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THE CENTRAL LIMIT QUESTION UNDER ρ-MIXING

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ABSTRACT. An earlier construction of a (non-trivial) strictly stationary ρ-mixing random sequence that fails to satisfy the central limit theorem, is refined here in order to try to have the fastest possible "mixing rate" for ρ-mixing, depending on the "moment properties" of the r.v.'s. In particular, the examples here show that when only finite second moments are assumed, for the central limit theorem the mixing rate $\sum \rho(2^n) < \infty$ used by Ibragimov is essentially as slow as permissible.

1. Introduction. First we define some notation. Log denotes the natural logarithm, and $\log^+ x := \max\{0, \log x\}$. The indicator function of a set S is denoted by I_S. The notation $a \ll b$ means $a = O(b)$. The notation $a \sim b$ means $\lim a/b = 1$. The greatest integer $\leqq x$ is denoted by $[x]$. A sequence $(a_n, n = 1, 2, \ldots)$ of positive numbers is said to be "slowly varying" as $n \to \infty$ if $\lim_{n\to\infty}[\sup_{n \leqq m \leqq 2n} a_m]/[\inf_{n \leqq m \leqq 2n} a_m] = 1$. When a subscript itself is of the form a_n, it will be written as $a(n)$. The notation $\mathscr{L}_2(\cdot)$ refers only to real-valued random variables (instead of general complex-valued ones).

Suppose $X := (X_k, k \in \mathbf{Z})$ is a strictly stationary sequence of real-valued random variables on a probability space $(\Omega, \mathscr{F}, P)$. For $-\infty \leqq J \leqq L \leqq \infty$ let $\mathscr{F}_J^L$ denote the σ-field of events generated by the random variables $(X_k, J \leqq k \leqq L)$. For any two σ-fields $\mathscr{A}$ and $\mathscr{B} \subset \mathscr{F}$, define the "maximal correlation" **[8, 12]** by

$$\rho(\mathscr{A}, \mathscr{B}) := \sup|\mathrm{Corr}(f, g)| \qquad f \in \mathscr{L}_2(\mathscr{A}),\ g \in \mathscr{L}_2(\mathscr{B}).$$

For each $n = 1, 2, 3, \ldots$ define the dependence coefficient $\rho(n) := \rho(\mathscr{F}_{-\infty}^0, \mathscr{F}_n^\infty)$. By our assumption of stationarity, $\rho(n) = \rho(\mathscr{F}_{-\infty}^J, \mathscr{F}_{J+n}^\infty)$ $\forall J \in \mathbf{Z}$. Also, obviously the sequence $\rho(n), n = 1, 2, \ldots$ is non-increasing as n increases. The stationary random sequence $X := (X_k)$ is said to be "ρ-mixing" **[18]** if $\rho(n) \to 0$ as $n \to \infty$.

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For each $n = 1, 2, \ldots$ define the partial sum $S_n := X_1 + X_2 + \cdots + X_n$. Ibragimov [**14**, Theorem 2.1 and 2.2] proved the following central limit theorem (or "CLT" for short).

THEOREM 0 (IBRAGIMOV). *Suppose $X := (X_k, k \in \mathbf{Z})$ is a strictly stationary sequence of real-valued random variables such that $\mathrm{E}X_0 = 0, \mathrm{E}X_0^2 < \infty$, $\mathrm{Var}\ S_n \to \infty$ as $n \to \infty$, and $\rho(n) \to 0$ as $n \to \infty$. Then $\mathrm{Var}\ S_n = n \cdot h(n)$, where $(h(n), n = 1, 2, \ldots)$ is slowly varying as $n \to \infty$. Suppose that in addition at least one of the following two conditions is satisfied:*

(i) $\mathrm{E}|X_0|^{2+\delta} < \infty$, *for some* $\delta > 0$, *or*

(ii) $\sum_{n=1}^{\infty} \rho(2^n) < \infty$.

Then $S_n/(\mathrm{Var}\ S_n)^{1/2} \to N(0, 1)$ in distribution as $n \to \infty$.

Here $N(0, 1)$ denotes the standard normal distribution. (In checking Ibragimov's [**14**] paper for these results, a look at [**2**, Theorem 1] or [**19**, Theorem 4.1] might be helpful.)

Some background comments are in order. The assumption $EX_k^2 < \infty$ is standard in central limit theory, because of its role in the CLT for i.i.d. (independent, identically distributed) r.v.'s. The assumption $\mathrm{Var}\ S_n \to \infty$ (which holds automatically if $X := (X_k)$ is a non-degenerate i.i.d. sequence) is made, explicitly or implicitly, in CLT's for dependent r.v.'s in order to avoid some totally trivial counterexamples. One such example is the sequence $X := (X_k)$ defined by $X_k := W_k - W_{k-1}$, where the r.v.'s $(W_k, k \in \mathbf{Z})$ are i.i.d. with $P(W_0 = 0) = P(W_0 = 1) = 1/2$; this sequence X is 1-dependent (and therefore satisfies $\rho(n) = 0\ \forall n \geqq 2$), but $\forall n \geqq 1$, $S_n = W_n - W_0$ (a telescoping sum), which obviously fails to be asymptotically normally distributed as $n \to \infty$. To avoid such examples we shall henceforth consider only stationary sequences X satisfying $\mathrm{Var}\ S_n \to \infty$ (and $\mathrm{E}X_0^2 < \infty$).

In Theorem 0, conditions (i) and (ii) cannot be omitted altogether. To show this, the author [**1**] constructed a strictly stationary sequence $X := (X_k)$ such that $\mathrm{E}X_0 = 0$, $\mathrm{E}X_0^2 < \infty$, $\mathrm{Var}\ S_n \to \infty$, $\rho(n) \to 0$, and subsequences of the S_n's (suitably normalized) converge in distribution to other limit laws besides the normal distribution. In [**3**, Theorem 1] this construction was refined so as to enlarge the class of partial limit laws of S_n and also to have the sequence X satisfy some additional strong mixing properties. The purpose of the present paper is to refine the construction in [**1**] again, in a different direction, in order to obtain some insight into the following general question: Suppose $X := (X_k)$ is strictly stationary, with finite second moments, and satisfies $\mathrm{Var}\ S_n \to \infty$ and $\rho(n) \to 0$; suppose $\mathrm{E}q(|X_0|) < \infty$ where $q : [0, \infty) \to [0, \infty)$ is an increasing function, presumably satisfying $x^2 \ll q(x) \ll x^{2+\delta}$ as $x \to \infty$ for every $\delta > 0$; then under these conditions, what is the slowest "mixing rate" for $\rho(n)$ (i.e., rate of convergence of $\rho(n)$ to 0) that will still imply that S_n is asymptotically normally distributed as $n \to \infty$? The approach to such a question consists of two parts, for a given function q: (1) to prove as "efficiently" as possible a CLT for stationary sequences $X := (X_k)$ satisfying $\mathrm{E}q(|X_0|) < \infty$, with the slowest mixing rate on $\rho(n)$ that one can get by with; and (2) to construct as "efficiently" as possible a counterexample, a strictly stationary sequence $X := (X_k)$ satisfying $\mathrm{E}q(|X_0|) < \infty$, where S_n fails to be asymptotically normally distributed, with the fastest mixing rate on $\rho(n)$ that one can achieve. If the mixing rate in the counterexample is only slightly slower than in the theorem, then (for the given function q) one has fairly well pinned down the answer to the above question. In this paper we seek to construct counterexamples as "efficiently" as possible, given the known techniques. The opposite task, "efficiently" proving CLT's, apparently has not been carried out yet beyond Theorem 0 (and will not be pursued in this paper).

This work is motivated by recent research of two different kinds. First, for the "strong mixing" [**20**] condition (which is similar to but weaker than ρ-mixing and whose definition need not be given here), Herrndorf [**11**] has attacked (the analog of) the question above for a fairly general class of functions q. For functions q of the form $q(x) := x^{2+\delta}$, where $0 < \delta < \infty$, fairly precise answers to that question (for strong mixing) were already provided by a CLT of Ibragimov [**13**, Theorem 1.7] and some counterexamples by Davydov [**6**, Example 1]. For the function $q(x) := x^2$, counterexamples with an arbitrarily fast mixing rate for strong mixing were constructed by the author [**3**, Theorem 2] and Herrndorf [**10**]. In his recent paper, Herrndorf [**11**] gave an (apparently quite "efficient") proof of the CLT (in fact, of the weak invariance principle) under strong mixing, for his broad class of functions q, and gave a (non-stationary) "efficient" counterexample for the function $q(x) := x^2 \log^+ x$. The above question, for ρ-mixing, comes as an analogy to this work of Herrndorf. Comparing Theorem 0 to the results under strong mixing alluded to above, one sees that the answer to the question for ρ-mixing is quite different from that for strong mixing (for a given function q).

