

SMOOTHNESS TO THE BOUNDARY OF CONFORMAL MAPS

STEVEN R. BELL^{*†} AND STEVEN G. KRANTZ^{*}

1. Introduction. Usually a genuinely good theorem finds its way from the technical formulations and generally cumbersome proofs of the research literature into books. In the hands of generations of bookwriters, a once forbidding theorem becomes an old friend whose proof is, if not simple, then at least polished, elegant, and compelling. Examples of this phenomenon in analysis are the Riesz Representation Theorem, Picard's Little Theorem, and Mergelyan's Theorem.

The subject of the present paper is an anomaly when viewed in the above light. In spite of the fact that it is a central, significant, and easily formulated result which bears directly on any first course in complex analysis, few people seem to be familiar with it. Fewer still know how to prove it, and there seems to be no optimal or canonical proof. Before we state the result in question we recall that the Riemann Mapping Theorem asserts that if $D \subseteq \mathbb{C}$ is simply connected and not equal to all of \mathbb{C} and if $\Delta \subseteq \mathbb{C}$ is the unit disc then there exists a biholomorphic (one-to-one, onto, holomorphic) mapping $f: \Delta \rightarrow D$. Now we have

THEOREM A. *Let $D \subseteq \mathbb{C}$ be a bounded, simply connected domain which is bounded by a C^∞ smooth Jordan curve. If $f: \Delta \rightarrow D$ is any biholomorphic mapping then f and all its derivatives have continuous extensions to the closure of Δ . Furthermore, f^{-1} and all its derivatives have continuous extensions to the closure of D .*

COROLLARY. *If D_1, D_2 are bounded, simply connected domains with C^∞ boundary and $f: D_1 \rightarrow D_2$ is biholomorphic then f and all its derivatives extend continuously to \bar{D}_1 .*

The proof of the corollary is almost immediate. For the Riemann Mapping Theorem guarantees the existence of biholomorphic maps $\varphi_j: \Delta \rightarrow D_j$, $j = 1, 2$. Theorem A guarantees that each φ_j and all its derivatives extend continuously to $\bar{\Delta}$. Likewise, each φ_j^{-1} and all its derivatives extend continuously to \bar{D}_j . Finally

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$$h = \varphi_2^{-1} \circ f \circ \varphi_1: \Delta \rightarrow \Delta$$

is biholomorphic, hence (see [11], p. 271) is a Möbius transformation. In particular, h and all its derivatives extend continuously to $\bar{\Delta}$. Thus we may conclude that

$$f = \varphi_2 \circ h \circ \varphi_1^{-1}$$

has the desired properties.

More generally, it follows from Theorem A that if $f: D_1 \rightarrow D_2$ is a biholomorphic map between multiply connected domains in the plane with C^∞ smooth boundaries, then f and all its derivatives extend continuously to \bar{D}_1 . Indeed, to prove this, it suffices to find conformal maps ϕ_j of D_j onto domains Ω_j such that the boundaries of Ω_j are real analytic Jordan curves, and such that ϕ_j and ϕ_j^{-1} , together with their derivatives, extend continuously to the closures ($j = 1, 2$). Then $h = \phi_2 \circ f \circ \phi_1^{-1}$ is a biholomorphic map of Ω_1 onto Ω_2 which extends holomorphically past the boundaries via the Schwarz Reflection Principle. Hence, $f = \phi_2^{-1} \circ h \circ \phi_1$ and all its derivatives extend continuously to \bar{D}_1 . The domains Ω_j and the maps ϕ_j are constructed in a standard way as follows.

Suppose D is a multiply connected domain in the plane whose boundary curves are C^∞ smooth Jordan curves. Let D^0 be the (simply connected) domain obtained from D by filling in the holes of D . Let $g_0: D^0 \rightarrow \Delta$ be the inverse of the Riemann mapping function associated to D^0 . Let $\Omega^0 = g_0(D)$. The domain Ω^0 is biholomorphic to D and the outer boundary of Ω^0 is a real analytic curve. Choose a point b in one of the holes of Ω^0 . The transformation $(z - b)^{-1}$ maps Ω^0 to a bounded domain D^1 . Repeat the procedure above to get a conformal map $g_1: D^1 \rightarrow \Omega^1$ where Ω^1 is a domain with two boundary components that are real analytic curves. It is clear that this procedure can be continued until a map $\phi: D \rightarrow \Omega$ is obtained where Ω is a domain whose boundary curves are all real analytic. The map ϕ is a composition of maps g_i and $(z - b_i)^{-1}$. Since these maps and their inverses extend, together with their derivatives, continuously to the boundaries, so does ϕ .

Apart from its aesthetic interest, Theorem A and its corollary are fundamental in that they allow one to do function theory on a smoothly bounded, simply connected domain by pulling it back to the disc. An example of this technique is given in Section 7, where we discuss boundary regularity of the Dirichlet problem.

We now give some background to put Theorem A in perspective. A good complex analysis course usually contains the celebrated theorem of Caratheodory ([3], 1913) to the effect that a biholomorphic mapping between two regions D_1 and D_2 , each bounded by a (not necessarily smooth) Jordan curve, extends continuously and univalently to a homeo-

morphism between the closure of D_1 and the closure of D_2 . Less well known is that Painlevé, in his Paris thesis ([9], 1887), proved a version of Theorem A. The proof was doubted by Harnack, who questioned the assertion (made without proof by Painlevé in [9]) that a biholomorphic map $f: \Delta \rightarrow D$, with D having C^1 boundary, extends univalently to $\partial\Delta$. Painlevé defended his theorem in his paper [10] in which he proved that if D is a simply connected domain bounded by a C^1 curve and $f: \Delta \rightarrow D$ is biholomorphic then f has a continuous univalent extension to $\bar{\Delta}$.

Of course Painlevé's result in [10] has been forgotten since it is (dramatically) subsumed by Caratheodory's. However the result of Painlevé's thesis is quite independent of Caratheodory's, and involves different ideas. Other authors who pioneered the study of boundary smoothness, by considering Green's potentials, were Kellogg [5] and Seidel [12]. Results very rapidly became quite technical, and the theory became bound up with very delicate pointwise questions about regularity for Green's potentials and Dirichlet problems (for some of the best modern results along these lines we refer the reader to the papers of S. Warschawski; for instance, see [13]).

