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INFINITE SUMS OF PRODUCTS OF CONTINUOUS q-ULTRASPHERICAL FUNCTIONS

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ABSTRACT. Let $C_n(x;\beta|q)$ be the continuous q-ultra-spherical polynomial and $D_n(x;\beta|q)$ be the q-ultraspherical function of the second kind. By exploiting a special case of the recently found q-Feldheim bilinear sum, the following infinite sums are computed:

$$\sum_{n=0}^{\infty}\left(\frac{(q;q)_n}{(\beta^2;q)_n}\right)^2\frac{1-\beta q^n}{1-\beta}\beta^{n/2}C_n(x;\beta|q)C_n(y;\beta|q)C_n(z;\beta|q),$$

$0<\beta<1,\ 0<q<1$, and

$$\sum_{n=0}^{\infty}\frac{(q;q)_n}{(\beta^2;q)_n}\frac{1-\beta q^n}{1-\beta}(q/\beta)^{n/2}C_n(x;\beta|q)C_n(y;\beta|q)D_n(z;\beta|q)$$

$0<q<\beta<1$.

1. Introduction. Recently, Rahman [8] showed that

$$\begin{aligned} C_n(\cos\theta;\beta|q) = {}& \frac{(1+\beta q^n)(\beta^2;q)_n}{(1+\beta)(q;q)_n}\Big(Q_n(e^{i\theta};\beta^{\frac12},(\beta q)^{\frac12}, \\ & -\beta^{\frac12},-(\beta q)^{\frac12})+Q_n(e^{-i\theta};\beta^{\frac12},(\beta q)^{\frac12},-\beta^{\frac12},-(\beta q)^{\frac12})\Big), \\ & 0\le\theta\le\pi \end{aligned}\tag{1.1}$$

and

$$\begin{aligned} D_n(\cos\theta;\beta|q) = {}& i\frac{(1+\beta q^n)(\beta^2;q)_n}{(1+\beta)(q;q)_n}\Big(Q_n(e^{i\theta};\beta^{\frac12},(\beta q)^{\frac12},-\beta^{\frac12},-(\beta q)^{\frac12}) \\ & -Q_n(e^{-i\theta};\beta^{\frac12},(\beta q)^{\frac12},-\beta^{\frac12},-(\beta q)^{\frac12})\Big), 0<\theta<\pi, \end{aligned}\tag{1.2}$$

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where $C_n(x;\beta|q)$ is Rogers' q-ultraspherical polynomial [**3**], and $D_n(x;\beta|q)$ is the q-ultraspherical function of the second kind, see [**1**], defined by

$$D_n(\cos\theta;\beta|q) = 4\sin\theta\frac{(\beta e^{2i\theta},\beta e^{-2i\theta},\beta,\beta q;q)_\infty(\beta^2;q)_n}{(e^{2i\theta},e^{-2i\theta},q,\beta^2;q)_\infty(\beta q;q)_n} \sum_{k=0}^{\infty}\frac{(q/\beta,q^{n+1};q)_k}{(q,\beta q^{n+1};q)_k}\beta^k\cos(n+2k+1)\theta. \tag{1.3}$$

In addition to the standard finite series expression, $C_n(x;\beta|q)$ has an infinite series form which is obtained from (1.3) by simply replacing $\cos(n+2k+1)\theta$ by $\sin(n+2k+1)\theta$. The q-shifted factorials are defined by

$$(a;q)_n = \begin{cases} 1, & \text{if } n=0 \\ (1-a)(1-aq)\dots(1-aq^{n-1}), & \text{if } n=1,2,\dots. \end{cases} \tag{1.4}$$

We also use the notation $(a_1,a_2,\dots,a_k;q)_n$ to mean $(a_1;q)_n(a_2;q)_n,\dots$ $\dots(a_k;q)_n$. In (1.1) and (1.2), $Q_n(z;a,b,c,d)$ is the q-Wilson function of the second kind [**8**], defined by

$$\begin{aligned} &Q_n(z;a,b,c,d) \\ &= \frac{(abczq^n,bcdzq^n,bzq^{n+1},czq^{n+1},a/z,b/z,c/z,d/z;q)_\infty}{(bc,bd,cd,abq^n,acq^n,q^{n+1},bcz^2q^{n+1},z^{-2};q)_\infty}z^n \\ &{}_8W_7(bcz^2q^n;bz,cz,bcq^n,zq/a,zq/d;q,adq^n), \end{aligned} \tag{1.5}$$

where ${}_8W_7$ represents a special type of the basic hypergeometric series

$$\begin{aligned} {}_{r+1}\varphi_r\left(\begin{matrix} a_1, & a_2, & \dots, & a_{r+1} \\ & b_1, & \dots, & b_r \end{matrix};q,z\right) = \sum_{n=0}^{\infty}\frac{(a_1,a_2,\dots,a_{r+1};q)_n}{(q,b_1,\dots,b_r;q)_n}z^n. \end{aligned} \tag{1.6}$$

The series (1.6) is called balanced if $z = q$ and $b_1b_2\dots b_r = qa_1a_2\dots a_{r+1}$. It is called well-poised if $a_2b_1 = a_3b_2 = \dots = a_{r+1}b_r =$

qa_1; and very-well-poised if, in addition, $b_1 = a_1^{\frac{1}{2}}$, $b_2 = -a_1^{\frac{1}{2}}$. The series (1.6) is called nearly-poised of the first kind if $qa_1 \neq a_2 b_1 = a_3 b_2 = \ldots = a_{r+1} b_r$, and nearly-poised of the second kind if $qa_1 = a_2 b_1 = \ldots = a_r b_{r-1} \neq a_{r+1} b_r$. The W notation is for a very-well-poised series:

$$(1.7) \qquad {}_{r+3}W_{r+2}(a; a_1, a_2, \ldots, a_r; q, z) =$$

$${}_{r+3}\varphi_{r+2}\left(\begin{matrix} a, & qa^{\frac{1}{2}}, & -qa^{\frac{1}{2}}, & a_1, & a_2, & \ldots, & a_r \\ & a^{\frac{1}{2}}, & -a^{\frac{1}{2}}, & qa/a_1, & qa/a_2, & \ldots, & qa/a_r \end{matrix}; q, z\right).$$

