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MODULI OF CONTINUITY AND GENERALIZED BCH SETS

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1. Introduction. Let $\Lambda = \{|z| < 1\}$ and $C = \{|z| = 1\}$. For $\omega = \omega_h$, the modulus of continuity of a bounded complex-valued function h on $[0, 2\pi]$, the class of ω-sets is defined to be the subclass of closed sets K of linear measure 0 in C satisfying

$$\sum_k \omega(|I_k|) < \infty, \tag{1.1}$$

where (I_k) is an enumeration of the component arcs of $C\backslash K$ and $|I_k|$ is the length of I_k. When ω is *equivalent* to the modulus of continuity $\nu(t) = t\log(2\pi e/t)$ (that is, $m\omega \le \nu \le M\omega$ for some $m, M > 0$), then (1.1) is referred to as the Carleson condition and the ω-sets are called the BCH (Beurling–Carleson–Hayman) sets.

The BCH sets first arose in the characterization by Beurling [**4**] and Carleson [**5**] of the boundary zero sets of the functions in the class Λ_ω when $\omega(t) = t^\alpha, 0 < \alpha \le 1$. By definition, Λ_ω is the class of continuous functions f on $\overline{\Lambda} = \{|z| \le 1\}$ that are analytic in Λ and satisfy

$$|f(z) - f(w)| \le c\omega(|z - w|), \quad z, w \in \Lambda,$$

for some $c > 0$. Recently, Shirokov [**15**] generalized the result of Beurling and Carleson by characterizing the complete zero sets $Z(f)$ of functions f in Λ_ω for arbitrary moduli of continuity ω.

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THEOREM A. (Shirokov). *Let ω be a modulus of continuity such that $0 \not\equiv \omega < 1$, and let K be a closed subset of $\overline{\Lambda}$. Then there exists $f \in \Lambda_\omega, f \not\equiv 0$, such that $K = Z(f)$ if and only if*

$$\sum_{\xi \in \Lambda \cap K} (1 - |\xi|) < \infty \tag{1.2}$$

and

$$\int_C \log(\omega(\rho_K(z)))|dz| > -\infty \tag{1.3}$$

where $\rho_K(z)$ denotes the distance from z to K.

As a corollary of Theorem A, we have a characterization of the zero sets $Z(f^*)$ of the boundary functions f^* of $f \in \Lambda_\omega$. It turns out that when $K \subseteq C$, condition (1.3) defines a class of generalized BCH sets (see §3). Also, in this case the function f may be taken to be an outer function.

COROLLARY A. *There exists (an outer function) $f \in \Lambda_\omega$, $f \not\equiv 0$ such that $K = Z(f^*)$ if and only if K is a ϕ_ω–set, where ϕ_ω is the modulus of continuity defined by*

$$\phi_\omega(t) = \int_0^t \log \frac{1}{\omega(s)} ds, \quad t \in [0, 2\pi]. \tag{1.4}$$

A question that arises immediately from Theorem A is the following: For what moduli of continuity $\omega, 0 \not\equiv \omega < 1$, are the classes of zero sets

$$Z(\Lambda_\omega) = \{Z(f) : f \in \Lambda_\omega, f \not\equiv 0\}$$

the same? It is evident from Theorem A that if there exist $\alpha, \beta > 0$ such that

$$\omega^\alpha \le \nu \le \omega^\beta, \tag{1.5}$$

then $Z(\Lambda_\omega) = Z(\Lambda_\nu)$. In §3 we prove the converse. After some preliminary results concerning moduli of continuity in §2, we show in

§3 that the ϕ_ω–sets are the same as the ϕ_ν–sets if and only if (1.5) holds. Together with Corollary A, this gives the converse.

In §4, the relationship between generalized BCH sets and Hausdorff measure is considered. The main result is a generalization of the following theorem and corollary of J. Shapiro [**14**] (see also P. Ahern [**1**]).

THEOREM B. (Shapiro, Ahern). *Suppose that ω is a monotone non-decreasing continuous function defined on $(0, 2\pi]$ such that $\omega(t)/t$ is strictly decreasing. Then the following conditions on ω are equivalent.*

(a) $H_\omega(K) = 0$ *for every* BCH *set* K.

(b) $\int_0^{2\pi} \frac{1}{\omega(t)} dt = \infty$.

Here, H_ω denotes the Hausdorff measure generated by the determining function ω.

COROLLARY B. *The following assertions about ω are equivalent.*

(a) $\mu(K) = 0$ *for every* BCH *set* K *and every finite positive Borel measure* μ *on* C *with* $\omega_\mu(t) = O(\omega(t))$.

(b) $\int_0^{2\pi} \frac{1}{\omega(t)} dt = \infty$.

The proofs of the generalizations rely on various results of Ahern including one that was previously unpublished. At this point the authors wish to thank Professor Ahern for some helpful discussions and for providing them with his unpublished result.

In §5 we prove a boundary analogue of some theorems concerning radial descent to 0. The following theorem of A. Samuelsson [**13**; Theorem 5.7] provides a Hausdorff measure condition on the set of radii along which a bounded analytic function $f \not\equiv 0$ can descend to 0 at an asymptotic rate given in terms of ω.

THEOREM C. (Samuelsson). *Let ω be as in Theorem B and let $f \not\equiv 0$ be a bounded analytic function with* $\sup|f| \le 1$. Let

$$R(f, \omega, a) = \{\eta \in C : \limsup_{r\to 1} \frac{\omega(1-r)}{(1-r)\log(1/|f(r\eta)|)} < a\},\ a \in (0, \infty].$$

Then $H_\omega[R(f, \omega, a)] < \infty$ for each $a \in (0, \infty)$.

On the topological side, a result of K.F. Barth and W.J. Schneider [**2**] (see also [**3**]) shows that $R(f, \omega, a)$ must be a first–category set for each $a \in (0, \infty]$. For the boundary analogue of these results, the measure and topological conditions combine simply to give a necessary and sufficient condition involving generalized BCH sets. We conclude the section with a result based on Corollary A which provides a partial converse to the combined results of Samuelsson and of Barth and Schneider.

Throughout this paper, we shall use the basic results and terminology of H^p theory as presented in [**6**]. The authors wish to express their appreciation to the referee for several helpful suggestions.

2. Moduli of continuity. By definition, the modulus of continuity ω_h of a bounded complex–valued function h on $[0, 2\pi]$ is

$$\omega_h(t) = \sup\{|h(x)-h(y)| \ : \ |x-y| \le t\}, \quad t \in [0, 2\pi].$$

In the sequel, we shall call ω a modulus of continuity if $\omega = \omega_h$ for some bounded function h on $[0, 2\pi]$. When it is convenient, we shall assume without mention that a function defined on $[0, 2\pi]$ has been extended to $[0, \infty)$ so that it is constant on $[2\pi, \infty)$.

