i(\overline{\Omega}) \\ &= -\left(\sum_{i=1}^n \overline{\lambda}_i (\langle v_i^*, \chi_{\hat{\Omega}} - \chi_{\overline{\Omega}} \rangle + G^i(\overline{\Omega})) \right) \\ &\geq -\sum_{i=1}^n \overline{\lambda}_i G^i(\hat{\Omega}) \quad (\text{since } v_i^* \in \partial G^i_{\mathcal{D}}(\overline{\Omega})) \\ &\geq 0 \quad (\text{since } \hat{\Omega} \text{ is feasible to } (P)). \end{aligned}$$

This shows that $F(\hat{\Omega}) \geq L(\overline{\Omega}, \overline{\lambda})$.

THEOREM 4.2. (STRONG DUALITY). *Let everything be as in* Theorem 3.3. it If Ω^* solves (P), then there exist $\lambda_1^* \geq 0, i = 1, \ldots, n$, such that (Ω^*, λ^*) solves (D), and $F(\Omega^*) = L(\Omega^*, \lambda^*)$.

PROOF. Assume Ω^* solves (P); then, by Theorem 3.3, there exist $\lambda_i^* \geq 0, i = 1, \ldots, n$, such that (Ω^*, λ^*) is feasible to (D), and in addition, $\lambda_i^* G^i(\Omega^*) = 0$ for $i = 1, \ldots, n$, hence $F(\Omega^*) = F(\Omega^*) +$

$\sum_{i=1}^{n} \lambda_i^* G^i(\Omega^*) = L(\Omega^*, \lambda^*)$. It follows that (Ω^*, λ^*) solves (D) by Theorem 4.1.

We have thus proved the Wolfe's duality theorems [4] for set functions with subgradients in a more general setting that includes a constained subfamily $\mathcal{S}$.

References

1. J.H. Chou, Wei-Shen Hsia and Tan-Yu Lee, *On multiple objective programming problems with set functions*, J. of Math. Anal. and Appl. **105** (1985), 383-394.
2. ———, ——— and ———, *Epigraphs of convex set functions*, J. of Math. Anal. and Appl. **118** (1986), 247-254.
3. R.J.T. Morris, *Optimal constrained selection of a measurable subset*, J. of Math. Anal. and Appl. **70** (1979), 546-562.
4. P. Wolfe, *A duality theorem for nonlinear programming*, Quart. Appl. Math. **19** (1961), 239-244.
5. J. Zowe, *A duality theorem for a convex programming problem in order complete vector lattices*, J. of Math. Anal. and Appl. **50** (1975), 273-287.
6. K. Yosida and E. Hewitt, *Finitely additive measures*, Trans. Amer. Math. Soc. **72** (1952), 46-66.

Department of Mathematics, University of Alabama, University, AL 35486