The principal results of this paper are:

$$(1.8) \quad \sum_{n=0}^{\infty} \frac{(q;q)_n}{(\beta^2;q)_n} \frac{1-\beta q^n}{1-\beta} (q/\beta)^{\frac{n}{2}} C_n(\cos\theta;\beta|q) C_n(\cos\varphi;\beta|q)$$
$$\cdot D_n(\cos\psi;\beta|q) = \frac{(\beta,\beta q;q)_\infty}{2(q,\beta^2;q)_\infty}$$
$$\cdot \operatorname{Im}\left(\frac{(\beta e^{2i\psi}, qe^{-2i\psi}/\beta, (\beta q)^{\frac{1}{2}} e^{i\theta+i\varphi-i\psi}, (\beta q)^{\frac{1}{2}} e^{-i\theta-i\varphi-i\psi}; q)_\infty}{(e^{2i\psi}, qe^{-2i\psi}, (\frac{q}{\beta})^{\frac{1}{2}} e^{i\theta+i\varphi-i\psi}, (\frac{q}{\beta})^{\frac{1}{2}} e^{-i\theta-i\varphi-i\psi}; q)_\infty}\right.$$
$$\left.\cdot \frac{(\beta q)^{\frac{1}{2}} e^{i\theta-i\varphi-i\psi}, (\beta q)^{\frac{1}{2}} e^{i\varphi-i\theta-i\psi}; q)_\infty}{((\frac{q}{\beta})^{\frac{1}{2}} e^{i\theta-i\varphi-i\psi}, (\frac{q}{\beta})^{\frac{1}{2}} e^{i\varphi-i\theta-i\psi}; q)_\infty}\right),$$
$$0 \leq \theta,\ \varphi \leq \pi, \quad 0 < \psi < \pi,$$

and

$$(1.9) \quad \sum_{n=0}^{\infty} \left(\frac{(q;q)_n}{(\beta^2;q)_n}\right)^2 \frac{1-\beta q^n}{1-\beta} \beta^{\frac{n}{2}} C_n(\cos\theta;\beta|q) C_n(\cos\varphi;\beta|q)$$
$$\cdot C_n(\cos\psi;\beta|q) = \frac{(\beta q;q)_\infty (\beta;q)_\infty^3}{(\beta^2;q)_\infty^2}$$
$$\left|\frac{(\beta e^{2i\theta}, \beta e^{2i\varphi}, \beta e^{2i\psi}; q)_\infty}{(\beta^{\frac{1}{2}} e^{i\theta+i\varphi+i\psi}, \beta^{\frac{1}{2}} e^{i\theta-i\varphi+i\psi}, \beta^{\frac{1}{2}} e^{i\theta+i\varphi-i\psi}, \beta^{\frac{1}{2}} e^{i\psi+i\varphi-i\theta}; q)_\infty}\right|^2,$$

$0 \le \theta, \varphi, \psi \le \pi$. For convergence of the series (1.8), we need $|q/\beta| < 1, |\beta| < 1$, while for (1.9), we require $|q| < 1, 0 < \beta < 1$. To avoid unnecessary complications we shall assume throughout the paper that $0 < q, \beta < 1$. In [**11**] we stated (1.9) without proof. In the following sections we shall present detailed proofs of both (1.8) and (1.9).

2. The q-Feldheim formula. The fundamental formula that we need to evaluate the infinite sums in (1.8) and (1.9) is the q-Feldheim formula recently obtained by Rahman [**10**]:

$$
\begin{aligned}
(2.1)\quad &\sum_{n=0}^{\infty} \frac{\left(\frac{b^2c^2}{q};q\right)_n (1-b^2c^2q^{2n-1})(ab,ac,d,f,g,h;q)_n \left(\frac{dfgt}{bc};q\right)_{2n}}{(q;q)_n\left(1-\frac{b^2c^2}{q}\right)\left(\frac{bc^2}{a},\frac{b^2c}{a},\frac{dft}{bc},\frac{dgt}{bc},\frac{fgt}{bc},\frac{dfght}{b^3c^3};q\right)_n} \\
&\cdot \frac{\left(\frac{t}{a^2}\right)^n}{(b^2c^2;q)_{2n}}\, {}_8W_7\left(\frac{dfgtq^{2n-1}}{bc}; dq^n, fq^n, gq^n, \frac{dfgt}{b^3c^3}, \frac{b^2c^2q^n}{h}; q, \frac{ht}{bc}\right) \\
&\cdot p_n(x;a,b,c,bca^{-1})p_n(y;a,b,c,bca^{-1}) = \frac{\left(\frac{dt}{bc},\frac{ft}{bc},\frac{gt}{bc},\frac{dfgt}{bc};q\right)_\infty}{\left(\frac{dft}{bc},\frac{dgt}{bc},\frac{fgt}{bc},\frac{t}{bc};q\right)_\infty} \\
&\cdot \sum_{k=0}^{\infty} \frac{(d,f,g,h;q)_k}{\left(q,bc,bc,\frac{b^2c}{a};q\right)_k} \cdot \frac{\left|(bca^{-1}e^{i\theta}, bca^{-1}e^{i\varphi};q)_k\right|^2 q^k}{\left(\frac{bc^2}{a},\frac{bc}{a^2},\frac{bcq}{t},\frac{dfght}{b^3c^3};q\right)_k} \\
&{}_{10}W_9\left(\frac{a^2q^{-k}}{bc}; ae^{i\theta}, ae^{-i\theta}, ae^{i\varphi}, ae^{-i\varphi}, \frac{aq^{1-k}}{b^2c}, \frac{aq^{1-k}}{bc^2}, q^{-k}; q, q\right) \\
&+ \frac{\left(d,f,g,h,t,t,\frac{at}{c},\frac{ct}{a};q\right)_\infty}{q(1-q)\left(q,\frac{a}{c},\frac{qc}{a},bc,bc,ab,ac;q\right)_\infty} \\
&\cdot \frac{\left(\frac{bct}{q},\frac{dfgt}{bc},\frac{dfgt^2}{b^4c^4};q\right)_\infty \left|(ae^{i\theta},ce^{i\theta},be^{i\varphi},bca^{-1}e^{i\varphi};q)_\infty\right|^2}{\left(\frac{bc^2}{a},\frac{b^2c}{a},\frac{bc}{t},\frac{dft}{bc},\frac{dgt}{bc},\frac{fgt}{bc},\frac{ht}{bc},\frac{dfght}{b^3c^3};q\right)_\infty \left|(te^{i\theta+i\varphi},te^{i\theta-i\varphi};q)_\infty\right|^2} \\
&\cdot \sum_{k=0}^{\infty} \frac{\left(\frac{dt}{bc},\frac{ft}{bc},\frac{gt}{bc},\frac{ht}{bc};q\right)_k \left|(te^{i\theta+i\varphi},te^{i\theta-i\varphi};q)_k\right|^2 q^k}{\left(q,t,\frac{qt}{bc},t,\frac{at}{c},\frac{ct}{a},\frac{bct}{q},\frac{dfght^2}{b^4c^4};q\right)_k} \\
&\cdot \int_{\frac{aq}{c}}^{q} d_qu \frac{\left(u,\frac{cu}{a},\frac{uc}{q}\left(\frac{btq^{k-1}}{a}\right)^{\frac12}, -\frac{uc}{q}\left(\frac{btq^{k-1}}{a}\right)^{\frac12}, \frac{bc^2u}{aq}, \frac{bcu}{q};q\right)_\infty}{\left(bctuq^{k-2}, \frac{bc^2tuq^{k-2}}{a}, uc\left(\frac{btq^{k-1}}{a}\right)^{\frac12}, -uc\left(\frac{btq^{k-1}}{a}\right)^{\frac12}, utq^{k-1}, \frac{cutq^{k-1}}{a};q\right)}
\end{aligned}
$$