It is safe to say that most questions related to Theorem A are completely understood, and they were resolved sufficiently long ago that the subject may be perceived to not be of current interest. But analogous questions for domains in \mathbb{C}^n , $n > 1$, are of great current interest (indeed the major questions are still open), and have led us to re-examine the situation in \mathbb{C} . It is our intention here to present a fairly simple and self-contained proof of Theorem A, based on ideas developed by one of us (see [2] and references therein) for use in \mathbb{C}^n .

Before we begin discussing the ideas of the proof, a few more remarks of a heuristic nature are in order. First, it is unreasonable to expect smooth boundary behavior of a biholomorphic map $f: \Delta \rightarrow D$ if D itself does not have smooth boundary. For instance, if D is a simply connected domain whose boundary has a corner, then conformality forces any biholomorphic map of Δ to D not to have a continuously differentiable extension to the closure of Δ . Thus if we want the derivatives up to order k of a biholomorphic map $f: \Delta \rightarrow D$ to extend continuously to $\partial\Delta$, we would expect to hypothesize that the Jordan curve bounding D is C^k (actually one has to assume a bit more—see the remarks at the end of this article).

At the opposite end of the spectrum, we may consider what happens when ∂D is real analytic. In that case, the Schwarz reflection principle may be applied to see that f continues holomorphically past every point of $\partial\Delta$.

The paper of Kerzman [6] gives an elegant method of studying boundary regularity of biholomorphic maps using the regularity theory of the Dirichlet problem on smoothly bounded domains in \mathbb{C} . This regularity theory

unfortunately cannot be considered standard knowledge (for, say, a first year graduate student), and it is difficult to give a self-contained presentation of the necessary ideas. The proof which we give is instead based on ideas arising from the theory of the Bergman kernel (see [7]). However, in order to keep the presentation elementary, we have developed the necessary tools in a completely ad hoc fashion. Those familiar with the theory of Hilbert spaces with reproducing kernels will see more elegant or natural ways to do some of the work (see Section 2 for more on the Hilbert space point of view), but our goal here has been to give a presentation which is self-contained and accessible to a student with just one semester's background in complex analysis.

2. The Idea of the Proof. In this section, we sketch the proof of Theorem A. Striving in this section for elegance and clarity, we shall (in this section only!) use some Hilbert space language and Lebesgue measure theory.

The Bergman projection plays a key role in our proof. For now, we shall define the Bergman projection in terms of Hilbert space. In § 4, we shall construct the Bergman projection in an ad hoc fashion using only basic facts from complex analysis. If D is a bounded domain in \mathbf{C} , then the space $L^2(D)$ of square integrable complex valued function on D forms a Hilbert space with inner product given by $\langle u, v \rangle = \int_D u \bar{v} dm$. Here, dm denotes Lebesgue measure (or area measure) on $\mathbf{R}^2 \cong \mathbf{C}$. The space $A^2(D)$ of holomorphic functions on D contained in $L^2(D)$ forms a closed subspace of $L^2(D)$. Indeed, it is a relatively simple consequence of the Mean Value Property that if h_i are holomorphic functions that converge to h in $L^2(D)$, then h must also be holomorphic on D . Since $A^2(D)$ is a closed subspace of $L^2(D)$, the orthogonal projection P of $L^2(D)$ onto $A^2(D)$ is defined. This operator P is the Bergman Projection associated to D . It is very useful in the study of holomorphic mappings because of the following transformation formula.

If $f: D_1 \rightarrow D_2$ is a biholomorphic mapping between bounded domains in \mathbf{C} , then the Bergman Projections P_1 and P_2 associated to D_1 and D_2 transform according to

$$f' \cdot ((P_2 \varphi) \circ f) = P_1(f' \cdot (\varphi \circ f))$$

for all φ in $L^2(D_2)$.

Here, $f' \cdot (\varphi \circ f)$ is shorthand notation for the function $f'(z) \cdot \varphi(f(z))$ defined on D_1 . See Proposition 4.3.1 below.

The idea of the proof of Theorem A is now fairly easy to describe. We wish to prove that if D_1 and D_2 are bounded by C^∞ smooth closed Jordan curves, then f and all its derivatives extend continuously to \bar{D}_1 . We might hope to find a function φ with the following properties.

(2.1) $P_2 \varphi = 1$

(2.2) φ belongs to $C^\infty(\bar{D}_2)$ and vanishes to high order on the boundary of D_2 .

If we have such a function, then (see § 5)

$$f' = f' \cdot (1 \circ f) = f' \cdot ((P_2\varphi) \circ f) = P_1(f' \cdot (\varphi \circ f)).$$

Now $f'(\varphi \circ f)$ should be well behaved near the boundary of D_1 since φ vanishes to high order on the boundary of D_2 ; and initially we have some control (by Cauchy estimates) on the boundary behavior of f' . Furthermore, we might hope that P_1 projects functions that are well behaved near the boundary to similar functions. If this is the case, then $f' = P_1(f'(\varphi \circ f))$ should also be well behaved.

The remainder of this paper is devoted to showing that this wishful thinking can be solidified into a proof.

3. Review of Some Facts about Infinitely Differentiable Functions. In this section we review some notions from real variable theory which will already be familiar to experts in analysis but may be less familiar to others. A good reference for all these matters is [4].

If $U \subseteq \mathbf{C}$ is open, $k \in \{0, 1, 2, \dots\}$, and $f: U \rightarrow \mathbf{C}$, then we say that f is k times continuously differentiable on U and write $f \in C^k(U)$ if all partial derivatives of f , of orders up to and including k , exist on U and are continuous. We say that f is infinitely differentiable on U and write $f \in C^\infty(U)$ if $f \in C^k(U)$ for every $k \geq 0$. Finally, if $E \subseteq \mathbf{C}$ is not necessarily open and $f: E \rightarrow \mathbf{C}$ then we say that $f \in C^k(E)$ (resp. $C^\infty(E)$) if there is an open $U \supseteq E$ and an $\tilde{f} \in C^k(U)$ (resp. $C^\infty(U)$) such that $\tilde{f}|_E = f$.