PROPOSITION 2.1. *A real–valued function ω on $[0, 2\pi]$ is a modulus of continuity if and only if $\omega(0) = 0$, it is monotone nondecreasing, and it is subadditive, that is,*

$$\omega(t+s) \le \omega(t) + \omega(s), \quad t, s \ge 0. \tag{2.1}$$

In this case, ω is its own modulus of continuity and satisfies

$$\frac{\Lambda}{\Lambda+1}\omega(t) \le \omega(\Lambda t) \le (\Lambda+1)\omega(t), \ \ \Lambda, t \ge 0, \tag{1}$$

and

$$\frac{\omega(s)}{s} < 2\frac{\omega(t)}{t}, \quad 0 < t < s. \tag{2}$$

PROOF. We omit the proof of the characterizing properties of the modulus of continuity and use these to prove (1) and (2).

(1). For the right–hand inequality, let n be the nonnegative integer such that $n \le \Lambda < n+1$. Then

$$\omega(\lambda t) \le \omega[(n+1)t] \le (n+1)\omega(t) \le (\lambda+1)\omega(t).$$

Putting aside the trivial case when $\Lambda = 0$, the left–hand inequality follows from the right–hand inequality after replacing Λ by $1/\Lambda$ and t by Λt. (2). Taking $\lambda = s/t$ in (1), we have

$$\frac{\omega(s)}{s} = \frac{\omega(\lambda t)}{\lambda t} \leq (\frac{\lambda + 1}{\lambda})\frac{\omega(t)}{t} < 2\frac{\omega(t)}{t}.$$

This completes the proof.

Note that (2) says that $\omega(t)/t$ is "almost decreasing" when ω is a modulus of continuity.

COROLLARY 2.1. *If ω is a modulus of continuity and $L \equiv \liminf_{t\to 0} \omega(t)/t < \infty$, then $\sup_{t>0} \omega(t)/t \leq 2L$.*

PROOF. Let $s, \varepsilon > 0$. Choose a strictly decreasing sequence $(t_n)_1^\infty$ converging to 0 such that $\lim_{n\to\infty} \omega(t_n)/t_n = L$. Select a positive integer N such that $n \geq N$ implies $t_n < s$ and $\omega(t_n)/t_n \leq L + \varepsilon$. Then Proposition 2.1(2) implies that

$$\frac{\omega(s)}{s} < \frac{2\omega(t_N)}{t_N} \leq 2L + 2\varepsilon.$$

The required conclusion follows.

Recall that a bounded function h on $[0, 2\pi]$ (along with ω_h) is said to be Lipschitz of order $\alpha \in (0, 1]$ if $\omega_h(t) \leq Mt^\alpha$ for some $M > 0$. The next two corollaries follow from Corollary 2.1.

COROLLARY 2.2. *A bounded function h on $[0, 2\pi]$ is Lipschitz of order 1 (identically 0) if and only if $L < \infty$ $(= 0)$, respectively.*

COROLLARY 2.3. *If ω is a modulus of continuity, then either ω is Lipschitz of order 1 or else $\omega'(0) = \infty$.*

The primary class of moduli of continuity consists of those that are concave downward. For this class, Proposition 2.1(2) can be improved.

PROPOSITION 2.2. *Let ω be a monotone nondecreasing function on $[0, 2\pi]$ with $\omega(0) = 0$. If $\omega(t)/t$ is a monotone nonincreasing function on $(0, 2\pi]$, then ω is a modulus of continuity. In particular, if ω is concave downward, then $\omega(t)/t$ is monotone nonincreasing on $(0, 2\pi]$ and ω is* *a modulus of continuity that is continuous on* $(0, 2\pi]$. *In addition, if* ω *is concave downward and* $\omega'(t_0)$ *exists, then* $\omega(t_0) \geq t_0\omega'(t_0)$ *with equality only if* $\omega(t) = \omega'(t_0)t$ *for all* $t \in [0, t_0]$.

PROOF. We shall prove only the first assertion by verifying (2.1). Putting aside the trivial case when t or s is 0, suppose without loss of generality that $s \geq t > 0$. Then by assumption,

$$\frac{\omega(t+s)}{t+s} \leq \frac{\omega(s)}{s} \leq \frac{\omega(t)}{t}.$$

Hence

$$\begin{aligned} \omega(t+s) &\leq \frac{t+s}{s}\omega(s) \\ &= \frac{t}{s}\omega(s) + \omega(s) \\ &\leq \omega(t) + \omega(s) \end{aligned}$$

as required.

Given a continuous modulus of continuity ω, Lebedev [**10, 16**; p. 100] introduced the function

$$\tilde{\omega}(t) = \sup_{y>0, 0\leq x\leq t} \left(\frac{x}{x+y}\omega(t+y) + \frac{y}{x+y}\omega(t-x)\right) \tag{2.2}$$

as the smallest concave–downward modulus of continuity ν such that $\nu \geq \omega$. In fact, the following proposition holds under a weaker hypothesis on ω.

PROPOSITION 2.3. *Let* ω *be a monotone nondecreasing function on* $[0, 2\pi]$ *satisfying* $\omega(0) = 0$. *The function* $\tilde{\omega}$ *defined by* (2.2) *is the smallest concave–downward modulus of continuity greater than or equal to* ω.

We shall omit the proof since it is somewhat lengthy and elementary. Though Proposition 2.3 does not require ω to be a modulus of continuity, this requirement is essential in the next result.

LEMMA 2.1. *If* ω *is a modulus of continuity, then* $\tilde{\omega} \leq 2\omega$.

PROOF. We need only consider $t_0 \in (0, 2\pi)$ for which $\tilde{\omega}(t_0) > \omega(t_0)$. Then, for $\varepsilon > 0$, there exist $x \in (0, t_0)$ and $y > 0$ such that

$$\tilde{\omega}(t_0) \leq \frac{x}{x+y}\omega(t_0+y) + \frac{y}{x+y}\omega(t_0-x) + \varepsilon.$$

By Proposition 2.1(1), we have

$$\begin{aligned}\tilde{\omega}(t_0) &\leq \left(\frac{x}{x+y}(\frac{t_0+y}{t_0}+1)+\frac{y}{x+y}(\frac{t_0-x}{t_0}+1)\right)\omega(t_0)+\varepsilon\\ &= 2\omega(t_0)+\varepsilon.\end{aligned}$$

The required conclusion now follows.