$$\cdot \frac{\left(bca^{-1}tue^{i\theta}q^{k-1}, bca^{-1}tue^{-i\theta}q^{k-1}, ctue^{i\varphi}q^{k-1}, ctue^{-i\varphi}q^{k-1}; q\right)_\infty}{\left(\frac{cue^{i\theta}}{q}, \frac{cue^{-i\theta}}{q}, bca^{-1}\frac{ue^{i\varphi}}{q}, bca^{-1}\frac{ue^{-i\varphi}}{q}; q\right)_\infty},$$

where the integral on the right hand side is a q-integral defined by

$$(2.2)\qquad \begin{aligned} &\int_0^a f(x)d_qx = a(1-q)\sum_{n=0}^{\infty} f(aq^n)q^n, \\ &\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx, \end{aligned}$$

and $x = \cos\theta,\ y = \cos\varphi$.

The q-integral notation makes the expression look a bit simpler than what it really is. In fact, it is the sum of multiples of two balanced, non-terminating and very-well-poised ${}_{10}\varphi_9$ series. Despite its long and cumbersome appearance, (2.1) is actually a very useful formula. In [**10**], it was shown how a special case of it leads to the Poisson kernel for continuous q-Jacobi polynomials in much the same way as Feldheim's formula [**7**] gives Bailey's Poisson kernel for the Jacobi polynomials as a special case, see [**4,9**]. The parameters a, b, c are assumed to be real and d, f, g, h are arbitrary complex numbers. We assume, for the moment, that t is real and $|t| < 1$. Convergence of the infinite series on both sides requires further restrictions on the parameters, but there is no need to mention them here since we only need some special cases of (2.1). The polynomials p_n on the left hand side of (2.1) are special cases of the Askey-Wilson polynomials [**2**] defined by

$$(2.3)\ p_n(x; a, b, c, d) = {}_4\varphi_3\left(\begin{matrix} q^{-n} & abcdq^{n-1}, & ae^{i\theta}, & ae^{-i\theta} \\ & ab, & ac, & ad \end{matrix}; q, q\right),$$

$$x = \cos\theta,$$

where $\max(|q|, |a|, |b|, |c|, |d|) < 1$. The connection of these polynomials with $C_n(x; \beta|q)$ was given by Askey and Ismail [**3**],

$$(2.4)\qquad p_n(x; a, aq^{\frac{1}{2}}, -a, -aq^{\frac{1}{2}}) = \frac{(q;q)_n}{(a^4;q)_n} a^n C_n(x; a^2|q).$$

Since we are interested only in the case of q-ultraspherical polynomials, we shall set $b = aq^{\frac{1}{2}}, c = -a$ in (2.1), and choose the other parameters as follows:

$$\text{(2.5)}\qquad \begin{aligned}&\text{Case I}: d=q, f=a^2q^{\frac{1}{2}}, g=-a^2q, h=-a^2q^{\frac{1}{2}}, t=az;\\ &\text{Case II}: d=a^4, f=a^2q^{\frac{1}{2}}, g=-a^2q, h=-a^2q^{\frac{1}{2}}, t=zq^{\frac{1}{2}}/a,\end{aligned}$$

where z is a complex number which we shall assume to be outside the basic interval [-1,1]. Note that when t is complex, the products like $|(te^{i\varphi};q)_n|^2$ are to be interpreted as $(te^{i\varphi}, te^{-i\varphi};q)_n$.

Using a limiting case of Bailey's transformation formula [**4**, 8.5(1)], namely,

$$\text{(2.6)}\qquad \begin{aligned}&\frac{(aq/e, aq/f;q)_\infty}{(aq, aq/ef;q)_\infty}{}_8W_7(a;b,c,d,e,f;q,\lambda q/ef)\\ &= (\lambda_{q/e}\frac{\lambda q/f;q)_\infty}{(\lambda q, \lambda q/ef;q)_\infty}{}_8W_7(\lambda;\lambda b/a, \lambda c/a, \lambda d/a, e, f;q, aq/ef),\end{aligned}$$

where $\lambda = qa^2/bcd$, we first express the ${}_8W_7$ series on the left hand side of (2.1) as a multiple of $Q_n(z;a,aq^{\frac{1}{2}},-a,-aq^{\frac{1}{2}})$ via (1.5), then simplify the coefficients to obtain the following special cases of (2.1):

(2.7)