All of the definitions in the preceding paragraph apply equally well to a function $f: U \rightarrow \mathbf{C}$ (resp. $f: E \rightarrow \mathbf{C}$) with $U \subseteq \mathbf{R}$ (resp. $E \subseteq \mathbf{R}$) provided partial derivative is replaced by ordinary derivative.

EXERCISE. Let $\Delta \subseteq \mathbf{C}$ be the unit disc and $\bar{\Delta} = \Delta \cup \partial\Delta$ be its closure. A function $f: \bar{\Delta} \rightarrow \mathbf{C}$ satisfies $f \in C^k(\bar{\Delta})$ if and only if all partial derivatives of $f|_\Delta$ of order not exceeding k extend continuously to $\bar{\Delta}$.

Notice that implicit in the above discussion is the convention that $C^0(U)$ (resp. $C^0(E)$) denotes the continuous functions on U (resp. E).

LEMMA 3.1. *There is a C^∞ function $u: \mathbf{R} \rightarrow \mathbf{R}$ such that*

- (i) $0 \leq u(x) \leq 1$ for all x
- (ii) $u(x) \equiv 0$ for $x \leq 0$
- (iii) $u(x) \equiv 1$ for $x \geq 1$

COROLLARY 3.2. Let $0 < \varepsilon < 1$. There is a C^∞ function $u_\varepsilon: \mathbf{R} \rightarrow \mathbf{R}$ such that

$$\begin{aligned}
(3.2.1) \quad & -1 \leq u_\varepsilon \leq 1 \quad \forall x \in \mathbf{R} \\
(3.2.2) \quad & u_\varepsilon(x) \equiv -1 \text{ if } x \leq -\varepsilon/2 \\
(3.2.3) \quad & u_\varepsilon(x) \equiv 1 \text{ if } x \geq \varepsilon/2 \\
(3.2.4) \quad & u'_\varepsilon(0) \neq 0.
\end{aligned}$$

This paper deals with simply connected domains with C^∞ boundary. We should like to discuss these now in some detail.

DEFINITION 3.3. Let $D \subseteq \mathbf{C}$ be a bounded, simply connected domain with boundary curve consisting of the simple, closed curve $\gamma: [0, 1] \rightarrow \mathbf{C}$. We say that D has C^k boundary if $\gamma \in C^k [0, 1]$, $\gamma'(t) \neq 0$ for all t , and

$$\frac{d^j}{dt^j} \gamma(0) = \frac{d^j}{dt^j} \gamma(1) \text{ for all } j \leq k.$$

We define C^∞ boundary similarly.

If $D \subseteq \mathbf{C}$ is a given domain and $z \in \mathbf{C}$, it will be convenient to let

$$d_D(z) \equiv \inf_{w \in \partial D} |z - w| = \text{distance of } z \text{ to } \partial D.$$

Now we will state another nice, and much more practical, way to think about domains with C^∞ boundary.

PROPOSITION 3.4. *If D is a bounded, simply connected domain with C^∞ boundary, then there is a C^∞ function $r: \mathbf{C} \rightarrow \mathbf{R}$ such that*

$$(3.4.1) \quad D = \{z \in \mathbf{C}: r(z) < 0\}$$

$$(3.4.2) \quad \nabla r(z) \neq 0 \text{ for all } z \in \partial D.$$

Such an r is called a C^∞ defining function for D .

PROPOSITION 3.5. *Let $D \subseteq \mathbf{C}$ be bounded and simply connected with C^∞ boundary. Let r be any C^∞ defining function for D . Then there is a number $C_0 > 0$ such that*

$$|r(z)| \leq C_0 \cdot d_D(z), \text{ all } z \in D$$

A final definition is as follows. If $D \subseteq \mathbf{C}$ is a bounded, simply connected domain with C^∞ boundary and $f: \bar{D} \rightarrow \mathbf{C}$ is C^k , we say that f vanishes to order k on ∂D if any derivative of f on \bar{D} , up to and including $(k - 1)^{\text{th}}$ order, vanishes on ∂D .

EXERCISE. If $f \in C^k(\bar{D})$, r is a C^∞ defining function for D , and there is a constant C such that

$$|f(z)| \leq C|r(z)|^k$$

then f vanishes to order k on ∂D .

4. The Bergman Kernel and Projection. It is generally agreed that the space $L^2(-\pi, \pi)$ of square integrable functions on the interval $(-\pi, \pi)$ is the natural space in which to study Fourier series $\sum a_k e^{ikh}$. We shall see that an L^2 space of analytic functions plays an important role in the theory of conformal mappings. If D is a bounded domain in \mathbf{C} , we define $L_c^2(D)$ to be the space of *continuous*, complex valued functions φ on D such that $|\varphi|^2$ is Riemann integrable over D . We define an inner product for functions in $L_c^2(D)$ via

$$\langle \varphi, \psi \rangle_D = \int_D \varphi(z) \overline{\psi(z)} dV_z.$$

Here $dV_z = dx dy$ is the usual Riemann element of area. Note that the Schwarz inequality guarantees that $\langle \varphi, \psi \rangle_D$ is well-defined. The L^2 norm of φ is defined to be $\|\varphi\|_D = (\langle \varphi, \varphi \rangle_D)^{1/2}$. The subspace $A^2(D)$ of $L_c^2(D)$ consisting of analytic functions will be seen to have many important connections with conformal maps.

Suppose $f: D_1 \rightarrow D_2$ is a biholomorphic mapping of D_1 onto D_2 . If we let $u(x, y) = \operatorname{Re} f(x + iy)$ and $v(x, y) = \operatorname{Im} f(x + iy)$, then we can view f as a mapping $T(x, y) = (u(x, y), v(x, y))$ of $D_1 \subseteq \mathbf{R}^2$ into \mathbf{R}^2 . The Cauchy-Riemann equations are $(\partial u/\partial x) = (\partial v/\partial y)$ and $(\partial v/\partial x) = -(\partial u/\partial y)$; thus, the Jacobian Determinant $|T'|$ satisfies

$$\begin{aligned} |T'| &= \operatorname{Det} \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \operatorname{Det} \begin{bmatrix} u_x & -v_x \\ v_x & u_x \end{bmatrix} \\ &= u_x^2 + v_x^2 = |f'(x + iy)|^2. \end{aligned}$$

This fact gives rise to a particularly simple change of variables formula for conformal maps and is at the heart of much of the reasoning in this paper.