The other major motivation for this work is the recent research on strictly stationary ρ-mixing sequences $X := (X_k,\ k \in \mathbf{Z})$ satisfying the mixing rate $\sum_{n=1}^{\infty} \rho(2^n) < \infty$ used in Theorem 0 (ii). This mixing rate was introduced in 1970 by Ibragimov and Rozanov [**16**], who showed that this rate implies a continuous spectral density and derived bounds on the error of the approximation of this spectral density by trigonometric polynomials; see [**17**, 182, Lemma 17, and 190, Note 2]. Their result implies the existence of $\sigma^2 := \lim_{n\to\infty} n^{-1} \operatorname{Var} S_n$, $0 \leqq \sigma^2 < \infty$. More recently, using their ideas from these results, the author [**2**], under the assumption $\operatorname{Var} S_n \to \infty$, showed that $\sigma^2 > 0$ and derived a bound on the speed of convergence of $n^{-1} \operatorname{Var} S_n$ to σ^2. Peligrad [**19**, Theorem 4.1] confirmed the existence of such a limit σ^2 for some non-stationary sequences. In [**19**, Lemma 3.5 and Theorem 2.3] she also established uniform integrability of $(S_n^2/n, n = 1, 2, \ldots)$ and (with the aid of an extra assumption on another dependence coefficient) proved a weak invariance principle. (See also [**19**, Theorem 2.5].) In [**4**], asymptotic normality is proved for some estimators of probability density, with the same normalizing constants as in the i.i.d. case. Falk [**7**] proved uniform convergence of spectral densities in some stationary random arrays. All of these results seem to depend critically on the mixing rate $\sum_{n=1}^{\infty} \rho(2^n) < \infty$; and indeed some stationary Gaussian sequences constructed in [**17**, p.179–180] show that some of these results fail under the barely slower rate $\rho(n) \ll (\log n)^{-1}$. Because of this recent prominence of the rate $\sum \rho(2^n) < \infty$, it seems worthwhile to see whether it is in fact the slowest possible rate under which one has the CLT under just the assumption of finite second moments (as in Theorem 0 (ii)). This is essentially confirmed in Theorem 2 below.

In constructing our counterexamples we shall assume that a sequence $\tau := (\tau(n), n = 1, 2, \ldots)$ of positive numbers is given, and we shall endeavor to have $\rho(n) \ll \tau(n)$. The conditions that we shall impose on this sequence $\tau := (\tau(n), n = 1, 2, \ldots)$ are as follows:

(1.1) $$\tau(1) \geqq \tau(2) \geqq \tau(3) \geqq \cdots \downarrow 0,$$

(1.2) $$\sum_{k=1}^{\infty} k^{-1}\,\tau(k) = \infty \qquad \left(\text{equivalently } \sum_{n=1}^{\infty} \tau(2^n) = \infty\right),$$

(1.3) $$\lim_{n\to\infty} \tau(2n)/\tau(n) = 1.$$

The equivalence of the two equations in (1.2) is an easy consequence of (1.1). Also, (1.1) and (1.3) imply that τ is "slowly varying". Condition (1.3) is not very restrictive in our context. If $\limsup_{n\to\infty} \rho(2n)/\rho(n) < 1$, then $\sum_{n=1}^{\infty} \rho(2^n) < \infty$ and one has the CLT (Theorem 0 (ii)). Condition (1.3) is satisfied in natural "borderline" cases such as when $\tau(n) = (\log n)^{-1}$ $\forall n \geqq 2$ or when $\tau(n) = (\log n)^{-1} (\log\log n)^{-1}$ $\forall n \geqq 3$.

In constructing our counterexamples $X := (X_k)$ we shall assume that a function $q\colon [0, \infty) \to [0, \infty)$ is given, and we shall endeavor to have $Eq(|X_0|) < \infty$. The conditions that we shall impose on this function q are as follows:

(1.4)
q is continuous and non-decreasing;
$q(0) = 0$, $q(x) \geqq x^2$ $\forall x \geqq 0$;
$q(x) \ll x^3$ as $x \to \infty$;
$\forall a > 0$, $\lim_{x\to\infty} q(x + a)/q(x) = 1$; and
$\exists C > 0$ such that $\forall x \geqq C$, $\forall y \geqq C$, $q(xy) \leqq q(x)q(y)$.

Except perhaps for continuity, these conditions on q are all used in the construction of the counterexamples. However, these conditions are not particularly restrictive. Because of Theorem 0 (i) it would in fact seem reasonable to consider only functions q such that $q(x) = o(x^{2+\delta})$ as $x \to \infty\ \forall \delta > 0$. Equation (1.4) is satisfied by such functions as $q(x) := x^2(1 + \log^+ x)$ and $q(x) := x^2 \exp((\log^+ x)^{1/2})$.

Our main result is as follows.

THEOREM 1. *Suppose* $\tau := (\tau(n), n = 1, 2, \ldots)$ *is a sequence of positive numbers satisfying* (1.1), (1.2), *and* (1.3). *Suppose* q: $[0, \infty) \to [0, \infty)$ *is a function satisfying* (1.4), *such that for some positive number* d,

$$q\left(\left(n \cdot \exp\left(-d \sum_{k=1}^{n} k^{-1} \tau(k)\right)\right)^{1/2}\right) = o(n) \quad \text{as } n \to \infty. \tag{1.5}$$

Then there exists a strictly stationary sequence $X := (X_k, k \in \mathbf{Z})$ *of real-valued random variables such that*

$$\mathrm{E}X_0 = 0,\ \mathrm{E}X_0^2 < \infty, \text{ and } \operatorname{Var} S_n \to \infty \text{ as } n \to \infty, \tag{1.6}$$

$$\mathrm{E}q(|X_0|) < \infty, \tag{1.7}$$

$$\rho(n) \ll \tau(n) \text{ as } n \to \infty, \text{ and} \tag{1.8}$$

(1.9) *there exists an increasing sequence* $n(1) < n(2) < n(3) < \cdots$ *of positive integers such that,* $\forall x \neq 0$, $\lim_{\ell \to \infty} P(S_{n(\ell)}/(\operatorname{Var} S_{n(\ell)})^{1/2} \leq x) = F(x)$, *where*

$$F(x) := e^{-1} I_{[0,\infty)}(x) + \sum_{J=1}^{\infty} (1/J!) e^{-1} (2\pi J)^{-1/2} \int_{-\infty}^{x} e^{-u^2/(2J)}\, du.$$

Of course (1.7) implies the middle equation in (1.6), but this redundancy doesn't hurt.

The probability distribution function F in Theorem 1 is that of a Poisson mixture of normal distributions (including the point mass at 0). Other Poisson mixtures of normal distributions also arise as partial limit laws of S_n (suitably normalized). One can see this by examining the construction of the sequence X (in §3) or by combining (1.9) with a simple "blocking" argument similar to the proof of [**13**, 353, Theorem 1.1]. With a more complicated construction, one can achieve a broader class of partial limit laws, as in [**3**, Theorem 1]. But for simplicity we shall confine our attention to just the partial limit law represented by F.

In some special cases of Theorem 1 (including Theorem 2 below), one can also modify the construction in order to achieve the additional property of "information regularity" (and hence also "absolute regularity") as in [**3**, Theorem 1], by using some arguments from the proof of that result. This too we shall omit, in order to avoid extra complications.

A simple calculation will show that if $x^{2+\delta} \ll q(x)$ as $x \to \infty$ for some $\delta > 0$, then (1.5) cannot be satisfied (with $\tau(n) \downarrow 0$). Of course this is what would be expected from Theorem 0 (i).

Theorem 1 is the result of trying to construct the counterexample X as "efficiently" as possible, given the known techniques. Based on the hope that Theorem 1 is essentially sharp, one might conjecture that if $X := (X_k)$ is strictly stationary and satisfies (1.6) and (1.7) with q satisfying (1.4) and $n \ll q((n \cdot \exp(-d \sum_{k=1}^{n} k^{-1} \rho(k)))^{1/2})$ as $n \to \infty$ $\forall d > 0$, then S_n is asymptotically normally distributed as $n \to \infty$.

In the special case where $q(x) = x^2$, the equation $\rho(n) \leqq \tau(n)$ $\forall n \geqq 1$ (technically stronger than $\rho(n) \ll \tau(n)$) comes from the proof (in §3) at no extra cost. Because it shows that Theorem 0 (ii) is essentially sharp, this special case is worth stating explicitly anyhow.