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(q;q)_n}{(a^4;q)_n}\frac{1-a^4q^{2n}}{1-a^4}a^nC_n(x;a^2|q)C_n(y;a^2|q)Q_n(z;a,aq^{\frac{1}{2}},-a,-aq^{\frac{1}{2}})\\ &= h_1(a)\frac{(a^2z^2, a^2/z^2;q)_\infty}{(qz^2, z^{-2};q)_\infty}(1+z(aq^{\frac{1}{2}}+a^{-1}q^{\frac{-1}{2}})+z^2)^{-1}\\ &\sum_{k=0}^{\infty}\frac{|(-aq^{\frac{1}{2}}e^{i\phi}, -aq^{\frac{1}{2}}e^{i\varphi};q)_k|^2q^k}{(-q^{\frac{1}{2}}, -a^2q^{\frac{1}{2}}, -azq^{3/2}, -aq^{3/2}/z;q)_k}{}_{10}W_9(-q^{-k-\frac{1}{2}};ae^{i\theta}, ae^{-i\varphi},\\ &ae^{-i\theta}, ae^{-i\varphi}, -q^{-k}/a^2, q^{\frac{1}{2}-k}/a^2, q^{-k};q,q)\\ &+h_2(a)\frac{(a^2z^2;q^2)_\infty(a^2/z^2;q)_\infty(-a^3z/q^{\frac{1}{2}};q)_\infty}{q(1-q)(-1,-q,z^{-2},-aq^{\frac{1}{2}}/z;q)_\infty}\\ &\cdot\frac{|(a^2e^{2i\theta}, qa^2e^{2i\varphi};q^2)_\infty|^2}{(aze^{i\theta+i\varphi}, aze^{-i\theta-i\varphi}, aze^{i\theta-i\varphi}aze^{i\varphi-i\theta};q)_\infty}\\ &\cdot\sum_{k=0}^{\infty}\frac{(azq^{\frac{1}{2}}, aze^{i\theta+i\varphi}, aze^{-i\theta-i\varphi}, aze^{i\theta-i\varphi}, aze^{i\varphi-i\theta};q)^k}{(q, az, -az, -a^3zq^{-\frac{1}{2}}, qz^2;q)_k}q^k\int_{-q}^{q}d_quI_1(u)\end{aligned}$$

in case I, where

(2.8)
$$h_1(a) = \frac{(a^2;q)_\infty^2}{(1-a^2)(q,a^4;q)_\infty}, \quad h_2(a) = \frac{(a^2;q)_\infty^3}{(1-a^2)(q;q)_\infty(a^4;q)_\infty^2},$$

(2.9) $I_1(u) =$
$$\frac{(u,-u,\frac{ua}{q}(azq^{k-\frac{3}{2}})^{\frac{1}{2}},\frac{-ua}{q}(azq^{k-\frac{3}{2}})^{\frac{1}{2}},a^2uq^{\frac{-1}{2}},-a^2uq^{\frac{-1}{2}};q)_\infty}{a^3zuq^{k-\frac{3}{2}},-a^3zuq^{k-\frac{3}{2}},ua(azq^{k-\frac{3}{2}})^{\frac{1}{2}},-ua(azq^{k-\frac{3}{2}})^{\frac{1}{2}},azuq^{k-1},-azuq^{k-1};q)_\infty}$$
$$\cdot\frac{(-a^2zue^{i\theta}q^{k-\frac{1}{2}},-a^2zue^{-i\theta}q^{k-\frac{1}{2}},-a^2zue^{i\varphi}q^{k-1},-a^2zue^{-i\varphi}q^{k-1};q)_\infty}{(-auq^{-1}e^{i\theta},-auq^{-1}e^{-i\theta},-auq^{\frac{-1}{2}}e^{i\varphi},-auq^{\frac{-1}{2}}e^{-i\varphi};q)_\infty},$$

and

(2.10)
$$\sum_{n=0}^{\infty}\frac{1-a^4q^{2n}}{1-a^4}(q^{\frac{1}{2}}/a)^nC_n(x;a^2|q)C_n(y;a^2|q)Q_n(z;a,aq^{\frac{1}{2}},-a,-aq^{\frac{1}{2}})$$
$$= h_1(a)\frac{(a^2/z^2,qz^2/a^2,-az,-a^3zq;q)_\infty}{(qz^2,z^{-2},-zq/a,-z/a^3;q)_\infty}\sum_{k=0}^{\infty}\frac{(a^4;q)_k|(-aq^{\frac{1}{2}}e^{i\theta},-aq^{\frac{1}{2}}e^{i\varphi};q)_k|^2}{(q,-q^{\frac{1}{2}},-a^2q^{\frac{1}{2}},-a^3qz,-a^3q/z;q)_k}$$
$$\cdot q^k{}_{10}W_9(-q^{-k-\frac{1}{2}};ae^{i\theta},ae^{-i\theta},ae^{i\varphi},ae^{-i\varphi},-q^{-k}/a^2,q^{\frac{1}{2}-k}/a^2,q^{-k};q,q)$$
$$+ h_2(a)\frac{(\frac{a^2}{z^2},-az,a^4;q)_\infty(\frac{qz^2}{a^2};q^2)_\infty}{q(1-q)(q,-1,-q,z^{-2},\frac{-a^3}{z};q)_\infty}$$
$$\frac{|(a^2e^{2i\theta},qa^2e^{2i\varphi};q^2)_\infty|^2}{(\frac{zq^{\frac{1}{2}}e^{i\theta+i\varphi}}{a},\frac{zq^{\frac{1}{2}}e^{-i\theta-i\varphi}}{a},\frac{zq^{\frac{1}{2}}e^{i\theta-i\varphi}}{a},\frac{zq^{\frac{1}{2}}e^{i\varphi-i\theta}}{a};q)_\infty}$$
$$\cdot\sum_{k=0}^{\infty}\frac{(\frac{qz}{a},\frac{zq^{\frac{1}{2}}e^{i\theta+i\varphi}}{a},\frac{zq^{\frac{1}{2}}e^{-i\theta-i\varphi}}{a},\frac{zq^{\frac{1}{2}}e^{i\theta-i\varphi}}{a},\frac{zq^{\frac{1}{2}}e^{i\varphi-i\theta}}{a};q)_k}{(q,\frac{zq^{\frac{1}{2}}}{a},\frac{-zq^{\frac{1}{2}}}{a},\frac{-qz}{a^3},qz^2;q)_k}q^k\int_{-q}^{q}d_qu\,I_2(u),$$

where $I_2(u)$ is the same as $I_1(u)$ with z replaced by $zq^{\frac{1}{2}}/a^2$.

Evaluation of the expressions on the right hand side of (2.7) and (2.10) is clearly a formidable job. Fortunately, there do exist quadratic transformation formulas that enable us to simplify these expressions considerably.