4.1 CHANGE OF VARIABLES FORMULA. Suppose $f: D_1 \rightarrow D_2$ is a biholomorphic mapping of D_1 onto D_2 . Let $F: D_2 \rightarrow D_1$ denote the inverse of f . If φ is in $L_c^2(D_2)$ and ψ is in $L_c^2(D_1)$, then

$$(4.1.1.) \quad \int_{D_1} f'(z) \cdot \varphi(f(z)) \cdot \overline{\psi(z)} dV_z = \int_{D_2} \varphi(w) \cdot \overline{F'(w)} \cdot \overline{\psi(F(w))} dV_w.$$

To be more precise, if φ and ψ are as above, then the operators A_1 and A_2 defined via

$$A_1 \varphi(z) = f'(z) \cdot \varphi(f(z))$$

and

$$A_2 \psi(w) = F'(w) \cdot \psi(F(w))$$

$$\begin{aligned} \|A_1\varphi\|_{D_1} &= \|\varphi\|_{D_2}, \\ \|A_2\psi\|_{D_2} &= \|\psi\|_{D_1}, \text{ and} \\ \langle A_1\varphi, \psi \rangle_{D_1} &= \langle \varphi, A_2\psi \rangle_{D_2}. \end{aligned}$$

PROOF. Since $|f'|^2$ is equal to the real Jacobian determinant of f when viewed as a mapping from \mathbf{R}^2 into \mathbf{R}^2 , the standard change of variables formula yields

$$\int_{D_1} |f'|^2 |\varphi \circ f|^2 = \int_{D_2} |\varphi|^2.$$

Similarly, $\|A_2\psi\|_{D_2} = \|\psi\|_{D_1}$. Finally,

$$\begin{aligned} \langle A_1\varphi, \psi \rangle_{D_1} &= \int_{D_1} f'(z) \varphi(f(z)) \overline{\psi(z)} dV_z \\ &= \int_{D_1} |f'(z)|^2 \varphi(f(z)) \overline{F'(f(z)) \psi(F(f(z)))} dV_z \\ &= \int_{D_2} \varphi(w) \overline{F'(w) \psi(F(w))} dV_w = \langle \varphi, A_2\psi \rangle_{D_2}. \end{aligned}$$

(Note that here we have used the identity $f'(z) = 1/F'(f(z))$.)

4.2 THE BERGMAN KERNEL. Let Δ denote the unit disc and suppose $h(z)$ is in $A^2(\Delta)$. We shall now derive the Bergman integral formula

$$(4.2.1) \quad h(z) = \frac{1}{\pi} \int_{\Delta} \frac{1}{(1 - z\bar{w})^2} h(w) dV_w.$$

The function $K_{\Delta}(z, w) = 1/\pi(1 - z\bar{w})^2$ is known as the *Bergman kernel* for the unit disc.

PROOF OF (4.2.1). First, we note that the formula is true when $z = 0$. Indeed, the Cauchy integral formula yields that $h(0) = (1/2\pi) \int_0^{2\pi} h(re^{i\theta}) d\theta$. If we integrate this formula against rdr , we obtain

$$h(0) = 2 \int_0^1 h(0)rdr = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} h(re^{i\theta}) d\theta r dr.$$

But this last is just polar coordinates for

$$\frac{1}{\pi} \int_{\Delta} h(w) dV_w.$$

We shall use the formula $h(0) = \langle h, (1/\pi) \rangle_{\Delta}$ together with the change of variables formula (4.1.1) to prove (4.2.1).

If $a \in \Delta$, we let $\varphi_a(z) = ((z + a)/1 + \bar{a}az)$. This function is a biholomorphic mapping of Δ onto itself. Notice that $\varphi_a(0) = a$ and that $\varphi'_a(0) = 1 - |a|^2$. Let $\Phi_a(w) = \varphi_a^{-1}(w) = ((w - a)/(1 - \bar{a}w))$. Now

$$\begin{aligned} (1 - |a|^2)h(a) &= \varphi'_a(0) h(\varphi_a(0)) = \frac{1}{\pi} \int_{\Delta} \int_{\Delta} \varphi'_a(z) h(\varphi_a(z)) dV_z = \langle \varphi'_a \cdot (h \circ \varphi_a), \frac{1}{\pi} \rangle_{\Delta} \\ &= \langle h, (1/\pi) \Phi'_a \rangle_{\Delta} \end{aligned}$$

(by (4.1.1))

$$= \frac{1}{\pi} \int_{\Delta} h(w) \frac{1 - |a|^2}{(1 - a\bar{w})^2} dV_w.$$

Thus (4.2.1) is proved.

Let D be a bounded simply connected domain in \mathbf{C} . We now wish to derive a Bergman integral formula for D . To be precise, we seek a function $K_D(z, w)$ defined on $D \times D$ such that

- 1) $K_D(z, w)$ is analytic in z and conjugate-analytic in w ;
- 2) $\int_D |K_D(z, w)|^2 dV_w < \infty$ for each $z \in D$,
- 3) $K_D(z, w) = \overline{K_D(w, z)}$, and
- 4) If $h \in A^2(D)$, then

$$h(z) = \int_D K_D(z, w) h(w) dV_w.$$

If such a function exists, it is unique. Indeed, if $K_1(z, w)$ and $K_2(z, w)$ both satisfy properties (1)–(4), then consider

$$I \equiv \int_D K_1(a, w) K_2(w, b) dV_w = K_2(a, b).$$

Notice that

$$\bar{I} = \int_D K_1(w, a) K_2(b, w) dV_w = K_1(b, a).$$

Hence $K_1(a, b) = \overline{K_1(b, a)} = \bar{I} = I = K_2(a, b)$. The three main ingredients we shall use to construct $K_D(z, w)$ are

- A) The Bergman integral formula (4.2.1) for the disc,
- B) The existence of the Riemann mapping function associated to D , and
- C) The change of variables formula (4.1.1).