THEOREM 2. *Suppose* $\tau := (\tau(n), n = 1, 2, \ldots)$ *is a sequence of positive numbers satisfying* (1.1), (1.2), *and* (1.3). *Then there exists a strictly stationary sequence* $X := (X_k, k \in \mathbf{Z})$ *of real-valued r.v.'s such that* (1.6) *and* (1.9) *hold and* $\rho(n) \leqq \tau(n)$ $\forall n \geqq 1$.

The following three corollaries in essence list other special cases of Theorem 1. The derivation of these corollaries from Theorem 1 is elementary and is left to the reader.

COROLLARY 1. *For each* $\alpha > 0$ *there exists a strictly stationary seuqence* $X := (X_k)$ *satisfying* (1.6) *and* (1.9) *such that* $\mathrm{E}X_0^2(\log^+ |X_0|)^\alpha < \infty$ *and* $\rho(n) \ll (\log n)^{-1}$.

COROLLARY 2. *For each* $\alpha > 0$ *and* $\beta > 0$ *such that* $\alpha + \beta \leqq 1$ *there exists a strictly stationary sequence* $X := (X_k)$ *satisfying* (1.6) *and* (1.9) *such that* $\mathrm{E}X_0^2 \exp((\log^+ |X_0|)^\alpha) < \infty$ *and* $\rho(n) \ll (\log n)^{-\beta}$.

COROLLARY 3. *For each function* q: $[0, \infty) \to [0, \infty)$ *satisfying* (1.4) *such that* $\forall \delta > 0$, $q(x) = o(x^{2+\delta})$ *as* $x \to \infty$, *there exists a strictly stationary sequence* $X := (X_k)$ *satisfying* (1.6), (1.7), (1.9), *and* $\rho(n) \to 0$.

The proof of Theorems 1 and 2 is in §3. §2 is devoted to some preliminary work that will be used in that proof. Many of the arguments in §2 and §3 will follow the arguments in **[1]** closely, but they will nevertheless be given in full here because of the numerous extra details involved here.

With trivial modifications of certain arguments in §2 and §3, one can considerably relax the assumptions (1.3) and (1.4) in Theorem 1. But since (1.3) and (1.4) in their present form seem fairly natural in our context, we shall stick with them.

2. Preliminaries. This section contains the preliminary work for the proof of Theorems 1 and 2 given in §3.

Let **C** denote the complex numbers. For any continuous function f: $\mathbf{R} \to \mathbf{C}$ and any $t > 0$ define the "modulus of continuity" $w(f, t) := \sup_{x, y \in \mathbf{R}, |x-y| \leqq t} |f(x) - f(y)|$. For any continuous function f: $\mathbf{R} \to \mathbf{C}$ which is periodic with period 2π, and any integer $n \geqq 0$, define the quantity

$$E_n(f) := \inf_{\substack{a_{-n}, a_{-n+1}, \\ \ldots, a_n \in \mathbf{C}}} \left(\sup_{\lambda \in \mathbf{R}} \left| f(\lambda) - \sum_{k=-n}^{n} a_k e^{ik\lambda} \right| \right).$$

The following variant of Jackson's Theorem can be found in Timan [**22**, 257, eqn. 10] (and applies equally well to complex-valued functions on **R** as to real-valued functions). It can also be derived from [**24**, p. 115, eqn. (13.8)].

LEMMA 1 (JACKSON). *There exists a positive constant A such that for every continuous function f: $\mathbf{R} \to \mathbf{C}$ with period 2π and every integer $n \geqq 0$, $E_n(f) \leqq A \cdot w(f, 1/(n+1))$.*

The next three lemmas are a review of some elementary trigonometry. Proofs are sketched in some cases where a convenient reference seems hard to find.

LEMMA 2. *If $v_1 \geqq v_2 \geqq v_3 \geqq \cdots \downarrow 0$, $-\pi \leqq \lambda \leqq \pi$, and $\lambda \neq 0$, then $\sum_{k=1}^{\infty} v_k e^{ik\lambda} := \lim_{n\to\infty} \sum_{k=1}^{n} v_k e^{ik\lambda}$ exists in **C** and $|\sum_{k=1}^{\infty} v_k e^{ik\lambda}| \leqq \pi \cdot v_1/|\lambda|$.*

This follows from Zygmund [**24**, p. 3, Theorem (2.2)]. (First recall that in (2.3), if $u_k := e^{ik\lambda}$ and $0 < |\lambda| \leqq \pi$, then $\max_k |U_k| \leqq 2/|e^{i\lambda} - 1| \leqq \pi/|\lambda|$.)

LEMMA 3. *Suppose $\tau := (\tau(n), n = 1, 2, \ldots)$ is a sequence of positive numbers satisfying* (1.1), (1.2), *and* (1.3). *Then the following three statements hold.*

(i) $\sup_{n \geqq 1} (n^{-1} \sum_{k=1}^{n} \tau(k))/\tau(n) < \infty$.

(ii) *The function g defined by $g(\lambda) := \sum_{k=1}^{\infty} (\sin k\lambda) \cdot k^{-1} \tau(k)$ is continous, and defining $B := 4\pi + 3 \cdot \sup_{n \geqq 1} (n^{-1} \sum_{k=1}^{n} \tau(k))/\tau(n)$ one has that $\forall n \geqq 1$, $w(g, 1/n) \leqq B \cdot \tau(n)$.*

(iii) *The function h defined by $h(\lambda) := \sum_{k=1}^{\infty} (\cos k\lambda) \cdot k^{-1} \tau(k)$ satisfies $\exp h(\lambda) \sim \exp \sum_{k=1}^{(1/\lambda)} k^{-1} \tau(k)$ as $\lambda \to 0+$.*

Property (i) is simple. In (ii) the continuity of g is shown in Zygmund [**24**, p. 182, Theorem (1.3)]. Property (iii) is practically in [**24**, pp. 188–189]. In (iii), of course, $h(\lambda)$ will diverge if $\lambda \equiv 0 \bmod 2\pi$. For convenience the proofs of (ii) and (iii) are sketched here.

PROOF OF (ii). Using (i), define the positive number $b := \sup_{n \geqq 1} (n^{-1} \sum_{k=1}^{n} \tau(k))/\tau(n)$.

If $0 < \lambda \leqq 1$, then $0 \leqq \sum_{k=1}^{(1/\lambda)} (\sin k\lambda) \cdot k^{-1}\tau(k) \leqq \sum_{k=1}^{(1/\lambda)} k\lambda \cdot k^{-1}\tau(k) \leqq b \cdot \tau((1/\lambda))$, and by Lemma 2,

$$\left|\sum_{k=1+(1/\lambda)}^{\infty} (\sin k\lambda) \cdot k^{-1}\tau(k)\right| \leqq \pi \cdot \tau(1 + (1/\lambda)) \leqq \pi \cdot \tau([1/\lambda]),$$

and hence $|g(\lambda)| \leqq (b + \pi) \cdot \tau((1/\lambda))$.

If $n \in \mathbf{N}$ and $0 < \mu \leqq 1/n$, then $|g(\mu)| \leqq (b + \pi) \cdot \tau([1/\mu]) \leqq (b + \pi) \cdot \tau(n)$.

If $n \in \mathbf{N}$ and $1/n \leqq \mu, \nu \leqq \pi$ with $|\mu - \nu| \leqq 1/n$, then, using Lemma 2 again,

$$\begin{aligned} |g(\mu) - g(\nu)| &\leqq \sum_{k=1}^{n} |\sin k\mu - \sin k\nu| \cdot k^{-1}\tau(k) \\ &\quad + \left|\sum_{k=n+1}^{\infty} (\sin k\mu) \cdot k^{-1}\tau(k)\right| \\ &\quad + \left|\sum_{k=n+1}^{\infty} (\sin k\nu) \cdot k^{-1}\tau(k)\right| \\ &\leqq \sum_{k=1}^{n} k \cdot |\mu - \nu| \cdot k^{-1}\tau(k) + \pi \cdot n^{-1}\tau(n) \cdot (\mu^{-1} + \nu^{-1}) \\ &\leqq (1/n) \cdot \sum_{k=1}^{n} \tau(k) + 2\pi \cdot \tau(n) \leqq (b + 2\pi) \cdot \tau(n). \end{aligned}$$

Using the preceding two paragraphs and the fact that g is an odd function with period 2π, one can show that, for all $n \in \mathbf{N}$ and any two real numbers μ and ν with $|\mu - \nu| \leqq 1/n$, the inequality $|g(\mu) - g(\nu)| \leqq [2(b + \pi) + (b + 2\pi)] \cdot \tau(n)$ holds. Thus (ii) holds.