3. Partial reductions of (2.7) and (2.10). Jain and Verma [**14**] found the following extension of Bailey's transformation formula [**5,13**]:

$$\begin{aligned}(3.1)\quad &\int_{q^2/f}^{q} \frac{(u, \frac{fu}{q}, \frac{au}{b}, \frac{au}{c}, \frac{au}{d}; q)_\infty}{(\frac{au}{q}, \frac{bu}{q}, \frac{cu}{q}, \frac{du}{q}, \frac{eu}{q}; q)_\infty} d_q u \\ &= \frac{(\frac{\lambda q}{a}, \frac{\lambda b}{a}, \frac{\lambda c}{a}, \frac{\lambda d}{a}, f, \frac{q}{f}; q)_\infty}{(b, c, d, \frac{\lambda f}{a}, \frac{aq}{\lambda f}, \frac{eq}{f}; q)_\infty} \\ &\int_{aq^2/\lambda f}^{q} d_q u \frac{(u, \frac{u\lambda^{\frac{1}{2}}}{q}, \frac{-u\lambda^{\frac{1}{2}}}{q}, \frac{\lambda f u}{aq}; q)_\infty}{(\frac{au}{f}, u\lambda^{\frac{1}{2}}, -u\lambda^{\frac{1}{2}}, \frac{\lambda u}{q}; q)_\infty} \\ &\cdot \frac{(\lambda u(aq)^{\frac{-1}{2}}, -\lambda u(aq)^{\frac{-1}{2}}, \lambda u a^{\frac{-1}{2}}, -\lambda u a^{\frac{-1}{2}}, \frac{au}{b}, \frac{au}{c}, \frac{au}{d}, \frac{au}{e}; q)_\infty}{(\frac{ua^{\frac{1}{2}}}{q}, \frac{-ua^{\frac{1}{2}}}{q}, u(\frac{a}{q})^{\frac{1}{2}}, -u(\frac{a}{q})^{\frac{1}{2}}, \frac{\lambda bu}{a}, \frac{\lambda cu}{a}, \frac{\lambda du}{a}, \frac{\lambda eu}{a}; q)_\infty},\end{aligned}$$

where $\lambda = \frac{qa^2}{bcd}$ and $f = \frac{eb^2c^2d^2}{a^2q^2} = \frac{ea^2}{\lambda^2}$. Using $d = -(aq)^{\frac{1}{2}}$ in Bailey's formula, that is, the terminating case of (3.1), we obtain a transformation of the first ${}_{10}\varphi_9$ series on the right hand sides of (2.7) and (2.10) as a balanced ${}_4\varphi_3$ series:

$$\begin{aligned}(3.2)\quad &{}_{10}W_9\left(-q^{-k-\frac{1}{2}}; ae^{i\theta}, ae^{-i\theta}, ae^{i\varphi}, ae^{-i\varphi}, \frac{-q^k}{a^2}, \frac{q^{\frac{1}{2}-k}}{a^2}, q^{-k}; q, q\right) \\ &= \frac{\left(-q^{\frac{1}{2}}, -a^2q^{\frac{1}{2}}, a^2e^{2i\theta}, -a^2q^{\frac{1}{2}}e^{-i\theta-i\varphi}, -a^2q^{\frac{1}{2}}e^{i\varphi-i\theta}; q\right)_k}{\left(a^4, -aq^{\frac{1}{2}}e^{i\theta}, -aq^{\frac{1}{2}}e^{-i\theta}, -aq^{\frac{1}{2}}e^{i\varphi}, -aq^{\frac{1}{2}}e^{-i\varphi}; q\right)_k} \\ &\cdot {}_4\varphi_3\left(\begin{matrix} a^2e^{-2i\theta}, & -q^{\frac{1}{2}}e^{-i(\theta+\varphi)}, & -q^{\frac{1}{2}}e^{i(\varphi-\theta)}, & q^{-k} \\ & -a^2q^{\frac{1}{2}}e^{i(\varphi-\theta)}, & -a^2q^{\frac{1}{2}}e^{-i(\theta+\varphi)}, & q^{1-k}\frac{e^{-2i\theta}}{a^2}\end{matrix}; q, q\right).\end{aligned}$$

For the q-integral parts of (2.7) and (2.10), we again use $d = -(aq)^{\frac{1}{2}}$ in (3.1), apply Bailey's 4-term transformation formula [**6**] for balanced and nonterminating ${}_{10}W_9$ series and obtain the needed result: (3.3)

$$\int_{-q}^{q} \frac{(u, -u, \lambda u a^{\frac{-1}{2}}, -\lambda u a^{\frac{-1}{2}}, iu(\frac{a}{fq})^{\frac{1}{2}}, -iu(\frac{a}{fq})^{\frac{1}{2}}, \frac{\lambda u}{e}, -(aq)^{\frac{1}{2}}\frac{u}{f}; q)_\infty}{(\frac{au}{f}, \frac{-au}{f}, \frac{\lambda u a^{\frac{-1}{2}}}{e}, \frac{-\lambda u a^{\frac{-1}{2}}}{e}, iu(\frac{aq}{f})^{\frac{1}{2}}, -iu(\frac{aq}{f})^{\frac{1}{2}}, \frac{-\lambda u}{a}, u(\frac{a}{q})^{\frac{1}{2}}; q)_\infty}$$
$$\cdot \frac{((aq)^{\frac{1}{2}}\frac{bu}{f}, (aq)^{\frac{1}{2}}\frac{cu}{f}; q)_\infty}{(-\frac{u}{b}(\frac{a}{q})^{\frac{1}{2}}, -\frac{u}{c}(\frac{a}{q})^{\frac{1}{2}}; q)_\infty} d_q u$$
$$= \frac{(-1, -q, b, c, \frac{bq}{f}, \frac{cq}{f}, \frac{eq}{f}, \frac{-aq^2}{bce}; q)_\infty (a; q^2)_\infty}{(\frac{qa^{\frac{1}{2}}}{bc}, f, \frac{q}{f}; q)_\infty (\frac{\lambda^2 q^2}{a^2}, \frac{\lambda^2 q^2}{ae^2}, \frac{aq}{b^2}, \frac{aq}{c^2}; q^2)_\infty}$$
$$\cdot \int_{q^2/f}^{q} \frac{(u, \frac{fu}{q}, \frac{au}{b}, \frac{au}{c}; q)_\infty}{(\frac{au}{q}, \frac{bu}{q}, \frac{cu}{q}, \frac{eu}{q}; q)_\infty} d_q u,$$

where $\lambda = -a^{\frac{3}{2}}q^{\frac{1}{2}}/bc$, $f = ea^2/\lambda^2$. If we now replace $a^{\frac{1}{2}}, b, c, e$ by $-ae^{-i\varphi}, -q^{\frac{1}{2}}e^{-i(\theta+\varphi)}, -q^{\frac{1}{2}}e^{i(\theta-\varphi)}$ and $-aq^{\frac{1}{2}-k}/z$, respectively, we get an expression for $\int_{-q}^{q} I_1(u)d_q u$ in terms of two balanced ${}_4\varphi_3$ series. Combination of (3.2) and (3.3) then give, after some simplifications,