The Riemann mapping theorem asserts that there is a one to one conformal mapping F of D onto Δ . Let $f = F^{-1}$. Let z be a point in D and let $a = F(z)$. Observe that for any $h \in A^2(D)$ the change of variables formula yields:

$$\begin{aligned} f'(a)h(f(a)) &= \int_{\Delta} K_{\Delta}(a, w) f'(w) h(f(w)) dV_w \\ &= \int_D K_{\Delta}(a, F(\zeta)) \overline{F'(\zeta)} h(\zeta) dV_{\zeta}. \end{aligned}$$

If we multiply this equation by $F'(z) = 1/f'(a)$ and replace a by $F(z)$, we obtain

$$h(z) = \int_D F'(z) K_D(F(z), F(\zeta)) \overline{F'(\zeta)} h(\zeta) dV_\zeta.$$

The function $K_D(z, w) = F'(z)K_D(F(z), F(w))\overline{F'(w)}$ is easily seen to satisfy (1)–(4) and is therefore the unique Bergman kernel for D .

4.3 THE BERGMAN PROJECTION. If $\varphi \in L^2_c(D)$, then we define a function $P_D\varphi$ on D via

$$P_D\varphi(z) = \int_D K_D(z, w)\varphi(w)dV_w.$$

Notice that if $h \in A^2(D)$, then $P_Dh = h$. The operator P_D is called the Bergman projection for D . If we retrace some of the steps in the construction of $K_D(z, w)$, we can derive an important transformation rule for the Bergman Projection.

PROPOSITION 4.3.1. *Let $F: D \rightarrow \Delta$ be a one to one conformal mapping of the simply connected domain D onto Δ and let $f = F^{-1}$. Then $P_\Delta(f'(f \circ \varphi)) = f' \cdot ((P_D\varphi) \circ f)$ for all φ in $L^2_c(D)$.*

PROOF.

$$\begin{aligned} P_\Delta(f'(\varphi \circ f))(z) &= \\ &= \int_\Delta K_\Delta(z, w) \cdot f'(w) \cdot \varphi(f(w))dV_w \\ &= \int_D K_D(z, F(\zeta)) \cdot \overline{F'(\zeta)} \cdot \varphi(\zeta)d\zeta \\ &= \frac{1}{F'(f(z))} \cdot (P_D\varphi)(f(z)) \\ &= f'(z) \cdot (P_D\varphi)(f(z)). \end{aligned}$$

5. Some Key Lemmas. Throughout this section, D will denote a simply connected domain in \mathbb{C} bounded by a C^∞ smooth closed Jordan curve. By 3.4, we may suppose that D has C^∞ defining function $r: D = \{z \in \mathbb{C}: r(z) < 0\}$. Most of the wishful thinking in § 2 hinged on the possibility of cooking up a function satisfying properties (2.1) and (2.2). We shall now show how such a φ can be constructed on D .

The construction of φ rests on the following lemma (in this lemma $\Delta = (\partial^2/\partial x^2) + (\partial^2/\partial y^2)$ is the usual Laplace operator):

LEMMA 5.1. $P_D(\Delta(\theta r^2)) \equiv 0$ for any function θ in $C^\infty(\bar{D})$.

PROOF. We shall show that

$$\int_D \bar{h} \Delta(\theta r^2) dV = 0$$

for each function h in $A^2(D)$. Setting $\bar{h}(\bar{w}) = K_D(z, w)$ then yields the Lemma. The idea of the proof is to apply Green's identity

$$\int_D u \Delta v dV = \int_D v \Delta u dV + \int_{\partial D} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) d\sigma$$

with $u = \bar{h}$ and $v = \theta \cdot r^2$. Since h is holomorphic, \bar{h} is harmonic and $\int_D v \Delta u = 0$. Furthermore, since θr^2 vanishes to second order on ∂D , the boundary integral also vanishes. But because h may not be in $C^2(\bar{D})$, we must be careful. Let $r_\varepsilon(z)$ be equal to $r(z) + \varepsilon$ if $r(z) \leq -\varepsilon$ and be equal to zero if $r(z) > -\varepsilon$. If $\varepsilon > 0$ is small, then the set $D_\varepsilon = \{z: r_\varepsilon(z) < 0\}$ is a domain bounded by a C^∞ smooth Jordan curve; the closure of D_ε is contained in D . Note that θr_ε^2 vanishes to second order on ∂D_ε and that $\Delta(\theta r_\varepsilon^2)$ converges uniformly on \bar{D} to $\Delta(\theta r^2)$. Thus, if $h \in A^2(D)$, then

$$\begin{aligned} \int_D \bar{h} \Delta(\theta r^2) dV &= \lim_{\varepsilon \rightarrow 0} \int_{D_\varepsilon} \bar{h} \Delta(\theta r_\varepsilon^2) dV \\ &= \lim_{\varepsilon \rightarrow 0} \int_{D_\varepsilon} \bar{h} \Delta(\theta r_\varepsilon^2) dV = 0, \end{aligned}$$

where the last equality is by the Green's Identity argument. Thus $P_D(\Delta(\theta r^2)) \equiv 0$.

Now we can construct the function φ .

LEMMA 5.2. *For each positive integer s , there is a function φ_s in $C^\infty(\bar{D})$ such that $P_D \varphi_s \equiv 1$ and a constant C_s such that $|\varphi_s(z)| \leq C_s d_D(z)^s$ for all z in D .*

PROOF. We shall use induction. The first function φ_1 will be given by $\varphi_1 = 1 - \Delta(\theta_1 r^2)$ where θ_1 is to be chosen. Note that $P_D \varphi_1 = P_D 1 - P_D \Delta(\theta_1 r^2) = 1$ by Lemma 5.1. Therefore, (by 3.5) we need only concern ourselves with choosing θ_1 in such a way that $|\varphi_1(z)| \leq c|r(z)|$. Now $\varphi_1 = 1 - \Delta(\theta_1 r^2) = 1 - r^2 \Delta \theta_1 - 4r \nabla \theta_1 \cdot \nabla r - 2\theta_1 |\nabla r|^2 - 2r \theta_1 \Delta r$. We would like to set $\theta_1 = (2|\nabla r|^2)^{-1}$. Then, using 3.5, we have

$$|1 - \Delta(\theta_1 r^2)| = |r(-r \Delta \theta_1 - 4 \nabla \theta_1 \cdot \nabla r - 2 \theta_1 \cdot \Delta r)| \leq K_1 |r(z)| \leq C_1 d_D(z).$$

However, the fact that ∇r vanishes at points inside D (where r takes its minimum value, for example) presents a problem. To solve the problem we multiply by a C^∞ function χ on \bar{D} which is identically 1 near the boundary of D and is identically 0 away from the boundary of D . Now set $\theta_1 = \chi / (2|\nabla r|^2)$. Then $\varphi_1 = 1 - \Delta(\theta_1 r^2) = r \Phi_1$ where Φ_1 is in $C^\infty(\bar{D})$. Thus, because of 3.5, we have constructed a function which satisfies the conclusion of the Lemma when $s = 1$.