PROOF OF (iii). First recall that $\forall x \in \mathbf{R}$, $|1 - \cos x| \leqq (1/2) \cdot x^2$.

For each λ, $0 < \lambda < 1$,

$$\begin{aligned} &\left|\sum_{k=1}^{(1/\lambda)} (\cos k\lambda) \cdot k^{-1}\tau(k) - \sum_{k=1}^{(1/\lambda)} k^{-1}\tau(k)\right| \\ &\quad \leqq \sum_{k=1}^{(1/\lambda)} (1/2) \cdot (k\lambda)^2 k^{-1}\tau(k) = \sum_{k=1}^{(1/\lambda)} (1/2) \cdot \lambda^2 k \cdot \tau(k) \\ &\quad \leqq (1/2) \cdot [1/\lambda]^{-1} \cdot \sum_{k=1}^{(1/\lambda)} \tau(k) \end{aligned}$$

and this converges to 0 as $\lambda \to 0+$, by (i); also, by Lemma 2,

$$\left|\sum_{k=(1/\lambda)+1}^{\infty} (\cos k\lambda) \cdot k^{-1}\tau(k)\right| \leqq \pi \cdot [1/\lambda]^{-1} \cdot \tau([1/\lambda]) \cdot \lambda^{-1}$$

which approaches 0 as $\lambda \to 0+$. Hence $\lim_{\lambda\to 0+} |h(\lambda) - \sum_{k=1}^{(1/\lambda)} k^{-1}\tau(k)| = 0$. Now it is clear that (iii) holds.

DEFINITION 1. A function f: $(0, r] \to (0, \infty)$ (where r is any positive number) is "slowly varying" as $\lambda \to 0+$ if $\lim_{\lambda\to 0+} [\sup_{\mu\in[\lambda,2\lambda]} f(\mu)]/[\inf_{\mu\in[\lambda,2\lambda]} f(\mu)] = 1$.

LEMMA 4. *Suppose* f: $[-\pi, \pi] \to [0, \infty)$ *is a continuous even function,* $f(\lambda) > 0\ \forall \lambda \in [-\pi, \pi] - \{0\}$, *and* f *is slowly varying as* $\lambda \to 0+$. *Then*

$$\int_{-\pi}^{\pi} n^{-1}[(\sin^2(n\lambda/2))/(\sin^2(\lambda/2))] \cdot f(\lambda)d\lambda \sim 2\pi \cdot f(1/n) \quad \text{as } n \to \infty.$$

This follows from careful but elementary calculations, using simple properties of the Fejer kernel.

Now some more notation is needed. In what follows, we shall sometimes be dealing with more than one strictly stationary sequence at the same time. In order to avoid confusion we introduce the following notation for a given strictly stationary sequence $X := (X_k, k \in \mathbf{Z})$: for $-\infty \leqq J \leqq L \leqq \infty$, the σ-field $\mathscr{F}_J^L$ will be denoted $\mathscr{F}_J^L(X)$, and for $n = 1, 2, \ldots$ the dependence coefficient $\rho(n)$ will be denoted $\rho_n(X)$.

Also, for any sequence $\tau := (\tau(n), n = 1, 2, \ldots)$ of numbers satisfying (1.1), (1.2), and (1.3), define the constant $B_\tau := 4\pi + 3 \cdot \sup_{n\geqq 1}(n^{-1} \sum_{k=1}^{n} \tau(k))/\tau(n)$. (This is the constant B in Lemma 3 (ii).)

Finally, to choose one specific value for A in Lemma 1, define A_J to be the least positive number such that $E_n(f) \leqq A_J \cdot w(f, 1/(n+1))$ for every $n \geqq 0$ and every continuous function f: $\mathbf{R} \to \mathbf{C}$ with period 2π. (The letter J in A_J stands for "Jackson".)

LEMMA 5. *Suppose* $\tau := (\tau(n), n = 1, 2, \ldots)$ *satisfies* (1.1), (1.2), *and* (1.3). *Then there exists a stationary real Gaussian sequence* $U := (U_k, k \in \mathbf{Z})$ *such that*

(i) $\rho_n(U) \leqq \tau(n)\ \forall n \geqq 1$, *and*

(ii) $\mathrm{Var}\ (U_1 + \cdots + U_n) \sim 2\pi n \cdot \exp[-(A_J B_\tau)^{-1} \sum_{k=1}^{n} k^{-1} \tau(k)] \to \infty$ *as* $n \to \infty$.

This is a refinement of [**1**, Lemmas 2 and 3]. The key role played in the proof of [**1**, Lemma 2] by the results in Helson and Sarason [**9**, **21**] was suggested to the author by M. Rosenblatt. A similar role is played here by a trick from [**17**, p. 179] closely related to [**9**, **21**].

PROOF OF LEMMA 5. On $[-\pi, \pi]$ define the functions h and f by $h(\lambda) := -(A_J B_\tau)^{-1} \sum_{k=1}^{\infty} (\cos k\lambda) \cdot k^{-1} \tau(k)$ and $f(\lambda) := \exp h(\lambda)$.

By Lemma 3 (iii), $f(\lambda) \sim \exp(-(A_J B_\tau)^{-1} \sum_{k=1}^{1/(\lambda)} k^{-1} \tau(k))$ as $\lambda \to 0+$. Also, f is an even function; and by Zygmund [**24**, p. 184, Theorem (1.8); p. 188, Theorem (2.15)], f is a continuous function provided one defines $f(0) = 0$. Also, $\lim_{n\to\infty} \sum_{k=n+1}^{2n} k^{-1} \tau(k) = 0$, and it follows that the function $\exp[-(A_J B_\tau)^{-1} \sum_{k=1}^{(1/\lambda)} k^{-1}\tau(k)]$ is slowly varying as $\lambda \to 0+$, and hence the function f is also slowly varying as $\lambda \to 0+$.

Let $U := (U_k, k \in \mathbf{Z})$ be a stationary real Gaussian sequence with spectral density f. By Lemma 4 and [**15**, p. 322, Theorem 18.2.1], $\mathrm{Var}(U_1 + \cdots + U_n) \sim 2\pi n \cdot f(1/n)$ as $n \to \infty$. Property (ii) in Lemma 5 can now be seen easily from the above arguments.

To verify property (i) in Lemma 5, first note that the conjugate function of h is $\tilde{h}$ defined by $\tilde{h}(\lambda) := -(A_J B_\tau)^{-1} \sum_{k=1}^{\infty} (\sin k\lambda) \cdot k^{-1} \tau(k)$ (which is continuous by Lemma 3). For each $n \geqq 1$,

$$\begin{aligned}\rho_n(U) &\leqq E_{n-1}(\exp(i\tilde{h})) \leqq A_J \cdot w(\exp(i\tilde{h}), 1/n)\\ &\leqq A_J \cdot w(\tilde{h}, 1/n) \leqq A_J \cdot [(A_J B_\tau)^{-1} \cdot B_\tau \cdot \tau(n)] = \tau(n).\end{aligned}$$

Here the first inequality comes from [**17**, p. 179, lines -7 to -5], the second from the definition of A_J, the third from some simple arithmetic in [17, page 132, lines 4 to 6], and the fourth from Lemma 3 (ii), the definition of B_τ, and elementary properties of $w(\cdot, \cdot)$. This completes the proof of Lemma 5.

The next lemma is due to Csaki and Fischer [**5**, p. 40, Theorem 6.2]. (See Witsenhausen [**23**, p. 105, Theorem 1] for a short proof.)