(3.4)

$$\sum_{n=0}^{\infty} \frac{(q;q)_n}{(a^4;q)_n} \frac{1-a^4q^{2n}}{1-a^4} a^n C_n(x;a^2|q) C_n(y;a^2|q) Q_n$$
$$\cdot (z; a, aq^{\frac{1}{2}}, -a, -aq^{\frac{1}{2}})$$
$$= h_1(a) \frac{(a^2z^2, \frac{a^2}{z^2}; q)_\infty}{(qz^2, z^{-2}; q)_\infty} (1 + z(aq^{\frac{1}{2}} + a^{-1}q^{\frac{-1}{2}}) + z^2)^{-1}$$
$$\cdot \sum_{k=0}^{\infty} \frac{(a^2e^{-2i\varphi}, -q^{\frac{1}{2}}e^{-i\theta-i\varphi}, -q^{\frac{1}{2}}e^{i\theta-i\varphi}; q)_k}{(a^4, -azq^{\frac{3}{2}}, \frac{-aq^{\frac{3}{2}}}{z}; q)_k} (qa^2e^{2i\varphi})^k$$

$${}_4\varphi_3\left(\begin{matrix} q^{k+1}, & -a^2q^{k+\frac{1}{2}}e^{-i\theta-i\varphi}, & -a^2q^{k+\frac{1}{2}}e^{i\theta-i\varphi}, & a^2e^{2i\varphi} \\ & a^4q^k, & -azq^{k+\frac{3}{2}}, -aq^{k+\frac{3}{2}}/z & \end{matrix}; q, q\right)$$

$$+ h_1(a)\frac{(a^2z^2, \frac{a^2}{z^2}, -a^3zq^{\frac{-1}{2}}; q)_\infty}{(z^{-2}, aze^{i\theta-i\varphi}, aze^{-i\theta-i\varphi}, -azq^{\frac{-1}{2}}e^{2i\varphi}; q)_\infty}$$
$$\cdot \frac{(q, a^2e^{2i\varphi}, -a^2q^{\frac{1}{2}}e^{i\theta-i\varphi}, -a^2q^{\frac{1}{2}}e^{-i\theta-i\varphi}, ; q)_\infty}{(a^4; q)_\infty}$$
$$\cdot \sum_{k=0}^{\infty} \frac{(a^2e^{-2i\varphi}, -q^{\frac{1}{2}}e^{i\theta-i\varphi}, -q^{\frac{1}{2}}e^{-i\theta-i\varphi}, \frac{-aq^{\frac{1}{2}}}{z}; q)_k}{(q, -a^2q^{\frac{1}{2}}e^{i\theta-i\varphi}, -a^2q^{\frac{1}{2}}e^{-i\theta-i\varphi}, -q^{\frac{3}{2}}\frac{e^{-2i\varphi}}{az}; q)_k} q^k$$

$$_4\varphi_3\left(\begin{matrix} aze^{i\theta-i\varphi}, & aze^{-i\theta-i\varphi}, & -azq^{-k-\frac{1}{2}}e^{2i\varphi}, & -zq^{\frac{1}{2}}/a \\ & qz^2 & -a^3zq^{\frac{-1}{2}}, & -zq^{\frac{1}{2}-k}/a \end{matrix} ; q, q\right)$$

$$+ h_1(a)\frac{(a^2z^2, \frac{a^2}{z^2}; q)_\infty (q, a^2e^{-2i\varphi}, -q^{\frac{1}{2}}e^{i\theta-i\varphi},}{(z^{-2}, -azq^{\frac{1}{2}}, aze^{i\theta+i\varphi}, aze^{-i\theta-i\varphi}, aze^{i\theta-i\varphi}, aze^{i\varphi-i\theta}; q)_\infty}$$
$$\cdot \frac{-q^{\frac{1}{2}}e^{-i\theta-i\varphi}; q)_\infty}{(a^4; q)_\infty (1 + q^{\frac{1}{2}}e^{-2i\varphi}/az)}$$
$$\cdot \sum_{k=0}^{\infty} \frac{a^2e^{2i\varphi}, aze^{i\theta+i\varphi}, aze^{i\varphi-i\theta}, -a^3zq^{\frac{-1}{2}}; q)_k}{(q, a^3ze^{i\theta+i\varphi}, a^3ze^{i\varphi-i\theta}, -azq^{\frac{1}{2}}e^{2i\varphi}; q)_k} q^k$$

$$\cdot {}_4\varphi_3\left(\begin{matrix} q^{-k}, & aze^{i\theta-i\varphi}, & aze^{-i\theta-i\varphi}, & -zq^{\frac{1}{2}}/a \\ & qz^2 & -a^3zq^{\frac{-1}{2}}, & q^{1-k}e^{-2i\varphi}/a^2 \end{matrix} ; q, q\right).$$

Similarly, replacing $a^{\frac{1}{2}}, b, c, e$ in (3.3) by $-ae^{-i\varphi}, -q^{\frac{1}{2}}e^{-i(\theta+\varphi)}, -q^{\frac{1}{2}}e^{i(\theta-\varphi)}$ and $-a^3q^{-k}/z$, respectively, we obtain a transformation of $\int_{-q}^{q} I_2(u)\, d_qu$. (3.2) and (3.3) then lead to the formula