Suppose we have constructed functions $\varphi_1, \varphi_2, \dots, \varphi_{s-1}$ which satisfy

- i) $P\varphi_i \equiv 1, i = 1, 2, \dots, s - 1$
- ii) $\varphi_i = r^i \Phi_i$ for Φ_i in $C^\infty(\bar{D})$.

We shall use the same idea used to construct φ_1 to construct a suitable φ_s . We shall choose a function θ_s in $C^\infty(\bar{D})$ so that $\varphi_s = \varphi_{s-1} - \Delta(r^{s+1}\theta_s)$ satisfies (i) and (ii). Once again, (i) is automatically satisfied because of Lemma 5.1. Now

$$\begin{aligned} \varphi_{s-1} - \Delta(r^{s+1}\theta_s) &= r^{s-1}\Phi_{s-1} - r^{s+1}\Delta\theta_s - 2(s+1)r^s\nabla r \cdot \nabla\theta_s \\ &\quad - s(s+1)r^{s-1}\theta_s|\nabla r|^2 - (s+1)r^s\theta_s\Delta r. \end{aligned}$$

If we choose

$$\theta_s = \frac{\chi\Phi_{s-1}}{s(s+1)|\nabla r|^2},$$

then $\varphi_s = r^s\Phi_s$ where Φ_s is in $C^\infty(\bar{D})$. Once again, 3.5 finishes the proof. The induction is complete.

Next, we prove a simple estimate for the Bergman kernel associated to the unit disc Δ . Let s be a positive integer.

LEMMA 5.3. *The Bergman kernel for the disc, $K_\Delta(z, w) = (1/\pi)(1 - z\bar{w})^{-2}$, satisfies*

$$\max_{z \in \Delta} \left| \frac{\partial^s}{\partial z^s} K_\Delta(z, w) \right| \leq \frac{(s+1)!}{\pi} d_\Delta(w)^{-s-2}.$$

PROOF. A simple induction reveals that $\partial^s/\partial z^s K_\Delta(z, w) = ((s+1)!/\pi)(\bar{w}^s/(1 - z\bar{w})^{s+2})$. The maximum principle allows us to deduce that

$$\begin{aligned} \max_{z \in \Delta} \left| \frac{\partial^s}{\partial z^s} K_D(z, w) \right| &= \frac{(s+1)!}{\pi} \max_{z \in \partial\Delta} \frac{|w|^s}{|1 - z\bar{w}|^{s+2}} \\ &\leq \frac{(s+1)!}{\pi} \left(\min_{z \in \partial\Delta} |1 - z\bar{w}| \right)^{-s-2}. \end{aligned}$$

Now $|(z - w)/(1 - z\bar{w})| = 1$ whenever z is in $\partial\Delta$ and w is in Δ .

Hence $\min_{z \in \partial\Delta} |1 - z\bar{w}| = \min_{z \in \partial\Delta} |z - w| = d_\Delta(w)$, and this completes the proof.

We must prove one last lemma. This lemma will allow us to conclude that $f' \cdot (\varphi \circ f)$ behaves well near $\partial\Delta$ when φ vanishes to high order on ∂D .

LEMMA 5.4. *Suppose $f: \Delta \rightarrow D$ is a biholomorphic mapping of the unit disc onto a domain in \mathbb{C} bounded by a C^∞ smooth Jordan curve. There is a constant c such that*

$$d_D(f(z)) \leq cd_\Delta(z)$$

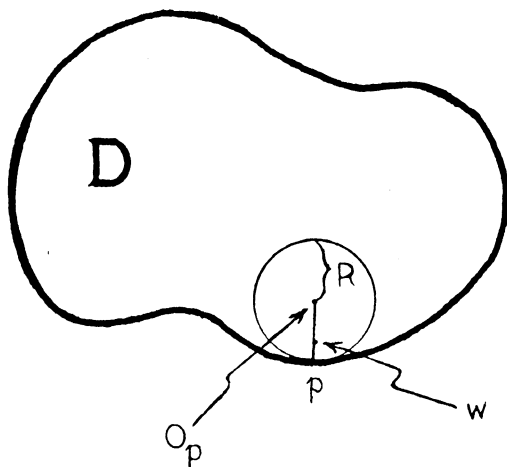


FIG. 1.

for all z in Δ ,

PROOF. Choose a positive real number R such that a disc of radius R can be rolled around the inside boundary of D without ever touching more than one boundary point. Let p be a point in ∂D and let C_p denote a circle of radius R contained in \bar{D} that is internally tangent to ∂D at p with center O_p . We shall consider points w that lie along the segment joining p to O_p . See Figure 1.

The Poisson Kernel $P(z, \cdot)$ for the disc enclosed by C_p is given by

$$P(z, \zeta) = \frac{R^2 - |z - O_p|^2}{2\pi|\zeta - z|^2}.$$

Notice that if ζ is on C_p , then

$$P(w, \zeta) \geq \frac{1}{2\pi} \frac{(R - |w - O_p|)(R + |w - O_p|)}{(2R)^2} \geq \frac{1}{8\pi R} d_D(w).$$

Let $F = f^{-1}$. We shall now apply the Poisson integral formula to the function $F(z)$ on the disc enclosed by C_p . We obtain

$$F(w) = \int_{C_p} P(w, \zeta) F(\zeta) d\theta_\zeta$$

(see [1], p. 165–166). Hence, because $P(z, \zeta)$ is a positive function,

$$|F(w)| \leq \int_{C_p} P(w, \zeta) |F(\zeta)| d\theta_\zeta.$$

Furthermore, since $\int_{C_p} P(w, \zeta) d\theta_\zeta = 1$, we find that

$$\begin{aligned} d_{\mathcal{A}}(F(w)) &= 1 - |F(w)| \\ &\geq \int_{C_p} P(w, \zeta)(1 - |F(\zeta)|)d\theta_{\zeta} \\ &\geq \frac{d_D(w)}{8\pi R} \int_{C_p} (1 - |F(\zeta)|)d\theta_{\zeta}. \end{aligned}$$