LEMMA 6 (CSAKI and FISCHER). *Suppose $\mathscr{A}_n$ and $\mathscr{B}_n$, $n = 1, 2, \ldots$, are σ-fields and the σ-fields $(\mathscr{A}_n \vee \mathscr{B}_n)$, $n = 1, 2, \ldots$, are independent. Then*

$$\rho\Big(\bigvee_{n=1}^{\infty} \mathscr{A}_n, \bigvee_{n=1}^{\infty} \mathscr{B}_n\Big) = \sup_{n \geqq 1} \rho(\mathscr{A}_n, \mathscr{B}_n).$$

LEMMA 7. *Suppose $N \geqq 2$ is an integer and $\tau := (\tau(n), n = 1, 2, \ldots)$ satisfies* (1.1), (1.2), *and* (1.3). *Then there exists a stationary real Gaussian sequence $W := (W_k, k \in \mathbf{Z})$ such that*

(i) $\rho_n(W) \leqq \tau(n)\ \forall n \geqq 1$;

(ii) $\mathrm{Var}(W_1 + \cdots + W_n) \to \infty$ *as* $n \to \infty$;

(iii) $\mathrm{Var}(W_1 + \cdots + W_n) \ll n \cdot \exp(-(A_J B_\tau)^{-1} \sum_{k=1}^{n} k^{-1} \tau(k))$ *as* $n \to \infty$; *and*

(iv) *the r.v.'s $W_1, W_2, \ldots, W_N$ are independent.*

PROOF. Define the sequence $\gamma := (\gamma(n), n = 1, 2, \ldots)$ of positive numbers by $\gamma(n) := \tau(Nn)$. Note that (1.1), (1.2), and (1.3) still hold with τ replaced by γ. Hence, by Lemma 5, there exists a stationary real Gaussian sequence $U := (U_k, k \in \mathbf{Z})$ such that

$$\rho_n(U) \leqq \gamma(n)\ \forall n \geqq 1 \tag{2.1}$$

and

$$\mathrm{Var}(U_1 + \cdots + U_n) \sim 2\pi n \cdot \exp\Big(-(A_J B_\gamma)^{-1} \sum_{k=1}^{n} k^{-1} \gamma(k)\Big) \to \infty \text{ as } n \to \infty. \tag{2.2}$$

For each $J = 1, 2, \ldots, N$ let $(W_{J+kN}, k \in \mathbf{Z})$ be a stationary real

Gaussian sequence with the same distribution as U, with these sequences $(W_{J+kN}, k \in \mathbf{Z})$, $J = 1, \ldots, N$, being independent of each other. By an elementary argument, the sequence $W := (W_k, k \in \mathbf{Z})$ is a stationary real Gaussian sequence.

By a simple argument using (2.1) and Lemma 6, one can show that $\rho_n(W) \leqq \tau(n)\ \forall n \geqq 1$. Also, it is clear that $\operatorname{Var}(W_1 + \cdots + W_n) \to \infty$ as $n \to \infty$ (by (2.2)), and that $W_1, \ldots, W_N$ are independent r.v.'s. Thus, in Lemma 7, only property (iii) remains to be verified.

First note that, for each $n \geqq 1$,

$$\left(n^{-1}\sum_{k=1}^{n}\gamma(k)\right)/\gamma(n) = \left((nN)^{-1}\sum_{k=1}^{n} N \cdot \tau(Nk)\right)/\tau(Nn)$$
$$\leqq \left((nN)^{-1}\sum_{j=1}^{Nn}\tau(j)\right)/\tau(Nn).$$

It follows that $B_\gamma \leqq B_\tau$. Hence, by (2.2), as $n \to \infty$,

$$\begin{aligned}\operatorname{Var}(W_1 + \cdots + W_n) &\sim N \cdot \operatorname{Var}(U_1 + \cdots + U_{(n/N)}) \\ &\sim 2\pi n \cdot \exp\left(-(A_J B_\gamma)^{-1}\sum_{k=1}^{(n/N)} k^{-1}\gamma(k)\right) \\ &\ll 2\pi n \cdot \exp\left(-(A_J B_\tau)^{-1}\sum_{k=1}^{(n/N)} k^{-1}\gamma(k)\right).\end{aligned} \tag{2.3}$$

Also, $\forall n \geqq N$,

$$\begin{aligned}\sum_{k=1}^{(n/N)} k^{-1}\gamma(k) &= N \cdot \sum_{k=1}^{(n/N)} (Nk)^{-1}\tau(Nk) \\ &\geqq \sum_{k=1}^{(n/N)}\sum_{j=0}^{N-1}(Nk+j)^{-1}\tau(Nk+j) \\ &\geqq \sum_{\ell=N}^{n}\ell^{-1}\tau(\ell) \\ &= \left(\sum_{\ell=1}^{n}\ell^{-1}\tau(\ell)\right) - \left(\sum_{\ell=1}^{N-1}\ell^{-1}\tau(\ell)\right).\end{aligned}$$

Since the very last sum is a constant (i.e., not depending on n), property (iii) in Lemma 7 now follows from (2.3). This completes the proof of Lemma 7.

DEFINITION 2. Suppose $N \geqq 2$ is an integer and $\tau := (\tau(n), n = 1, 2, \ldots)$ is a sequence of positive numbers satisfying (1.1), (1.2), and (1.3). A sequence $Y := (Y_k, k \in \mathbf{Z})$ of real-valued r.v.'s is said to satisfy "Condition $\mathscr{S}(N, \tau)$" if Y is strictly stationary and has the following properties:

(i) $\mathrm{E}Y_0 = 0$ and $\mathrm{E}Y_0^2 = 1/N$;

(ii) $\rho_n(Y) \leqq \tau(n)\ \forall n = 1, 2, \ldots$;

(iii) $\operatorname{Var}(Y_1 + \cdots + Y_n) \to \infty$ as $n \to \infty$;

(iv) $\mathrm{Var}(Y_1 + \cdots + Y_n) \ll n \cdot \exp[-(2A_J B_\tau)^{-1} \sum_{k=1}^n k^{-1}\tau(k)]$ as $n \to \infty$;

(v) $Y_1, Y_2, \ldots, Y_N$ are independent r.v.'s;

(vi) $\forall x \in \mathbf{R}$, $P(Y_0 \leqq x) = (1 - 1/N) I_{[0,\infty)}(x) + (1/N)(2\pi)^{-1/2} \int_{-\infty}^x e^{-u^2/2}\, du$; and

(vii) $\forall x \in \mathbf{R}$, $P(Y_1 + \cdots + Y_N \leqq x) = G_N(x)$, where

$$G_N(x) := \left(1 - \frac{1}{N}\right)^N I_{[0,\infty)}(x) + \sum_{J=1}^{N} \binom{N}{J} \left(\frac{1}{N}\right)^J \left(1 - \frac{1}{N}\right)^{N-J} (2\pi J)^{-1/2} \int_{-\infty}^{x} e^{-u^2/(2J)}\, du.$$

LEMMA 8. *Suppose $N \geqq 2$ is an integer and $\tau := (\tau(n), n = 1, 2, \ldots)$ is a sequence of positive numbers satisfying* (1.1), (1.2), *and* (1.3). *Then there exists a* (*strictly stationary*) *sequence $Y := (Y_k, k \in \mathbf{Z})$ of real-valued r.v.'s which satisfies Condition $\mathscr{S}(N, \tau)$.*

PROOF. Define the number $p := 1/N$. It follows from (1.3) that $\lim_{n\to\infty} \tau(1 + ((n-1)p/2))/\tau(n) = 1$. Let $n_0 \geqq 1$ be an integer such that $\forall n \geqq n_0$, $\tau(1 + ((n-1)p/2)) \leqq (4/3) \cdot \tau(n)$. Using (1.3) again, let $c > 0$ be sufficiently small that $\forall n \geqq 2$, $4((n-1)p)^{-1} c \leqq (1/3) \cdot \tau(n)$. Let the sequence $\gamma := (\gamma(n), n = 1, 2, \ldots)$ of positive numbers be defined by

$$\gamma(n) := \min\{c, (2/3) \cdot \tau(n_0), (1/2) \cdot \tau(n)\}. \tag{2.4}$$

We need several elementary properties of this sequence γ. First, (1.1), (1.2), and (1.3) hold with τ replaced by γ. Next, the following equations hold:

$$4 \cdot ((n-1)p)^{-1} \gamma(1) \leqq (1/3)\tau(n) \quad \forall n \geqq 2; \tag{2.5}$$

$$\gamma(1 + ((n-1)p/2)) \leqq (2/3)\tau(n) \quad \forall n \geqq 2; \tag{2.6}$$

and

$$B_\gamma \leqq B_\tau. \tag{2.7}$$

Here (2.5) holds by (2.4) and the definition of c. To see (2.6), note that (i) if $2 \leqq n \leqq n_0$, then $\gamma(1 + ((n-1)p/2)) \leqq (2/3)\tau(n_0) \leqq (2/3)\tau(n)$ by (2.4); and (ii) if instead $n > n_0$, then $\gamma(1 + ((n-1)p/2)) \leqq (1/2) \cdot \tau(1 + ((n-1)p/2)) \leqq (2/3\tau)(n)$ by (2.4) and the definition of n_0. To see (2.7), note that (i) if n satisfies $\gamma(n) < (1/2)\tau(n)$, then $\gamma(1) = \cdots = \gamma(n) = \min\{c, (2/3) \cdot \tau(n_0)\}$ and $(n^{-1} \sum_{k=1}^n \gamma(k))/\gamma(n) = 1 \leqq (n^{-1} \sum_{k=1}^n \tau(k))/\tau(n)$ by (1.1); and (ii) if instead n satisfies $\gamma(n) = (1/2)\tau(n)$, then

$$\left(n^{-1}\sum_{k=1}^{n}\gamma(k)\right)/\gamma(n) \leqq \left(n^{-1}\sum_{k=1}^{n}(1/2)\ \tau(k)\right)/((1/2)\tau(n))$$
$$= \left(n^{-1}\sum_{k=1}^{n}\tau(k)\right)/\tau(n).$$

Now (2.7) is clear from the definition of B_τ.