(3.5)
$$\sum_{n=0}^{\infty} \frac{1-a^4q^{2n}}{1-a^4}(q^{\frac{1}{2}}/a)^n C_n(x; a^2|q) C_n(y; a^2|q) Q_n(z; a, aq^{\frac{1}{2}}, -a, -aq^{\frac{1}{2}})$$
$$= h_1(a)\frac{(a^2/z^2, qz^2/a^2, -az, a^3zq; q)_\infty}{(-zq/a, -z/a^3, qz^2, z^{-2}; q)_\infty}$$
$$\cdot \sum_{k=0}^{\infty} \frac{(a^2e^{2i\theta}, -a^2q^{\frac{1}{2}}e^{-i\theta-i\varphi}; q)_k}{(q, -a^3zq; q)_k} \frac{(-a^2q^{\frac{1}{2}}e^{i\varphi-i\theta}; q)_k}{(-a^3q/z; q)_k} q^k$$

$$
{}_4\varphi_3\left(\begin{matrix} a^2e^{-2i\theta}, & -q^{\frac12}e^{-i\theta-i\varphi}, & -q^{\frac12}e^{i\varphi-i\theta}, & q^{-k} \\ -a^2q^{\frac12}e^{i\varphi-i\theta}, & -a^2q^{\frac12}e^{-i\theta-i\varphi}, & q^{1-k}e^{-2i\theta}/a^2 & \end{matrix};q,q\right)
$$

$$
+ h_1(a)\frac{(a^2z^2, \frac{qz^2}{a^2}, -az, a^2e^{2i\varphi}, a^2e^{-2i\varphi}, -q^{\frac12}e^{i\theta-i\varphi}, -q^{\frac12}e^{-i\theta-i\varphi}; q)_\infty}{q(1-q)(q, \frac{-zq}{a}, z^{-2}, \frac{-ze^{2i\varphi}}{a}, \frac{-aqe^{-2i\varphi}}{z}, \frac{zq^{\frac12}e^{i\theta-i\varphi}}{a}, \frac{zq^{\frac12}e^{-i\theta-i\varphi}}{a}; q)_\infty}
$$

$$
\cdot \sum_{k=0}^{\infty} \frac{(\frac{zq^{\frac12}e^{i\theta-i\varphi}}{a}, \frac{zq^{\frac12}e^{-i\theta-i\varphi}}{a}; q)_k}{(q, qz^2; q)_k}(a^2e^{2i\varphi})^k
$$

$$
\cdot \int_{-zq^{k+1}e^{2i\varphi}/a}^{q} \frac{(u, \frac{-auq^{-k}e^{-2i\varphi}}{z}, -a^2uq^{\frac{-1}{2}}e^{i\theta-i\varphi}, -a^2uq^{\frac{-1}{2}}e^{-i\theta-i\varphi}; q)_\infty}{(a^2q^{-1}e^{-2i\varphi}, -uq^{\frac{-1}{2}}e^{-i\theta-i\varphi}, -uq^{\frac{-1}{2}}e^{i\theta-i\varphi}, \frac{-a^3uq^{-k-1}}{z}; q)_\infty} d_qu.
$$

4. Proof of (1.8). We now use Watson's formula [**4**, 8.5(2)] to convert the ${}_4\varphi_3$ series in (3.5) to an ${}_8\varphi_7$ series, and Bailey's formula [**4**,8.5(3)] to convert the q-integral to another ${}_8\varphi_7$ series. We then interchange the order of summation in both terms of the right hand side of (3.5), use (2.6), simplify, and obtain

(4.1)

$$
\sum_{n=0}^{\infty} \frac{1-a^4q^{2n}}{1-a^4}(q^{\frac12}/a)^n C_n(x;a^2|q)C_n(y;a^2|q)Q_n(z;a,aq^{\frac12},-a,-aq^{\frac12})
$$

$$
= h_1(a)\frac{(\frac{a^2}{z^2}, \frac{qz^2}{a^2}, -az, -azq, -azqe^{-2i\theta}, -azqe^{-2i\varphi}, azq^{\frac12}e^{i\theta+i\varphi}; q)_\infty}{(z^{-2}, qz^2, azq^{\frac32}e^{-i\theta-i\varphi}, \frac{zq^{\frac12}e^{i\theta+i\varphi}}{a}, \frac{zq^{\frac12}e^{-i\theta-i\varphi}}{a}, \frac{zq^{\frac12}e^{i\theta-i\varphi}}{a}, \frac{zq^{\frac12}e^{i\varphi-i\theta}}{a}; q)_\infty}
$$

$$
\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{(a^2;q)_k(1-a^2q^{2k})}{(q;q)_k(1-a^2)}
$$

$$
\cdot \frac{(-q^{\frac12}e^{i\theta-i\varphi}, -q^{\frac12}e^{i\varphi-i\theta}, -q^{\frac12}e^{-i\theta-i\varphi}, -a^2q^{\frac12}e^{-i\theta-i\varphi}; q)_{k+j}}{(-azq, \frac{-zq}{a}, -azqe^{-2i\theta}, -azqe^{-2i\varphi}; q)_{k+j}}
$$

$$
\cdot \frac{(-q^{\frac12}e^{i\theta+i\varphi}; q)_k(\frac{zq^{\frac12}e^{-i\theta-i\varphi}}{a}; q)_j}{(-a^2q^{\frac12}e^{-i\theta-i\varphi}; q)_k(q;q)_j} \frac{(azq^{\frac12}e^{-i\theta-i\varphi}; q)_{2k+j}(1-azq^{2k+2j+\frac12}e^{-i\theta-i\varphi})}{(a^2q;q)_{2k+j}(1-azq^{\frac12}e^{-i\theta-i\varphi})}
$$

$$
\cdot (-az)^{k+j}(-q^{\frac12}e^{i\theta+i\varphi})^j.
$$

Setting $k + j = \ell$ and using the well-known formula for a very-well-poised ${}_6\varphi_5$ series [**13, IV.7**], we find that the double series above reduces to

$$\frac{(azq^{\frac{3}{2}}e^{-i\theta-i\varphi}, azq^{\frac{1}{2}}e^{-i\theta-i\varphi}, azq^{\frac{1}{2}}e^{i\theta-i\varphi}, azq^{\frac{1}{2}}e^{i\varphi-i\theta}; q)_\infty}{(-az, -azq, -azqe^{-2i\theta}, -azqe^{-2i\varphi}; q)_\infty}.$$

Thus, we get

(4.2)

$$\sum_{n=0}^{\infty} \frac{1-a^4q^{2n}}{1-a^4}(q^{\frac{1}{2}}/a)^n C_n(x;a^2|q)C_n(y;a^2|q)Q_n(z;a,aq^{\frac{1}{2}},-a,-aq^{\frac{1}{2}})$$
$$= h_1(a)\frac{(\frac{a^2}{z^2}, \frac{qz^2}{a^2}, azq^{\frac{1}{2}}e^{i\theta+i\varphi}, azq^{\frac{1}{2}}e^{-i\theta-i\varphi}, azq^{\frac{1}{2}}e^{i\theta-i\varphi}, azq^{\frac{1}{2}}e^{i\varphi-i\theta}; q)_\infty}{z^{-2}, zq^2, \frac{zq^{\frac{1}{2}}e^{i\theta+i\varphi}}{a}, \frac{zq^{\frac{1}{2}}e^{-i\theta-i\varphi}}{a} \frac{zq^{\frac{1}{2}}e^{i\theta-i\varphi}}{a}, \frac{zq^{\frac{1}{2}}e^{i\varphi-i\theta}}{a}; q)_\infty}$$

Use of (1.2) followed by replacement of a^2 by β completes the proof of (1.8).