Now $\lambda(p) = (1/8\pi R) \int_{C_p} (1 - |F(\zeta)|)d\theta_{\zeta}$ is a positive continuous function of the point p in the compact set ∂D . Hence, there is a constant $c > 0$ such that $\lambda(p) > (1/c)$ for all p in ∂D . Therefore $d_{\mathcal{A}}(F(w)) \geq (1/c)d_D(w)$ for all w in D with $d_D(w) < R$. The continuous function $d_{\mathcal{A}}(F(w))$ attains a minimum value on the compact set $\{w \in D: d_D(w) \geq R\}$. Thus, by enlarging c if necessary, we obtain that $d_{\mathcal{A}}(F(w)) \geq (1/c)d_D(w)$ for all w in D . Replacing w by $f(z)$ in this inequality yields

$$d_D(f(z)) \leq cd_{\mathcal{A}}(z).$$

6. The proof. All the ingredients for the proof are on the table. All we have left to do is to put them together.

Let s be a positive integer. We will show that $(d^s/dz^s) f(z)$ is a bounded function on \mathcal{A} . In § 4, we proved that $P_{\mathcal{A}}(f' \cdot (\varphi \circ f)) = f' \cdot ((P_D\varphi) \circ f)$. Lemma 5.2 allows us to choose a function $\varphi = \varphi_{s+2}$ such that $P_D\varphi \equiv 1$ and $|\varphi(z)| \leq cd_D(z)^{s+2}$. We now wish to show that

$$(6.1) \quad |f'(z)\varphi(f(z))| \leq (\text{constant})d_{\mathcal{A}}(z)^{s+1}.$$

We know that $|\varphi(f(z))| \leq cd_D(f(z))^{s+2} \leq (\text{constant})d_{\mathcal{A}}(z)^{s+2}$ by Lemma 5.4. Thus, if we show that $|f'(z)| < (\text{const.})d_{\mathcal{A}}(z)^{-1}$, then we will have the desired estimate.

Fix a point z in \mathcal{A} and let B_z denote the disc about z of radius $d_{\mathcal{A}}(z)$. Now $f'(z) = (1/\pi d_{\mathcal{A}}(z)^2) \int_{B_z} f' dV$ by the averaging property of holomorphic functions, and so $|f'(z)| \leq (1/\pi d_{\mathcal{A}}(z)^2) \|1\|_{B_z} \|f'\|_{B_z}$ via Schwarz's Inequality. Furthermore, $\|1\|_{B_z} = (\text{area } B_z)^{1/2} = \pi^{1/2} d_{\mathcal{A}}(z)$ and

$$\begin{aligned} \|f'\|_{B_z} &\leq \|f'\|_{\mathcal{A}} = \left(\int_{\mathcal{A}} |f'|^2 dV \right)^{1/2} \\ &= \left(\int_D 1 dV \right)^{1/2} = (\text{Area } D)^{1/2}. \end{aligned}$$

Thus $|f'(z)| \leq ((1/\pi) \text{Area } D)^{1/2} d_{\mathcal{A}}(z)^{-1}$ as desired. This establishes (6.1).

We now know that

$$\begin{aligned} f' &= f' \cdot (1 \circ f) = f' \cdot ((P_D\varphi) \circ f) \\ &= P_{\mathcal{A}}(f' \cdot (\varphi \circ f)) \end{aligned}$$

where $|f'(z)\varphi(f(z))| \leq (\text{constant})d_{\mathcal{A}}(z)^{s+1}$. For the last equality we have used Proposition 4.3.1. Writing out the projection, we see that

$$\begin{aligned} \frac{d^s}{dz^s} f(z) &= \frac{d^{s-1}}{dz^{s-1}} f' = \frac{d^{s-1}}{dz^{s-1}} (P_{\Delta}(f' \cdot (\varphi \circ f))) \\ &= \int_{\Delta} \frac{d^{s-1}}{dz^{s-1}} K_{\Delta}(z, w) f'(w) \varphi(f(w)) dV_w \end{aligned}$$

and we are able to conclude that $|d^s/dz^s f(z)| \leq (\text{constant})$ for $z \in \Delta$ because

$$\sup_{z \in \Delta} \left| \frac{d^{s-1}}{dz^{s-1}} K_{\Delta}(z, w) \right| \leq (\text{constant}) d_{\Delta}(w)^{-s-1}$$

(by Lemma 5.3) and because of (6.1).

We have proved that all the derivatives of f are bounded functions on Δ . But, if g is holomorphic on Δ , then

$$|g(z) - g(w)| \leq (\sup_{\zeta \in L} |g'(\zeta)|) |z - w|$$

where L denotes the line segment joining z to w . Hence, whenever $f^{(s+1)}(z)$ is bounded on Δ , we are able to conclude that $f^{(s)}(z)$ extends continuously to $\bar{\Delta}$.

We now know that $f \in C^{\infty}(\bar{\Delta})$. We would also like to know that $F = f^{-1} \in C^{\infty}(\bar{D})$. This will follow via the Inverse Function Theorem from the fact, which we are about to prove, that $\nabla f(z) \neq 0$ for $z \in \partial\Delta$.