Let $V := (V_k,\ k \in \mathbf{Z})$ be an i.i.d. sequence such that $P(V_0 = 1) = 1 - P(V_0 = 0) = p$.

Let the random integers $(k(\ell),\ \ell \in \mathbf{Z})$ be defined by the two conditions $\{k(\ell),\ \ell \in \mathbf{Z}\} = \{k : V_k = 1\}$ and $\ldots k(-2) < k(-1) < k\ (0) \leqq 0 < 1 \leqq k(1) < k(2) < k(3) < \ldots$. Deleting a set of probability zero from the probability space Ω if necessary, we assume that $k(\ell)\,(\omega)$ is defined $\forall \ell \in \mathbf{Z}$, $\forall \omega \in \Omega$.

Using Lemma 7, let $W := (W_k,\ k \in \mathbf{Z})$ be a stationary real Gaussian sequence independent of V such that

$$EW_0 = 0 \quad \text{and} \quad EW_0^2 = 1, \tag{2.8}$$

$$\rho_n(W) \leqq \gamma(n) \quad \forall n \geqq 1, \tag{2.9}$$

$$\mathrm{Var}(W_1 + \cdots + W_n) \to \infty \quad \text{as } n \to \infty, \tag{2.10}$$

$$\mathrm{Var}(W_1 + \cdots + W_n) \ll n \cdot \exp\Big[-(A_J B_\gamma)^{-1}\sum_{k=1}^{n}k^{-1}\gamma(k)\Big] \quad \text{as } n \to \infty, \tag{2.11}$$

and

$$W_1, \ldots, W_N \text{ are independent r.v.'s.} \tag{2.12}$$

(The normalization (2.8) does not affect any of the properties in Lemma 7.)

Define the random sequence $Y := (Y_k,\ k \in \mathbf{Z})$ as follows. For each $k \in \mathbf{Z}$ and each sample point $\omega \in \Omega$,

$$Y_k(\omega) := \begin{cases} W_\ell(\omega) & \text{if } k(\ell)(\omega) = k \text{ for some } \ell \\ 0 & \text{if } k \notin \{k(\ell)\,(\omega) : \ell \in \mathbf{Z}\}. \end{cases}$$

An elementary measure-theoretic argument will show that Y is strictly stationary. (Perhaps the easiest way to show this is to show that the sequence $((V_k,\ Y_k),\ k \in \mathbf{Z})$ of random vectors is strictly stationary.) Properties (i), (iii), (v), (vi), and (vii) in Definition 2 are all elementary consequences of the definition of Y and are left to the reader to verify. Properties (ii) and (iv) in Definition 2 will be verified here.

To start the proof of (ii), note first that

$$\rho_1(Y) = \rho(\mathscr{F}_{-\infty}^{0}(V) \vee \mathscr{F}_{-\infty}^{0}(W),\ \mathscr{F}_{1}^{\infty}(V) \vee \mathscr{F}_{1}^{\infty}(W))$$
$$= \rho_1(W) \leqq \gamma(1) \leqq \tau(1)$$

by Lemma 6, (2.9), and (2.4). Next, an elementary measure-theoretic argument will show that if $n \geqq 2, f \in \mathscr{L}_2(\mathscr{F}_{-\infty}^0(Y))$, $g \in \mathscr{L}_2(\mathscr{F}_n^\infty(Y))$, $Ef = Eg = 0$, $Ef^2 = Eg^2 = 1$, and $J \in \{0, 1, \ldots, n-1\}$, then

$$\begin{aligned}
&|E(fg|V_1 + \cdots + V_{n-1} = J)| \\
&\quad \leqq \rho(\mathscr{F}_{-\infty}^0(V) \vee \mathscr{F}_{-\infty}^0(W), \mathscr{F}_n^\infty(V) \vee (\mathscr{F}_{J+1}^\infty(W)) \\
&\quad = \rho_{J+1}(W).
\end{aligned}$$

It follows that, $\forall n \geqq 2$,

$$\begin{aligned}
\rho_n(Y) &\leqq \sum_{J=0}^{n-1} P(V_1 + \cdots + V_{n-1} = J) \cdot {}_{J+1}(W) \\
&\leqq P(V_1 + \cdots + V_{n-1} \leqq ((n-1)p/2)) \cdot \rho_1(W) \\
&\quad + P(V_1 + \cdots + V_{n-1} > ((n-1)p/2)) \cdot \rho_{1+((n-1)p/2)}(W) \\
&\leqq 4((n-1)p)^{-1} \cdot \rho_1(W) + \rho_{1+((n-1)p/2)}(W) \\
&\leqq 4((n-1)p)^{-1}\gamma(1) + \gamma(1 + ((n-1)p/2)) \\
&\leqq (1/3)\tau(n) + (2/3)\tau(n) = \tau(n),
\end{aligned}$$

where the last two inequalities come from (2.9), (2.5), and (2.6), and the third inequality comes from Chebyshev's inequality and the fact that the r.v. $V_1 + \cdots + V_{n-1}$ is binomial with parameters $n-1$ and p. This completes the proof of (ii) in Definition 2.

To verify (iv) there, first note that, by (1.1), for γ, one has that $\lim_{n\to\infty} \sum_{k=n+1}^{2n} k^{-1}\gamma(k) = 0$ and it follows that the quantity $\exp(-(A_J B_\gamma)^{-1} \sum_{k=1}^n k^{-1}\gamma(k))$ is slowly varying as $n \to \infty$. From this fact, (2.11), and the elementary equation

$$\operatorname{Var}(Y_1 + \cdots + Y_n) = \sum_{J=0}^{n} P(V_1 + \cdots + V_n = J) \cdot \operatorname{Var}(W_1 + \cdots + W_J),$$

it is easy to show that

$$\begin{aligned}
\operatorname{Var}(Y_1 + \cdots + Y_n) &\ll n \cdot \exp\Big(-(A_J B_\gamma)^{-1} \sum_{k=1}^n k^{-1}\gamma(k)\Big) \\
&\ll n \cdot \exp\Big(-(A_J B_\tau)^{-1} \sum_{k=1}^n k^{-1}\gamma(k)\Big) \\
&\ll n \cdot \exp\Big(-(2A_J B_\tau)^{-1} \sum_{k=1}^n k^{-1}\tau(k)\Big),
\end{aligned}$$

where the second $\ll$ comes from (2.7) and the third $\ll$ from the fact (see (2.4)) that $\gamma(k) = (1/2)\tau(k)$ for all except finitely many values of k. Thus (iv) in Definition 2 is verified. This completes the proof of Lemma 8.

Finally, when dealing with a function q satisfying (1.4), it will be handy to also work with another, closely related function.

LEMMA 9. *Suppose* q: $[0, \infty) \to [0, \infty)$ *is a function satisfying* (1.4). *Define the function* Q: $[0, \infty) \to [0, \infty)$ *by* $Q(x) := q(x^{1/2})$. *Then the following four statements hold*:

(i) Q *is continuous and non-decreasing*, $Q(0) = 0$, $Q(x) \geqq x \ \forall x \geqq 0$, *and* $Q(x) \ll x^{3/2}$ *as* $x \to \infty$.

(ii) $\forall a > 0, \lim_{x\to\infty} Q(x + a)/Q(x) = 1$.

(iii) *There exists* $\alpha > 0$ *such that*

$$Q(xy) \leqq Q(x)Q(y) \quad \forall x \geqq \alpha, y \geqq \alpha. \tag{2.13}$$

(iv) *If* $\alpha > 0$ *satisfies* (2.13), Z *is a* $N(0, 1)$ *r.v., and* $c \geqq \alpha$, *then* $Eq(|c^{1/2}Z|) \leqq Q(c) \cdot (Q(\alpha) + Eq(|Z|))$.