In the sequel, we shall call a modulus of continuity smooth (C^n–smooth, C^∞–smooth) if it has a continuous derivative (a continuous n^{th} derivative, infinitely many derivatives) on $(0, 2\pi]$ and satisfies

$$\lim_{t\to 0}\omega'(t)=\omega'(0)\in[0,\infty]. \tag{2.3}$$

THEOREM 2.1. *Given a continuous modulus of continuity ω and $\varepsilon > 0$ there exists a C^∞–smooth concave downward modulus of continuity $\overline{\omega}$ such that*

$$(\frac{1}{2}-\varepsilon)\omega \leq \overline{\omega} \leq \omega. \tag{2.4}$$

PROOF. By Lemma 2.1 we have $\tilde{\omega} \leq 2\omega$. Select a decreasing sequence $(t_n)_1^\infty$ converging to 0 with $t_1 = 2\pi$ such that the function ν, whose graph consists of the union of the origin and the line segments ℓ_n between $[t_n, \tilde{\omega}(t_n)]$ and $[t_{n+1}, \tilde{\omega}(t_{n+1})]$, for $n = 1, 2, ...,$ lies in the set

$$S = \{(t, s) : (1-\varepsilon)\tilde{\omega}(t) \leq s \leq \tilde{\omega}(t)\}.$$

We now define a function μ by "smoothing the corners" of ν at the points $[t_n, \tilde{\omega}(t_n)]$, for $n = 1, 2, ...,$ in such a way that the graph of μ is contained in S. This can be accomplished by smoothly interpolating C^∞–smooth concave downward function $\leq \nu$ to replace ν in suitably small neighborhoods of these points.

Define $\overline{\omega} = \mu/2$. Then (2.4) follows from the construction of $\overline{\omega}$ relative to S. In addition, (2.3) for $\overline{\omega}$ is a consequence of the fact that $\overline{\omega}$ is concave downward and C^∞–smooth on $(0, 2\pi]$. In fact, by the concavity of $\overline{\omega}$, we have $\overline{\omega}(t)/t$ is a monotone nonincreasing function on $(0, 2\pi]$. In particular, $\overline{\omega}'(0)$ exists in $[0, \infty]$. Also, by the smoothness of $\overline{\omega}$ on $(0, 2\pi]$ and L' Hôpital's rule, we have

$$\overline{\omega}'(0) = \lim_{t\to 0}\overline{\omega}'(t).$$

This completes the proof.

3. ω–sets. In this section we define ω–sets and prove some results concerning them. These lead to Corollary 3.1 which asserts that the classes of zero sets $Z(\Lambda_\omega)$ and $Z(\Lambda_\nu)$ are equal if and only if the moduli of continuity ω and ν ($0 \not\equiv \omega, \nu < 1$) satisfy (1.5), that is, there exist $\alpha, \beta > 0$ such that $\omega^\alpha \leq \nu \leq \omega^\beta$.

DEFINITION 3.1. *Let ω be a modulus of continuity. A subset K of C is called an ω–set if and only if K is closed, $|K| = 0$, and $\sum \omega(|I_k|) < \infty$, where (I_k) is an enumeration of the component arcs of $C \backslash K$.*

Here, we denote the linear measure of a measurable subset E of C by $|E|$. If ω is a modulus of continuity equivalent to $t \log(2\pi e/t)$, then the ω–sets are the well–known BCH (Beurling–Carleson–Hayman) sets and $\sum \omega(|I_k|) < \infty$ is the Carleson condition (cf. [**5**]). When ω is Lipschitz of order 1 (respectively discontinuous), then the ω–sets are the closed sets of measure 0 (respectively the finite sets). The latter are the extreme cases. The remaining one, in which ω is continuous and $\omega'(0) = \infty$ (for example, the defining modulus of continuity of the BCH sets given above) has the most interest.

THEOREM 3.1. *Let ω be a continuous modulus of continuity. Then the class of ω–sets is a subclass of the closed subsets of C having zero meausre that contains all finite sets and some infinite sets. It is closed under*

(1) *inclusion,*
(2) *finite unions, and*
(3) *homeomorphisms of C that are Lipschitz of order* 1.

PROOF. It is immediate from the definition that every finite set is an ω–set. To show that there are infinite ω–sets, select a sequence of positive numbers $(t_k)_1^\infty$ such that $\sum \omega(t_k) < \infty$ and $\sum t_k < 2\pi$. We can now easily construct a closed set K with a single cluster point such that the sequence of complementary component arcs $(I_k)_1^\infty$ satisfies $|I_{k+1}| = t_k$, for $k = 1, 2, \ldots$, and $|I_1| = 2\pi - \sum t_k$. It is also not difficult to construct perfect ω–sets using a Cantor–type construction.

(1). Let F be a closed subset of an ω–set K. Clearly, $|F| = 0$. Let (I_k) be an enumeration of the component arcs of $C \backslash F$. Denoting the components of $I_k \backslash K$ by I_{kj} for each k, it follows from the inclusion of closed sets $F \subseteq K$ that (I_{kj}) contains all of the component arcs of $C \backslash K$. By the subadditivity of ω, we have

$$\sum_k \omega(|I_k|) \leq \sum_{k,j} \omega(|I_{k,j}|) < \infty.$$

Thus F is an ω–set as required.

(2). We prove only that the union of two ω–sets K and F is again an ω–set. The general argument is an induction using this as the inductive step. Clearly, $K \cup F$ is a closed set of measure 0. Let (I_j) be an enumeration of the arcs of $C \backslash F$. Divide the sequences of component arcs of $C \backslash K$ into two subsequences (J_k) and (J_k^*), where (J_k) is an enumeration of those arcs having the property that J_k is contained in some I_j. Now every component arc of $C \backslash (K \cup F)$ is contained in one of the sequences (J_k) or $(J_k^* \bigcap I_j)$. Since K is an ω–set, we have $\sum \omega(|J_k|) < \infty$. On the other hand, each I_j has nonempty intersection with at most two arcs from the sequence (J_k^*), so that $\sum \omega(|J_k^* \bigcap I_j|) \leq 2 \sum \omega(|I_j|) < \infty$, using the assumption that F is an ω–set. We conclude that $K \cup F$ is an ω–set as required.

(3). Let K be an ω–set and (I_k) an enumeration of the component arcs of $C \backslash K$. If h is a homeomorphism that is Lipschitz of order 1 and $|K| = 0$, it follows that $h(K)$ is a closed set of measure 0. Furthermore, $(h(I_k))$ is an enumeration of the component arcs of $C \backslash h(K)$ and there exists $M > 0$ such that $|h(I_k)| \leq M|I_k|$, for each k. We can now apply Proposition 2.1(1) to conclude that $\sum \omega(|h(I_k)|) < \infty$ and hence, $h(K)$ is an ω–set.

In the sequel, we shall write $\mu \approx \nu$ and say that μ is *equivalent* to ν when μ and ν are real–valued functions defined on $(0, 2\pi]$ for which there exist constants $m, M > 0$ such that $m\mu \leq \nu \leq M\mu$ for positive t sufficiently near 0. In addition, we call a real-valued function μ on $[0, 2\pi]$ *allowed* if it is a monotone nondecreasing continuous function with $\mu(0) = 0, \mu \leq 1$, and $\log(1/\mu)$ integrable. Note that by Corollary 2.1, a continuous modulus of continuity $\omega \not\equiv 0$ satisfies $\omega(t) \geq ct$ for some $c > 0$ and hence is allowed if and only if $0 \not\equiv \omega \leq 1$.