5. Proof of (1.9). We now turn to (3.4) and observe that the coefficients of the first two ${}_4\varphi_3$ series on the right hand side are so matched that they can be combined into a single ${}_8\varphi_7$ series by Bailey's formula [**4, 8.5(3)**]. It can be easily seen that this series is ${}_8W_7(a^4z^2/q; a^4/q, aze^{i\theta-i\varphi}, aze^{-i\theta-i\varphi}, -azq^{\frac{1}{2}}, -azq^{-k-\frac{1}{2}}e^{2i\varphi}; q, q^{k+1})$. On the other hand, the terminating ${}_4\varphi_3$ series in (3.4) transforms to ${}_8W_7(a^4z^2; a^4/q, aze^{i\theta-i\varphi}, aze^{-i\theta-i\varphi}, -azq^{\frac{1}{2}}, q^{-k}; q, -azq^{k+\frac{1}{2}}e^{2i\varphi})$, by Watson's formula [**4, 8.5(2)**]. When we use them in (3.4), simplify the coefficients and interchange the order of summations, we find that the right hand side of (3.4) equals

(5.1)

$$\frac{h_1(a)(q,z^2;q)_\infty}{p(z)(a^4,-azq^{\frac{1}{2}};q)_\infty}\sum_{n=0}^{\infty}\frac{(\frac{a^4z^2}{q};q)_n(1-a^4z^2q^{2n-1})(\frac{a^4}{q},-azq^{\frac{1}{2}};q)_n}{(q;q)_n(1-\frac{a^4z^2}{q})(qz^2,-a^3zq^{-\frac{1}{2}};q)_n}$$
$$\cdot\frac{(aze^{i\theta-i\varphi}, aze^{-i\theta-i\varphi};q)_n}{(a^3ze^{i\varphi-i\theta}, a^3ze^{i\theta+i\varphi};q)_n}q^n$$
$$\cdot\Big(\frac{(-a^3zq^{\frac{-1}{2}}, a^3ze^{i\theta+i\varphi}, -a^3ze^{i\varphi-i\theta} - a^3zq^{n+\frac{1}{2}}e^{-2i\varphi};q)_\infty}{(a^4z^2, aze^{-i\theta-i\varphi}, aze^{i\theta-i\varphi}, -azq^{n-\frac{1}{2}}e^{2i\varphi};q)_\infty}$$

$$
{}_3\varphi_2\left(\begin{matrix} a^2e^{-2i\varphi}, & -q^{\frac{1}{2}}e^{i\theta-i\varphi}, & -q^{\frac{1}{2}}e^{-i\theta-i\varphi}, \\ & -a^3zq^{n+\frac{1}{2}}e^{-2i\varphi}, & -q^{\frac{3}{2}-n}\frac{e^{-2i\varphi}}{az} \end{matrix} ; q, q\right)
$$

$$
+ \frac{(a^2e^{-2i\varphi}, -q^{\frac{1}{2}}e^{i\theta-i\varphi}, -q^{\frac{1}{2}}e^{-i\theta-i\varphi}; q)_\infty}{(azq^ne^{i\theta+i\varphi}, azq^ne^{i\varphi-i\theta}, aze^{-i\theta-i\varphi}, aze^{i\theta-i\varphi}; q)_\infty(1+\frac{q^{\frac{1}{2}}e^{-2i\varphi}}{az})}
\cdot q^{\frac{n^2}{2}}\frac{(-a^3zq^{\frac{-1}{2}}; q)_n(-azqe^{2i\varphi})^n}{(-azq^{\frac{1}{2}}e^{2i\varphi}; q)_n(a^4z^2; q)_{2n}})
$$

$$
{}_3\varphi_2\left(\begin{matrix} azq^ne^{i\theta+i\varphi}, & azq^ne^{i\varphi-i\theta}, & -a^3zq^{n-\frac{1}{2}}; q, q \\ & a^4z^2q^{2n}, & -azq^{n+\frac{1}{2}}e^{2i\varphi} \end{matrix}\right)\Bigg),
$$

where

$$
p(z) = \frac{(z^2, z^{-2}; q)_\infty}{(a^2z^2, a^2/z^2; q)_\infty} = p(z^{-1}). \tag{5.2}
$$

The coefficients of the two balanced ${}_3\varphi_2$ series in (5.1) are so matched that Sears' summation formula [**12, 13**]is applicable. Thus, we obtain, after some simplifications,

(5.3)

$$
\sum_{n=0}^{\infty}\frac{(q;q)_n}{(a^4;q)_n}\frac{1-a^4q^{2n}}{1-a^4}a^nC_n(x;a^2|q)C_n(y;a^2|q)Q_n(z;a,aq^{\frac{1}{2}},-a,-aq^{\frac{1}{2}})
$$

$$
= \frac{h_1(a)(q,z^2;q)\infty}{p(z)(a^4,a^4z^2;q)_\infty}, \frac{(a^3ze^{i\theta+i\varphi}, a^3ze^{-i\theta-i\varphi}, a^3ze^{i\theta-i\varphi}, a^3ze^{i\varphi-i\theta}; q)_\infty}{(aze^{i\theta+i\varphi}, aze^{-i\theta-i\varphi}, aze^{i\theta-i\varphi}, aze^{i\varphi-i\theta}; q)_\infty}
$$

$$
\cdot {}_8W_7\left(\frac{a^4z^2}{q}; \frac{a^4}{q}, aze^{i\theta+i\varphi}, aze^{-i\theta-i\varphi}, aze^{i\theta-i\varphi}, aze^{i\varphi-i\theta}; q, q\right).
$$

Replacing a by $\beta^{\frac{1}{2}}$, and using (1.1) and Bailey's summation formula [**13**, IV.15] , we finally obtain (1.9).

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