PROOF. Statements (i) and (iii) are trivial and (ii) follows from the corresponding statement in (1.4) since $(x + a)^{1/2} \leqq x^{1/2} + a \ \forall x \geqq 1, a > 0$. To see (iv), note that

$$\begin{aligned} Eq(|c^{1/2}Z|) &= \int_{\{Z^2\leqq\alpha\}} Q(cZ^2)dP + \int_{\{Z^2>\alpha\}} Q(cZ^2)dP \\ &\leqq Q(c\alpha) + \int_\Omega Q(c)Q(Z^2)dP \\ &\leqq Q(c) \cdot Q(\alpha) + Q(c) \cdot Eq(|Z|). \end{aligned}$$

3. Proof of Theorems 1 and 2. What we shall prove is the following statement.

PROPOSITION 0. *Suppose* $\tau := (\tau(n), n = 1, 2, \ldots)$ *satisfies* (1.1), (1.2), *and* (1.3), *and* q *satisfies* (1.4) *and*

$$q((n\cdot\exp(-(2A_JB_\tau)^{-1}\sum_{k=1}^{n} k^{-1}\tau(k)))^{1/2}) = o(n) \quad as\ n \to \infty. \tag{1.5a}$$

Then there exists a strictly stationary sequence $X := (X_k)$ satisfying (1.6), (1.7), (1.9), and

$$\rho_n(X) \leqq \tau(n) \quad \forall n \geqq 1. \tag{1.8a}$$

Theorem 2 follows immediately from this statement (using $q(x) := x^2$). To derive Theorem 1, one simply has to replace the sequence $\tau := (\tau(n))$ in Proposition 0 by $c\tau := (c\tau(n), n = 1, 2, \ldots)$, where c is an appropriate positive constant, and use the elementary fact that $B_{c\tau} = B_\tau$.

PROOF OF PROPOSITION 0. Let the sequence $\tau := (\tau(n), n = 1, 2, \ldots)$ and the function q: $[0, \infty) \to [0, \infty)$ be arbitrary but fixed, satisfying the conditions in the hypothesis of Proposition 0. For each $n = 1, 2, \ldots$, define the quantity $\varepsilon(n) := \exp[-(2A_JB_\tau)^{-1}\sum_{k=1}^{n} k^{-1}\tau(k)]$. As in Lemma 9, define the function Q: $[0, \infty) \to [0, \infty)$ by $Q(x) := q(x^{1/2})$. Let α be such that (see Lemma 9)

$$\alpha \geqq 1 \tag{3.1}$$

and

$$Q(xy) \leqq Q(x) \cdot Q(y) \quad \forall x \geqq \alpha, y \geqq \alpha. \tag{3.2}$$

Since $\lim_{n\to\infty} \sum_{k=n+1}^{2n} k^{-1} \tau(k) = 0$, one has that $(\varepsilon(n), n = 1, 2, \ldots)$ is slowly varying as $n \to \infty$, and hence

$$n \cdot \varepsilon(n) \to \infty \quad \text{as } n \to \infty. \tag{3.3}$$

Let L be a positive integer such that

$$L \geqq 2 \text{ and } \forall \ell \geqq L, \ell \cdot \varepsilon(\ell) \geqq \alpha. \tag{3.4}$$

For each $m = 1, 2, \ldots$, we shall define a positive integer $N_m \geqq L$, a positive number $C_m \geqq \alpha$, a strictly stationary sequence $Y^{(m)} := (Y_k^{(m)}, k \in \mathbf{Z})$ satisfying condition $\mathscr{S}(N_m, \tau)$, and a positive number $d_m \geqq \alpha$ such that $\mathrm{Var}(Y_1^{(m)} + \cdots + Y_n^{(m)}) \leqq d_m \cdot n \cdot \varepsilon(n)\ \forall n \geqq 1$. The definition will be recursive and is as follows.

To start off, define $N_1 := L$ and $C_1 := \alpha$. Using Lemma 8, let $Y^{(1)} := (Y_k^{(1)}, k \in \mathbf{Z})$ be a strictly stationary sequence satisfying condition $\mathscr{S}(L, \tau)$. Using (iv) in Definition 2, let $d_1 \geqq \alpha$ be such that $\mathrm{Var}(Y_1^{(1)} + \cdots + Y_n^{(1)}) \leqq d_1 \cdot n \cdot \varepsilon(n)\ \forall n \geqq 1$.

Now suppose $M \geqq 2$ and that, for $m = 1, \ldots, M-1$, the following are already defined: the positive integer $N_m \geqq L$; the number $C_m \geqq \alpha$; the strictly stationary sequence $Y^{(m)} := (Y_k^{(m)}, k \in \mathbf{Z})$ satisfying condition $\mathscr{S}(N_m, \tau)$; and the number $d_m \geqq \alpha$ such that $\mathrm{Var}(Y_1^{(m)} + \cdots + Y_n^{(m)}) \leqq d_m \cdot n \cdot \varepsilon(n)\ \forall n \geqq 1$. Using Lemma 9, (3.3), and assumption (1.5a) in the hypothesis of Proposition 0, let N_M be a positive integer such that

$$N_M > N_{M-1}, \qquad N_M > L, \tag{3.5}$$

$$\begin{aligned} &Q(N_M \cdot \varepsilon(N_M))/N_M \\ &\leqq M^{-2} \cdot \left(Q(M \cdot \sum_{m=1}^{M-1} C_m d_m)\right)^{-1} \cdot \min_{1 \leq m \leq M-1} (C_m/N_m), \end{aligned} \tag{3.6}$$

$$\forall x \geqq N_M \cdot \varepsilon(N_M), \quad Q(x + C_1 + \cdots + C_{M-1}) \leqq 2Q(x). \tag{3.7}$$

By (3.1), (3.4), (3.5), and the above assumptions that $C_m \geqq \alpha$ and $d_m \geqq \alpha\ \forall m \leqq M-1$, one has that

$$N_M \cdot \varepsilon(N_M) \geqq \alpha \text{ and} \left(M \cdot \sum_{m=1}^{M-1} C_m d_m\right) \geqq \alpha. \tag{3.8}$$

Define $C_M := (N_M \cdot \varepsilon(N_M)) \cdot (M \cdot \sum_{m=1}^{M-1} C_m d_m)$. By (3.1) and (3.8), $C_M \geqq \alpha$. Using Lemma 8, let $Y^{(M)} := (Y_k^{(M)}, k \in \mathbf{Z})$ be a strictly stationary sequence which satisfies condition $\mathscr{S}(N_M, \tau)$ and is independent of the family of sequences $Y^{(1)}, Y^{(2)}, \ldots, Y^{(M-1)}$. Using (iv) in Definition 2, let $d_M \geqq \alpha$ be such that $\mathrm{Var}(Y_1^{(M)} + \cdots + Y_n^{(M)}) \leqq d_M \cdot n \cdot \varepsilon(n)\ \forall n \geqq 1$. This completes the recursive definition.

For each $M \geqq 2$,

$$Q(C_M) \leqq Q(N_M \cdot \varepsilon(N_M)) \cdot Q\Big(M \cdot \sum_{m=1}^{M-1} C_m d_m\Big), \tag{3.9}$$

$$C_M \geqq N_M \cdot \varepsilon(N_M) \geqq 1, \tag{3.10}$$

and

$$Q(C_1 + \cdots + C_M) \leqq 2Q(C_M). \tag{3.11}$$

Here, (3.9) and (3.10) follow from (3.2), (3.8), and the definition of C_M (and also (3.1)); (3.11) holds by (3.7) and (3.10). Also, for each $M \geqq 2$,

$$\frac{C_M}{N_M} \leqq \frac{Q(C_M)}{N_M} \leqq M^{-2} \cdot \min_{1 \leqq m \leqq M-1} \frac{C_m}{N_m}, \tag{3.12}$$

by (1.4) (which implies $Q(x) \geqq x\ \forall x \geqq 0$), (3.6), and (3.9). By (3.5) we also have that the sequence $(N_1, N_2, \ldots)$ of positive integers is strictly increasing.

By the above definitions, the random sequences $Y^{(1)}, Y^{(2)}, Y^{(3)}, \ldots$ are independent of each other, and (using Definition 2) for each $m \geqq 1$, $P(Y_0^{(m)} \neq 0) = 1/N_m$, $\mathrm{E}Y_0^{(m)} = 0$, and $\mathrm{Var}\ Y_0^{(m)} = 1/N_m$. By (3.12), $\sum_{m=1}^{\infty} C_m \cdot \mathrm{Var}\ Y_0^{(m)} = \sum_{m=1}^{\infty}(C_m/N_m) < \infty$, and, by (3.10), $\sum_{m=1}^{\infty}(1/N_m) < \infty$. By the Borel-Cantelli Lemma,

$$P(Y_0^{(m)} \neq 0 \text{ for infinitely many } m) = 0. \tag{3.13}$$

Define the random sequence $X := (X_k, k \in \mathbf{Z})$ as follows:

$$X_k := \sum_{m=1}^{\infty} C_m^{1/2} Y_k^{(m)} \quad \forall k \in \mathbf{Z}.$$

We have that for each $k \in \mathbf{Z}$ this sum converges a.s. and in $\mathscr{L}_2$.