PROPOSITION 3.1. *If μ is allowed, then*

$$\phi_\mu(t) = \int_0^t \log(1/\mu(s))ds, \quad t \in [0, 2\pi],$$

is a smooth concave–downward modulus of continuity with $\phi'_\mu(0) = \infty$. In the opposite direction, every continuous modulus of continuity ω

satisfying $\omega'(0) = \infty$ *is equivalent to* ϕ_μ *for some allowed* μ. *If* ω *is a smooth concave–downward modulus of continuity with* $\omega'(0) = \infty$, *then* $\omega = \phi_\mu$, *where* $\mu = \exp(-\omega')$.

PROOF. The assertions made in the first and third sentences present no difficulty. The assertion of the second sentence follows from Theorem 2.1 and the one made in the third.

The next proposition gives the generalized Beurling formulation of the definition of ω–sets for certain ω (cf. [4]). We denote the arclength distance between $\xi \in C$ and $K \subseteq C$ by $\rho(\xi, K)$ and use the convention that $\rho(\xi, \emptyset) = 2\pi$.

PROPOSITION 3.2. *Let* ω *be a smooth concave–downward modulus of continuity satisfying* $\omega'(0) = \infty$ *and let* K *be a closed subset of* C. *Then* K *is an* ω*–set if and only if*

$$\int_C \omega'(\rho(\xi, K))|d\xi| < \infty. \tag{3.2}$$

PROOF. Suppose that K is an ω–set. Then by definition, $|K| = 0$ and $\sum \omega(|I_k|) < \infty$, where (I_k) is an enumeration of the component arcs of $C\backslash K$. Writing the integral in (3.2) as a sum of integrals over half–arcs in the complement of K, we have

$$\int_C \omega'(\rho(\xi, K))|d\xi| = 2\sum \omega(\frac{|I_k|}{2}). \tag{3.3}$$

Necessity follows from the assumption that $\sum \omega(|I_k|) < \infty$ and the monotonicity of ω.

Conversely, suppose that (3.2) holds. Then $|K| = 0$ so that (3.3) is valid and $\sum \omega(\frac{|I_k|}{2}) < \infty$. This implies that $\sum \omega(|I_k|) < \infty$ by Proposition 2.1(1), completing the proof of sufficiency.

We note that somewhat shorter proofs of Theorem 3.1(1) and (2) can be given using Propositions 3.1 and 3.2. It is also to be noted that we have used the arclength distance $\rho(\xi, K)$ instead of the Euclidean distance $\rho_K(z)$ in Shirokov's Theorem A. This is done so that the calculations, such as (3.3), come out more simply. However, it is elementary to show, using a change of variables and Proposition 2.1(1), that Proposition 3.2 remains valid when $\rho(\xi, K)$ is replaced by $\rho_K(\xi)$.

PROPOSITION 3.3. *Let K be an infinite closed set of measure* 0.

(1) *There exists a continuous modulus of continuity ω such that K is not an ω–set.*

(2) *There is a modulus of continuity ω with $\omega'(0) = \infty$ such that K is an ω–set.*

PROOF. (1). Let $(J_k)_1^\infty$ be a sequence of component arcs of $C\backslash K$ such that $|J_{k+1}| < |J_k|$ for each k. Define ν to be the function on $[0, 2\pi]$ whose graph is the union of the origin with the line segments connecting $[|J_{k+1}|, 1/(k+1)]$ to $(|J_k|, 1/k)$ for $k = 1, 2, ...$, and $(|J_1|, 1)$ to $(2\pi, 1)$. Then $\omega_\nu \geq \nu$ and it is immediate that K is not an ω_ν–set.

(2). Though a direct proof can be given, we shall use a shorter one based on complex analytic methods. By a classical theorem of P. Fatou [**7**; p. 80] there exists a function f continuous on $\overline{\Lambda}$ and analytic in Λ such that $Z(f^*) = \{\eta \in C : f^*(\eta) = 0\} = K$. Then f must be one of the classes Λ_ν where ν is a continuous modulus of continuity such that $0 \not\equiv \nu < 1$. By necessity, in Theorem A and Proposition 3.2, it follows that K is a ϕ_ν–set. From Proposition 3.1, we have $\omega = \phi_\nu$ is a smooth concave downward modulus of continuity such that $\omega'(0) = \infty$. This completes the proof.

It is evident that equivalent moduli of continuity generate the same class of generalized BCH sets. The next proposition asserts that the converse also holds for nontrivial moduli of continuity.

PROPOSITION 3.4. *Let ω and ν be moduli of continuity that are not identically* 0. *Then the classes of ω–sets and ν–sets are the same if and only if $\omega \approx \nu$.*

PROOF. We prove only necessity. Suppose that $\omega \not\approx \nu$. Then at least one of ω and ν is continuous. If one is discontinuous, say ω, then the ω–sets are the finite sets. However, Theorem 3.1 insures that the ν–sets contain some infinite sets, so that the ν–sets are not the same as the ω–sets. Suppose now that both ω and ν are continuous and $\omega \not\approx \nu$, say $\limsup_{t\to 0}(\nu/\omega) = \infty$. Then there exists a decreasing sequence $(t_n)_1^\infty$ converging to 0 such that $\nu(t_n) \geq n\omega(t_n)$ for each n, and $\sum n^2\omega(t_n) < \infty$. Let k_n be the greatest integer less than or equal to $1/(\omega(t_n)n^2)$ for each n. It follows that

$$\frac{1}{\omega(t_n)n^2} - 1 < k_n \leq \frac{1}{\omega(t_n)n^2}$$

for $n = 1, 2, \ldots$, so that

$$\sum_1^\infty k_n \omega(t_n) \le \sum_1^\infty \frac{1}{n^2} < \infty, \tag{3.4}$$

but

$$\sum_1^\infty n k_n \omega(t_n) \ge \sum_1^\infty (\frac{1}{n} - n\omega(t_n)) = \infty. \tag{3.5}$$

In particular, since $\omega(t) \ge ct$ for some $c > 0$ (by Corollary 2.1 and the assumption that $\omega \not\equiv 0$), we have from (3.4) that $\sum k_n t_n < \infty$. We assume as we may that $\sum k_n t_n < 2\pi$. Now define a countable closed set with a single cluster point for which there is one complementary arc of length $2\pi - \sum k_n t_n$ and k_n of length t_n, for $n = 1, 2, \ldots$. By (3.4) and (3.5) along with the fact that $\nu(t_n) \ge n\omega(t_n)$ for each n, it follows that K is an ω–set but not a ν–set. If $\liminf_{t\to 0}(\nu/\omega) = 0$, then $\limsup_{t\to 0}(\omega/\nu) = \infty$ and by what was just proved (with ν and ω interchanged), there is a ν-set that is not an ω-set. This completes the proof.