Let $z_0 \in \partial\Delta$. We may assume that $z_0 = 1$. Let $w_0 = f(z_0)$. Choose a point p_0 which lies along the outward pointing normal vector to ∂D at w_0 so that the circle with center p_0 and radius $|p_0 - w_0|$ touches \bar{D} only at w_0 . See Figure 2. Let $a \in \mathbb{C}$ satisfy $|a| = 1$ and $a/(w_0 - p_0) = 1/|w_0 - p_0|$. The function

$$h(w) = \text{Re} \left(\frac{a}{w - p_0} \right)$$

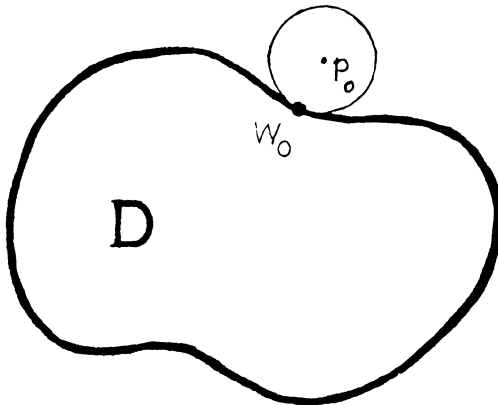


FIG. 2.

is harmonic on D , is in $C^\infty(\bar{D})$, and attains its maximum value M on \bar{D} at w_0 . Thus

$$g(z) = h(f(z)) - M$$

is harmonic on Δ , is in $C^\infty(\bar{\Delta})$, $g(z) < 0$ for $z \in \bar{\Delta} \setminus \{1\}$, and $g(1) = 0$. We will now show that $(\partial g / \partial r)(1) > 0$. Then the Chain Rule implies that $\nabla f(1) \neq 0$, as desired. The Poisson Integral Formula for the disc yields that

$$g(r) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(1-r^2)}{|r - e^{i\theta}|^2} g(e^{i\theta}) d\theta.$$

Hence

$$\frac{g(1) - g(r)}{1 - r} = \frac{1}{2\pi} \int_0^{2\pi} \frac{1+r}{|r - e^{i\theta}|^2} (-g(e^{i\theta})) d\theta \geq \frac{1}{8\pi} \int_0^{2\pi} -g(e^{i\theta}) d\theta$$

because

$$\frac{1+r}{|r - e^{i\theta}|^2} \geq \frac{1}{|r - e^{i\theta}|^2} \geq \frac{1}{4}.$$

Therefore

$$\frac{\partial g}{\partial r}(1) \geq \frac{1}{8\pi} \int_0^{2\pi} -g(e^{i\theta}) d\theta > 0$$

because $-g(e^{i\theta}) > 0$ for $\theta \neq 0$. This completes the proof of Theorem A.

7. An Application of Theorem A. Theorem A has considerable intrinsic interest. However, it is also a fundamental tool when putting into practice the dictum that doing function theory on a bounded simply connected domain is equivalent to doing function theory on the disc. In this section we provide an example of how this dictum is applied.

We noted in the introduction that the regularity theory for the Dirichlet problem may be used to give a short proof of Theorem A. Here we provide a converse:

THEOREM B. *Let D be a bounded, simply connected domain in \mathbb{C} with C^∞ boundary. Let $f: \partial D \rightarrow \mathbb{C}$ be a C^∞ function, and let $u \in C(\bar{D})$ be the solution to the Dirichlet problem on \bar{D} with boundary data f :*

$$\begin{aligned} \Delta u &= 0 \text{ on } D, \\ u|_{\partial D} &= f. \end{aligned}$$

Then in fact $u \in C^\infty(\bar{D})$.

PROOF. Suppose we already knew the assertion to be true for $D = \Delta$, the disc. Then we would proceed as follows: let

$$\Phi: \Delta \rightarrow D$$

be a conformal map. By Theorem A, Φ has a C^∞ continuation to $\bar{\Delta}$, which we continue to denote by Φ . Then $\check{f} \equiv f \circ (\Phi|_{\partial\Delta}) \in C^\infty(\partial\Delta)$. Since we are assuming that the theorem has been proved on the disc, we then know that the solution to the Dirichlet problem on the disc given by

$$\begin{aligned} \Delta \check{u} &= 0 \\ \check{u}|_{\partial\Delta} &= \check{f} \end{aligned}$$

is C^∞ on $\bar{\Delta}$. Since (by Theorem A) $\Phi^{-1} \in C^\infty(\bar{D})$, it follows that $u \equiv \check{u} \circ \Phi^{-1} \in C^\infty(\bar{D})$. Also

$$\Delta u = 0 \text{ on } D$$

and

$$u|_{\partial D} = f.$$

This proves the theorem, modulo our assumption. So we are reduced to proving Theorem B' below.

THEOREM B' . *If $f \in C^\infty(\partial\Delta)$ then the Poisson integral of f (the solution to the Dirichlet problem with boundary data f) is C^∞ on $\bar{\Delta}$.*

The proof of Theorem B' is accomplished by direct estimation, since the Poisson kernel for the disc is explicit. See [15] for details.

8. Concluding Remarks. We have already commented that if ∂D is C^k and $f: \Delta \rightarrow D$ is conformal then it does not necessarily follow that f extends to be C^k on $\bar{\Delta}$. A nice counterexample for $k = 1$ has been provided by Webster [14]. Let $f(z) = (z - 1)/\log(z - 1)$, $D = f(\Delta)$. Then it is straightforward to check that ∂D is C^1 but f does not extend C^1 to $\partial\Delta$.

It is true, and follows for instance from elliptic regularity theory, that if ∂D is C^k then a conformal mapping $f: \Delta \rightarrow D$ will extend C^{k-1} to \bar{D} and moreover that any derivative $f_{j,\ell} \equiv (\partial/\partial x)^j (\partial/\partial y)^\ell f$ with $j + \ell \leq k - 1$ will satisfy a Lipschitz condition of the form

$$\sup_{z, w \in \Delta} |f_{j,\ell}(z) - f_{j,\ell}(w)| \leq C_\delta |z - w|^{1-\delta}$$

for any $\delta > 0$. Even more refined results may be obtained if one introduces more sensitive measures of smoothness than C^k (see the theory of integral order Lipschitz-Zygmund spaces in [8]).

As Warschawski points out in [13], the hypothesis that ∂D be smooth along a whole arc is somewhat restrictive. In [13], he considers the situation where ∂D is smooth at just one point and studies smoothness to the boundary of a conformal map. This is very delicate and can be described no further here.

While the proof of our Theorem A is elementary, it is not simple. There

would be real merit in a genuinely simple proof, both for expository reasons and because it might lend ideas to the several variable situation where much less is known. We hope that this paper will inspire someone to find an easy proof of Theorem A.

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DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, W. LAFAYETTE, IN 47907

DEPARTMENT OF MATHEMATICS, WASHINGTON UNIVERSITY, ST. LOUIS, MO. 63139