Now let us verify the properties of the sequence X listed in Proposition 0. The facts that X is strictly stationary, $\mathrm{E}X_0 = 0$, and $\mathrm{E}X_0^2 < \infty$ are elementary. The fact that $\mathrm{Var}(X_1 + \cdots + X_n) \to \infty$ as $n \to \infty$ follows from Definition 2 (iii) and the elementary equation

$$\mathrm{Var}(X_1 + \cdots + X_n) = \sum_{m=1}^{\infty} C_m \mathrm{Var}(Y_1^{(m)} + \cdots + Y_n^{(m)}) \quad \forall n \geqq 1. \tag{3.14}$$

Also, by Lemma 6 and Definition 2 (ii), $\rho_n(X) \leqq \sup_{m \geqq 1} \rho_n(Y^{(m)}) \leqq \tau(n)$ $\forall n \geqq 1$; that is, (1.8a) in Proposition 0 holds. Now we only need to prove statements (1.7) and (1.9).

PROOF OF (1.7). For each finite subset S of $\{1, 2, 3, \ldots\}$ (including the empty set $\varnothing$) let H_S denote the event $\{\{m: Y_0^{(m)} \neq 0\} = S\}$. In what follows, it is understood that S ranges over just the finite subsets of $\{1, 2, \ldots\}$. By (3.13), the events H_S form a countable partition of the probability space Ω, except perhaps for an event of probability 0. Similarly, for each $m = 1, 2, 3, \ldots$ the events H_S with $m \in S$ form a countable partition of the event $\{Y_0^{(m)} \neq 0\}$ except perhaps for an event of probability 0. From (3.4), (3.5), and Definition 2, as noted above, $0 < P(Y_0^{(m)} \neq 0) = 1/N_m < 1\ \forall m \geqq 1$, and hence, by using (3.13), one can show that $P(H_S) > 0$ for every finite subset $S \subset \{1, 2, \ldots\}$. Since q is a non-negative function, one has that for every S, $E(q(|X_0|)|H_S)$ is well defined in $[0, \infty) \cup \{\infty\}$, and that

$$\text{(3.15)} \qquad Eq(|X_0|) = \textstyle\sum_S E(q(|X_0|)|H_S) \cdot P(H_S)$$

(which at this point is not assumed to be finite).

Define the quantity $r = Q(\alpha) + Eq(|Z|)$, where Z is any $N(0, 1)$ r.v. Then $r < \infty$ by (1.4) since $E|Z|^3 < \infty$. For each non-empty finite subset $S \subset \{1, 2, \ldots\}$, the conditional distribution of X_0 given H_S, is that of a normal r.v. with mean 0 and variance $\sum_{m\in S} C_m$, by Definition 2 (vi) and the fact $Y_0^{(1)}, Y_0^{(2)}, \ldots$ are independent r.v.'s, and hence $E(q(|X_0|)|H_S) \leqq r \cdot Q(\sum_{m\in S} C_m)$ by Lemma 9; and since $Q(\sum_{m\in S} C_m) \leqq 2Q(\max_{m\in S} C_m) \leqq 2\sum_{m\in S} Q(C_m)$ by (3.11) (and the fact that Q is non-decreasing), one has that $E(q(|X_0|)\,|H_S) \leqq 2r \cdot \sum_{m\in S} Q(C_m)$. For the empty set $\emptyset$ one has $E(q(|X_0|)|H_\emptyset) = 0$ since $q(0) = 0$ by (1.4). Hence, by (3.15),

$$\begin{aligned} Eq(|X_0|) &= \sum_{S\neq\emptyset} E(q(|X_0|)|H_S) \cdot P(H_S) \\ &\leqq \sum_{S\neq\emptyset} \sum_{m\in S} 2r \cdot Q(C_m) \cdot P(H_S) \\ &= \sum_{m=1}^{\infty} \sum_{\{S: m\in S\}} 2r \cdot Q(C_m) \cdot P(H_S) \\ &= 2r \cdot \sum_{m=1}^{\infty} Q(C_m) \cdot \sum_{\{S: m\in S\}} P(H_S) \\ &= 2r \cdot \sum_{m=1}^{\infty} Q(C_m) \cdot P(Y_0^{(m)} \neq 0) \\ &= 2r \cdot \sum_{m=1}^{\infty} Q(C_m)/N_m \\ &< \infty \end{aligned}$$

(where the last step holds by (3.12)). Thus (1.7) holds.

PROOF OF (1.9). For each $M \geqq 2$ the following statements hold:

$$\text{(3.16)} \qquad C_M \cdot \operatorname{Var}(Y_1^{(M)} + \cdots + Y_{N(M)}^{(M)}) = C_M,$$

$$\text{(3.17)} \quad \begin{aligned} &\sum_{m=1}^{M-1} C_m \operatorname{Var}(Y_1^{(m)} + \cdots + Y_{N(M)}^{(m)}) \\ &\leqq \sum_{m=1}^{M-1} C_m d_m \cdot N_M \cdot \varepsilon(N_M) = C_M/M, \end{aligned}$$

and

$$\text{(3.18)} \quad \begin{aligned} &\sum_{m=M+1}^{\infty} C_m \cdot \operatorname{Var}(Y_1^{(m)} + \cdots + Y_{N(M)}^{(m)}) \\ &= \sum_{m=M+1}^{\infty} (C_m/N_m) \cdot N_M \leqq \sum_{m=M+1}^{\infty} (1/m^2) \cdot C_M. \end{aligned}$$

Here (3.16) comes from Definition 2 (i) (v), (3.17) comes from the definitions of d_m, $1 \leqq m \leqq M - 1$, and of C_M, and (3.18) comes from Definition 2 (i) (v), the fact that $(N_m, m = 1, 2, \ldots)$ is increasing, and (3.12). Now, by elementary arguments, we have that, as $M \to \infty$,

$$\text{(3.19)} \qquad \operatorname{Var}(X_1 + \cdots + X_{N(M)}) \sim C_M,$$

$$\text{(3.20)} \quad \operatorname{Var}(C_M^{-1/2}(X_1 + \cdots + X_{N(M)}) - (Y_1^{(M)} + \cdots + Y_{N(M)}^{(M)})) \to 0,$$

and

$$\text{(3.21)} \qquad C_M^{-1/2}(X_1 + \cdots + X_{N(M)}) - (Y_1^{(M)} + \cdots + Y_{N(M)}^{(M)}) \to 0$$

in probability.

Now, $\forall J = 0, 1, 2, \ldots$, $\lim_{N\to\infty}\binom{N}{J}(1/N)^J(1 - 1/N)^{N-J} = (1/J!)e^{-1}$ by Poisson's classic limit theorem; and it follows that $\forall x \in \mathbf{R}$, $\lim_{N\to\infty} G_N(x) = F(x)$ where G_N and F are as in Definition 2 and the statement of Theorem 1. By Definition 2 (vii) and (3.21) and an elementary theorem on convergence in distribution, we have that $(X_1 + \cdots + X_{N(M)})/C_M^{1/2} \to F$ in distribution as $M \to \infty$; and, by (3.19) and another elementary theorem, $(X_1 + \cdots + X_{N(M)})/(\operatorname{Var}(X_1 + \cdots + X_{N(M)}))^{1/2} \to F$ in distribution as $M \to \infty$. This completes the proof of (1.9) and of Proposition 0.

REMARK. Proposition 0 remains valid, if in (1.5a), the number $2A_J B_\tau$ is replaced by a well chosen positive absolute constant (i.e., not depending on τ). To see this, one can apply Proposition 0 itself with τ replaced by $\gamma := (\gamma(n), n = 1, 2, \ldots)$ defined by $\gamma(n) := \min\{c, \tau(n)\}$, the constant $c > 0$ being chosen small (depending on τ) so that $\sup_{n\geqq 1} (n^{-1} \sum_{k=1}^{n} \gamma(k))/\gamma(n)$ is close to 1; the rest of the argument is elementary.

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Added in proof. M. Peligrad has proved the conjecture in § 1.

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