We turn now to results concerning ϕ_ω.

THEOREM 3.2. *Let ω, ν be continuous moduli of continuity such that $0 \not\equiv \omega, \nu < 1$. Then*

$$\limsup_{t\to 0} \phi_\omega(t)/(t \log \frac{1}{\omega(t)}) < \infty. \tag{3.6}$$

Furthermore, if $\phi_\omega \approx \phi_\nu$, then $\log(1/\omega) \approx \log(1/\nu)$ and there exist $\alpha, \beta > 0$ such that

$$\omega^\alpha \le \nu \le \omega^\beta. \tag{3.7}$$

PROOF. Using Proposition 2.1(2), we see that

$$\begin{aligned}\phi_\omega(t) &= \int_0^t \log \frac{s}{\omega(s)} ds - t\log t + t \\ &\le t \log \frac{2t}{\omega(t)} - t\log t + t \\ &= t\log \frac{1}{\omega(t)} + t(1 + \log 2), \quad t \in [0, 2\pi],\end{aligned}$$

and the first assertion follows. Suppose now that $\phi_\omega \approx \phi_\nu$ but

$$\limsup_{t\to 0} \frac{\log(1/\nu(t))}{\log(1/\omega(t))} = \infty.$$

Then there exists a decreasing sequence $(t_n)_1^\infty$ converging to 0 such that $n\log\frac{1}{\omega(t_n)} \le \log\frac{1}{\nu(t_n)}$. By the concavity of ϕ_ν (see Proposition 3.1), we have $t\log(1/\nu(t)) \le \phi_\nu(t)$. Also, the assumption that $\phi_\omega \approx \phi_\nu$ implies that there exists a constant $\delta > 0$ such that $\phi_\omega/\phi_\nu \ge \delta$ on $(0, 2\pi]$. Thus, for sufficiently large n, we have

$$\frac{\phi_\omega(t_n)}{t_n\log(1/\omega(t_n))} \ge \frac{n\phi_\omega(t_n)}{t_n\log(1/\nu(t_n))} \ge \frac{n\phi_\omega(t_n)}{\phi_\nu(t_n)} \ge n\delta.$$

This contradicts (3.6). The proof of the first part of the second assertion is completed by interchanging the roles of ν and ω. Finally, (3.7) follows immediately from what was just proved. The proof of the lemma is complete.

Recall that $Z(\Lambda_\omega) = \{Z(f) : f \in \Lambda_\omega, f \not\equiv 0\}$ denotes the class of zero sets (in $\overline{\Lambda}$) of the nontrivial functions in Λ_ω. The following is a corollary of Theorem A and Theorem 3.2.

COROLLARY 3.1. *Let ω and ν be moduli of continuity such that $0 \not\equiv \omega, \nu < 1$. Then $Z(\Lambda_\omega) = Z(\Lambda_\nu)$ if and only if $\{Z(f^*) : f \in \Lambda_\omega, f \not\equiv 0\} = \{Z(f^*) : f \in \Lambda_\nu, f \not\equiv 0\}$ which occurs if and only if* (3.7) *holds.*

4. Hausdorff measure and ω–sets. We start by recalling some basic facts about Hausdorff measure (see, for example [**12**]). A nonnegative monotone nondecreasing function μ continuous on $(0, 2\pi]$ is the determining function for a Hausdorff measure H_μ defined on the Borel subsets of C as follows. For E a Borel set,

$$H_\mu(E) = \lim_{r\to 0^+} (\inf \sum \mu(|A_j|)),$$

where the infimum is taken over all countable covers (A_j) of E by open arcs A_j with $|A_j| \le r$. If $L = \lim_{t\to 0^+} \mu(t) > 0$, then $H_\mu(\{\eta\}) = L$ for each $\eta \in C$, and $H_\mu(E)$ essentially counts the points of E. The following theorem relates Hausdorff measure to ω–sets and generalizes Theorem B.

THEOREM 4.1. *Let $\nu, \omega \not\equiv 0$ be moduli of continuity, where ν is continuous on $(0, 2\pi]$ and ω is C^2-smooth and concave downward. Then the following are equivalent.*

(1) $H_\nu(K) = 0$ *for all ω-sets K.*

(2) $\int_0^{2\pi} \frac{t\omega''(t)}{\nu(t)} dt = -\infty$.

Before giving the proof of Theorem 4.1, we state several corollaries. The first is a generalization of Corollary B. For this corollary, recall that every finite positive Borel measure μ on C can be identified with a monotone nondecreasing function $\tilde{\mu}$ on $[0, 2\pi]$ defined by

$$\tilde{\mu}(t) = \mu\{e^{i\theta} : 0 \leq \theta \leq t\}.$$

In what follows, ω_μ will denote the modulus of continuity of $\tilde{\mu}$.

COROLLARY 4.1. *Let ν and ω be as in Theorem 4.1. Then the following are equivalent.*

(1) $\mu(K) = 0$ *for all ω-sets K and every finite positive Borel measure μ satisfying $\omega_\mu(t) = O(\nu(t))$*

(2) $\int_0^{2\pi} \frac{t\omega''(t)}{\nu(t)} dt = -\infty$.

In fact, Corollary 4.1 is an immediate consequence of Theorem 4.1 and the following theorem (see [**8** Theoreme III, Chapitre II, p. 27].

THEOREM D. *Let ν be as in Theorem 4.1 and let E be a Borel subset of C. Then the following are equivalent.*

(1) $H_\nu(E) > 0$.

(2) *E supports a finite positive Borel measure μ with $\omega_\mu(t) = O(\nu(t))$.*

We now give the second corollary which, in light of Theorem D, generalizes a result proved in [**11**] asserting that a singular measure μ cannot put mass on a BCH set if its modulus of continuity $\omega_\mu(t) = O(t \log(2\pi e/t))$.

COROLLARY 4.2. *Let ω be a continuous modulus of continuity. Then $H_\omega(K) = 0$ for all ω-sets K.*

PROOF OF COROLLARY 4.2. By Theorem 2.1, we can assume without loss of generality that ω is C^2-smooth and concave downward. In the trivial case when $\omega \equiv 0$, the measure H_ω is the zero measure so the assertion holds. When $\omega \not\equiv 0$ but is Lipschitz of order 1, then H_ω is linear measure and the ω–sets are the closed sets of zero measure, so the assertion is valid.

Suppose now that $\omega'(0) = \infty$ and let $\varepsilon \in (0, 2\pi]$. An integration by parts yields

$$\int_\varepsilon^{2\pi} \frac{t\omega''(t)}{\omega(t)} dt = \frac{t\omega'(t)}{\omega(t)}\Big|_\varepsilon^{2\pi} + \int_\varepsilon^{2\pi} \frac{\omega'(t)}{\omega(t)} \left(\frac{t\omega'(t)}{\omega(t)} - 1 \right) dt. \tag{4.1}$$

Case 1. $\lim_{t\to 0} \frac{\omega'(t)t}{\omega(t)}$ does not exist. Since ω is concave downward, we have $0 \le t\omega'/\omega \le 1$, so that the integrands in (4.1) are nonpositive. Therefore the integrals are defined and nonpositive. Let ε approach 0. If the right–hand integral approaches a finite limit, then the assumption of this case and (4.1) imply the left–hand integral does not approach a limit, a contradiction. Thus the limit of the right–hand integral is $-\infty$, and, hence, so is the limit of the left–hand integral. Thus the desired result holds in this case.

Case 2. $\lim_{t\to 0} \frac{(t\omega')}{\omega} = \alpha \in [0, 1)$. Then, by the last assertion of Proposition 2.2, there exists a constant $c > 0$ such that $1 - (t\omega')/\omega \ge c$ and

$$\begin{aligned}\lim_{\varepsilon\to 1} \int_\varepsilon^{2\pi} \frac{\omega'(t)}{\omega(t)} \left(\frac{t\omega'(t)}{\omega(t)} - 1 \right) dt &\le \lim_{\varepsilon\to 0} (-c) \int_\varepsilon^{2\pi} \frac{\omega'(t)}{\omega(t)} dt \\ &= \lim_{\varepsilon\to 0} c \log \frac{\omega(\varepsilon)}{\omega(2\pi)} = -\infty.\end{aligned}$$

Since $(t\omega')/\omega$ is bounded, we conclude that the left–hand integral in (4.1) approaches $-\infty$ as ε goes to 0.

Case 3. $\lim_{t\to 0} \frac{(t\omega')}{\omega} = 1$. Then, for some $\delta > 0$, we have $(t\omega')/\omega \ge \frac{1}{2}$ for $t \in (0, \delta)$ and

$$\begin{aligned}\int_\varepsilon^{2\pi} \frac{t\omega''(t)}{\omega(t)} dt &= \int_\varepsilon^{2\pi} \frac{\omega''(t)}{\omega'(t)} \frac{\omega'(t)}{\omega(t)} t dt \\ &\le \int_\varepsilon^{\delta} \frac{\omega''(t)}{\omega'(t)} (\frac{1}{2}) dt \\ &= \frac{1}{2} \log \left(\frac{\omega'(\delta)}{\omega'(\varepsilon)} \right).\end{aligned}$$

Letting ε go to 0, the required conclusion again follows.

The proof of Corollary 4.2 is complete.

For the proof of Theorem 4.1, we shall need an unpublished result and proof of P. Ahern. The following notation will be used. If K is a closed subset of C of measure 0, then

$$T_K(t) = |\{\eta \in C : \rho(K,\eta) \le t\}|, \quad t \in [0, 2\pi].$$

The function T_K is called the type function for K.

LEMMA 4.1 (Ahern). *Let ω be a C^2–smooth concave downward modulus of continuity and let K be a closed set of measure 0. Then K is an ω–set if and only if*

$$\int_0^{2\pi} T_K(t)\omega''(t)dt > -\infty \tag{4.2}$$

PROOF. Let (I_k) be an enumeration of the component arcs of $C \backslash K$. Then

$$T_K(t) \ge \sum_{|I_k| \le t} |I_k| + t \sum_{|I_k| > t} 1. \tag{4.3}$$

Also, if I_k is an arc such that $|I_k| > t$, we have

$$|\{\eta \in C : \rho(K,\eta) \le t\} \bigcap I_k| \le 2t$$

and it follows that

$$T_K(t) \le \sum_{|I_k| \le t} |I_k| + 2t \sum_{|I_k| > t} 1. \tag{4.4}$$

Suppose that (4.2) holds. Then, by (4.3), we have

$$\begin{aligned} -\infty < \int_0^{2\pi} (\sum_{|I_k| \le t} |I_k|)\omega''(t)dt &= \sum_k |I_k| \int_{|I_k|}^{2\pi} \omega''(t)dt \\ &= \sum_k |I_k|[\omega'(2\pi) - \omega'(|I_k|)]. \end{aligned} \tag{4.5}$$

Since $\sum |I_k| = 2\pi < \infty$, it follows that

$$\sum |I_k|\omega'(|I_k|) < \infty. \tag{4.6}$$

Also, (4.3) implies that

$$
\begin{aligned}
& -\infty < \int_0^{2\pi} \left(t(\sum_{|I_k|>t} 1)\omega''(t) \right) dt \\
(4.7) \qquad & = \sum_k \int_0^{|I_k|} t\omega''(t)dt \\
& = \sum_k \left(t\omega'(t)|_0^{|I_k|} - \omega(|I_k|) \right) \\
& = \sum_k |I_k|\omega'(|I_k|) - \sum_k \omega(|I_k|),
\end{aligned}
$$

where the last two equalities are obtained using integration by parts and (4.6). It follows that $\sum \omega(|I_k|) < \infty$ and hence K is an ω–set.

Conversely, suppose that $\sum \omega(|I_k|) < \infty$. Then by the concavity of ω, we have $t\omega' \leq \omega$ so that $\sum |I_k|\omega'(|I_k|) < \infty$. Therefore, reading the equalities of (4.5) and (4.7) in reverse order, it follows that

$$
(4.8) \qquad -\infty < \int_0^{2\pi} \sum_{|I_k|\leq t} |I_k|\omega''(t)dt
$$

and

$$
(4.9) \qquad -\infty < \int_0^{2\pi} 2 \left(t(\sum_{|I_k|\geq t} 1)\omega''(t) \right) dt,
$$

respectively. We conclude from (4.4), (4.8), and (4.9) that (4.2) holds. This completes the proof.

The next two lemmas are also due to Ahern and are contained in [**1**; p. 324].

LEMMA 4.2 (Ahern). *Suppose that K is a closed subset of C having zero measure and μ is a finite positive Borel measure supported on K. Then*

$$T_K(t)\omega_\mu(t) \geq \mu(K)t.$$

LEMMA 4.3 (Ahern). *Let ν be a modulus of continuity such that $\nu'(0) = \infty$. Then there exists a finite positive Borel measure μ*

supported on a closed subset K of C having zero measure such that $\omega_\mu \approx \nu$ and $t/\nu \approx T_K$.

We turn now to the proof of Theorem 4.1.

PROOF. Suppose that (1) does not hold. Then by Theorem D there exists an ω–set K and a finite positive Borel measure μ supported on K such that $\omega_\mu(t) = 0\,(\nu(t))$. Thus by Lemma 4.2, it follows that

$$\frac{t}{\nu(t)} \leq \frac{cT_K(t)}{\mu(K)}$$

for some $c > 0$. Since K is an ω–set, it follows from Lemma 4.1 that (2) does not hold.

Conversely, if ν is Lipschitz, then (1) clearly holds. Suppose now that $\nu'(0) = \infty$ and (2) fails. By Lemma 4.3, there exists a finite positive Borel measure μ supported on a closed subset K of C having zero measure, such that $t/\nu \approx T_K$ and $\omega_\mu \approx \nu$. From the first equivalence and Lemma 4.1, we conclude that K is an ω–set. On the other hand, $\omega_\mu(t) = O\,(\nu(t))$ by the second equivalence, and it follows from Theorem D that $H_\nu(K) > 0$. Hence, not all ω–sets have H_ν–measure 0.

Theorem 4.1 is established.

5. Uniform and asymptotic boundary descent to 0. For $f \in H^\infty$ (the class of bounded analytic functions on Λ), the radial limit function $f^*(\eta) = \lim_{r\to 1} f(r\eta)$ is defined for almost all $\eta \in C$ and satisfies

$$\int_C \log|f^*(\eta)|\;|d\eta| > -\infty \tag{5.1}$$

if $f \not\equiv 0$. In this section we give a result which is a boundary analogue of both Theorem C and the topological result of Barth and Schneider alluded to in §1. We also use Theorem A to obtain a partial converse to a result that combines Theorem C and the result of Barth and Schneider.

We continue to call μ allowed if it is a monotone nondecreasing continuous function on $[0, 2\pi]$ for which $\mu(0) = 0$, $\mu \leq 1$, and $\log(1/\mu)$ is integrable, and we use the notation ϕ_μ for the modulus of continuity defined in Proposition 3.1. Recall also that $\rho(\eta, \xi)$ denotes the arclength distance between η and ξ in C.

DEFINITION 5.1. *Let μ be an allowed function and let $f \in H^\infty$. Then f is said to descend to* 0 *at a rate μ as η approaches ξ in C if*

$$|f^*(\eta)| \leq \mu(\rho(\eta, \xi)) \tag{5.2}$$

for each η in C where f^* is defined. The set of all such ξ is denoted $B(f, \mu)$.

Note that if $\log(1/\mu)$ is not integrable, then f cannot descend to 0 at a rate μ as η approaches any point ξ unless $f \equiv 0$. This is immediate from (5.1).

PROPOSITION 5.1. *If $f \in H^\infty$, $f \not\equiv 0$, and μ is allowed, then $B(f, \mu)$ is a closed set of measure* 0.

The proof is elementary, depending only on Definition 5.1 and (5.1). The following theorem characterizes the sets $B(f, \mu)$ for $f \not\equiv 0$.

THEOREM 5.1. *Let μ be allowed and let $K \subseteq C$. Then $K = B(f, \mu)$, for some $f \in H^\infty$, $f \not\equiv 0$, if and only if K is a ϕ_μ–set.*

PROOF. Suppose that $K = B(f, \mu)$ for some $f \in H^\infty$, $f \not\equiv 0$. Then K is a closed set of measure 0 by Proposition 5.1. It follows from (5.1) and (5.2) by integrating over half–arcs, that

$$\begin{aligned}
\infty &> \int_C \log(1/|f^*(\eta)|)\, |d\eta| \\
&\geq 2 \sum_k \int_0^{\frac{|I_k|}{2}} \log(1/\mu(t))\, dt \\
&= 2 \sum_k \phi_\mu\left(\frac{|I_k|}{2}\right) \\
&\geq \frac{2}{3} \sum_k \phi_\mu(|I_k|),
\end{aligned}$$

using Proposition 2.1(1).

In the opposite direction, if K is a ϕ_μ–set, then by Proposition 3.2 we have

$$\int_C \log((1/\mu)(\rho(\eta, K)))\, |d\eta| < \infty.$$

Then

$$f(z) = \exp\left(\int_C \frac{\eta + z}{\eta - z} \log\left(\mu\left(\rho(\eta, K)\right)\right) |d\eta|\right)$$

is a well-defined outer function in H^∞ with

$$|f^*(\eta)| = \mu\left(\rho(\eta, K)\right)$$

for each η in C where f^* is defined. Thus f is as required and the proof is complete.

Note that Theorem A could have been used to prove sufficiency in Theorem 5.1 for the case when μ is a nontrivial continuous modulus of continuity. In fact, if $f \in \Lambda_\mu$ with $c = 1$ in the defining inequality of Λ_μ, then $B(f, \mu)$ is equal to the boundary zero set $Z(f^*)$ of f.

We turn now to asymptotic boundary descent to 0.

DEFINITION 5.2. *Let μ be an allowed function and let $f \in H^\infty, \sup|f| < 1$. Then the set of ξ in C such that*

$$\limsup_{\eta \to \xi} \frac{\log(1/\mu(\rho(\eta, \xi)))}{\log(1/|f^*(\eta)|)} < \infty \tag{5.3}$$

is denoted $A(f, \mu)$.

Note that the quotient in (5.3) (defined to be 0 whenever $f^*(\eta) = 0$) makes sense at each point of $C \backslash \{\xi\}$ where f^* is defined.

THEOREM 5.2. *Let μ be an allowed function. If $f \in H^\infty, 0 \neq \sup|f| < 1$, then $A(f, \mu)$ is a countable union of ϕ_μ sets. In the opposite direction, if E is a countable union of ϕ_μ-sets, then there exists $f \in H^\infty, f \not\equiv 0$, such that $E \subseteq A(f, \mu)$.*

PROOF. Let $f \in H^\infty, 0 \neq \sup|f| < 1$. Then $A(f, \mu) = \cup_1^\infty F_n$, where

$$F_n = \{\xi \in C : \log\left(1/\mu(\rho(\eta, \xi))\right) \leq n \log(1/|f^*(\eta)|)\}$$

for each n. But $F_n = B(f, \mu^{1/n})$, so by Theorem 5.1, the set F_n is a $\phi_{\mu^{1/n}}$ set for each n. Since $\phi_{\mu^{1/n}} \approx \phi_\mu$ for each n, the first assertion follows from Proposition 3.4.

For the second assertion, suppose that $E = \cup_1^\infty F_n$, where F_n is a ϕ_μ-set for each n. By Theorem 3.1(2), we can assume without loss of generality that $(F_n)_1^\infty$ is a monotone nondescending sequence. Furthermore, we assume as we may that $\mu(2\pi) < 1$. For each positive integer n, let $g_n(\eta) = \log(1/\mu(\rho(\eta, F_n)))$ for $\eta \in C$. Then $(g_n)_1^\infty$ is a monotone nondescending sequence of positive functions bounded away from 0 and, by Proposition 3.2, we have

$$\int_C g_n(\eta)|d\eta| = \alpha_n < \infty,$$

for each n. By the monotone convergence theorem,

$$f(z) = \exp\left(-\int_C \frac{\eta + z}{\eta - z} \sum \left(\frac{1}{2^n \alpha_n} g_n(\eta)\right) |d\eta|\right)$$

is a well-defined outer function (with $0 < \sup|f| < 1$).

Suppose $\xi \in E$. Then there exists a positive integer n such that $\xi \in F_n$ and

$$\log \frac{1}{|f(z)|} \geq \frac{1}{2^n \alpha_n} \int_C \frac{1 - |z|^2}{|\eta - z|^2} g_n(\eta)|d\eta|.$$

It follows that

$$\begin{aligned} \log \frac{1}{|f^*(\eta)|} &\geq \frac{1}{2^n \alpha_n} \log \frac{1}{\mu(\rho(\eta, F_n))} \\ &\geq \frac{1}{2^n \alpha_n} \log \frac{1}{\mu(\rho(\eta, \xi))} \end{aligned}$$

for each $\eta \in C$ for which $f^*(\eta)$ is defined. Thus, $\xi \in A(f, \mu)$, and we conclude that $E \subseteq A(f, \mu)$ as required. The proof of Theorem 5.2 is thereby completed.

In §1, we quoted Samuelsson's Theorem C and noted a topological result of Barth and Schneider. Based on Theorem C and an elementary proof of the Barth–Schneider theorem given in [**3**], we have the following.

THEOREM C′. *Let $\omega \not\equiv 0$ be a continuous modulus of continuity and let $f \in H^\infty$, $0 < \sup|f| \leq 1$. Then, for each $a \in (0, \infty)$, the set $R(f, \omega, a)$ (defined in Theorem C) is the union of a monotone nondescending sequence of closed sets $(F_n)_1^\infty$ such that*

$$H_\omega(F_n) \leq M < \infty$$

for each n.

PROOF. Observe that, by Theorem 2.1, the assumption on $\omega(t)/t$ in Theorem C can be replaced by the weaker assumption that ω is a modulus of continuity such that $\omega \not\equiv 0$. Next, it is easily seen that $R(f,\omega,a) = \cup_1^\infty F_n$, where

$$F_n = \{\eta \in C : \omega(1-r) \le a(1-\frac{1}{n})(1-r)\log\frac{1}{|f(r\eta)|}, \text{ for } 1-\frac{1}{n} \le r < 1\}$$

for each n. By the continuity of f and ω, each F_n is closed. Thus $(F_n)_1^\infty$ is a monotone nondescending sequence of closed sets which, by Theorem C, has the required property.

COROLLARY C′. *The set $R(f,\omega,\infty)$ is a union of a monotone nondescending sequence of closed sets $(F_n)_1^\infty$ such that $H_\omega(F_n) < \infty$ for each n.*

We do not know if there is a complete converse to either Theorem C′ or Corollary C′ for non–Lipschitz ω. However the following result follows from Theorem A.

THEOREM 5.3. *Let $\omega = \phi_\nu$ where ν is a continuous modulus of continuity such that $0 \not\equiv \nu < 1$. Let E be a countable union of ω–sets. There exists an outer function $f \in H^\infty$ such that $E \subseteq A(f,\nu) \bigcap R(f,\omega,\infty)$.*

PROOF. By assumption, $E = \bigcup_1^\infty F_n$, where each F_n is an ω–set. Corollary A implies that there exists, for each positive integer n, an outer function $f_n \in \Lambda_\nu$ with $c = 1$ in the defined inequality of Λ_ν such that $\sup|f_n| < 1$ and $Z(f^*) = F_n$. For $\alpha_n > 0$ sufficiently small, the analytic α_n^{th} power $f_n^{\alpha_n}$ of f_n, for which $f_n^{\alpha_n}(0) > 0$, satisfies

$$(5.4) \qquad |1 - f_n^{\alpha_n}(z)| \le \frac{1}{2^n}, \quad |z| \le 1 - \frac{1}{n}.$$

Let $f = \prod_{n=1}^\infty f_n^{\alpha_n}$. By (5.4), the infinite product converges uniformly on compact subsets of Λ, and f is a well–defined outer function with $0 < \sup|f| < 1$. Also $|f| \le |f_n^{\alpha_n}| = |f_n|^{\alpha_n}$ for each n.

If $\xi \in E$, then $\xi \in F_n$ for some n. Since $f_n \in \Lambda_\nu$ (with $c = 1$) and $f_n(\xi) = 0$, we have

$$|f(z)| \le |f_n^{\alpha_n}(z)| \le \nu^{\alpha_n}(|z-\xi|), \; z \in \overline{\Lambda}$$

and hence

$$\alpha_n \log \frac{1}{\nu\left(\rho(\eta,\xi)\right)} \le \log \frac{1}{|f^*(\eta)|}, \tag{5.5}$$

for each $\eta \in C$ where f^* is defined and

$$\alpha_n \log \frac{1}{\nu(1-r)} \le \log \frac{1}{|f(r\xi)|}, \quad r \in [0,1). \tag{5.6}$$

It follows directly from (5.5) that $\xi \in A(f,\nu)$. On the other hand by Theorem 3.2 there exists a positive constant c such that $\omega(t)/t \le c \log(1/\nu(t))$ for $t \in (0, 2\pi]$. Taken together with (5.6), this implies that $\xi \in K_N$ for some positive integer N, where

$$K_n = \{\eta \in C : \omega(1-r) \le n(1-r) \log \frac{1}{|f(r\eta)|}, \text{ for } 1 - \frac{1}{n} \le r < 1\},$$

for $n = 1, 2, \ldots$. But $R(f, \omega, \infty) = \bigcup_1^\infty K_n$ so that $\xi \in R(f, \omega, \infty)$. We conclude that $E \subseteq A(f,\nu) \bigcap R(f,\omega,\infty)$ and the proof is complete.

In the proof of Theorem C′, it is shown that, for $a \in (0, \infty]$, the set $R(f, \omega, a)$ is a countable union of closed sets such that f converges to 0 *uniformly* along the radii ending in each closed set. Thus the problem of characterizing the sets $R(f, \omega, a)$ is cognate to the problem of characterizing the sets

$$\{\eta \in C : |f(r\eta)| \le \mu(1-r) \text{ for } 0 \le r < 1\} \tag{5.7}$$

for $f \in H^\infty$, $f \not\equiv 0$, where μ is a monotone nondecreasing continuous function defined on $[0,1]$ satisfying $\mu(0) = 0$. E.M. Kegejan [**9**] considered the latter problem prior to Samuelsson's work [**13**] and gave primarily necessary conditions on the sets of type (5.7). Kegejan's conditions are of a different kind than Samuelsson's Hausdorff measure conditions, and seem difficult to relate. In particular, it is not clear whether they lead to stronger conditions than are given in Theorem C′ and Corollary C′, and therefore leave the question of whether these results have converses somewhat in doubt.

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