ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 17, Number 1, Winter 1987

THE HOLONOMIC IMPERATIVE AND THE HOMOTOPY GROUPOID OF A FOLIATED MANIFOLD

JOHN PHILLIPS

Since A. Connes in [1, 2] defined the C^*-algebra of a foliation for the purpose of studying index theory on foliated manifolds a number of C^*-algebraists have worked to obtain the necessary geometric prerequisites in order to understand these new examples of C^*-algebras. One of the first conceptual difficulties encountered is the notion of holonomy as defined by C. Ehresmann. Connes based his operator algebra constructions on the holonomy groupoid (or graph) of the foliation as defined by H. E. Winkelnkemper in [9]. In fact this groupoid (and its topology) had already been defined in the very general setting of topological foliations by Ehresmann himself [3, pp. 130–132]. The C^∞ structure of this groupoid was introduced by J. Pradines in [6] although no details or proofs have appeared. Besides the timeliness of his rediscovery of the holonomy groupoid for C^∞ foliations, one of the merits of Winkelnkemper's paper [9] is its concrete constructions and concise proofs.

Now, even though holonomy is natural enough from a differential equations point of view [5, p. 377] and is useful in the study of foliations *per se*, one wonders whether it is really necessary for the groupoid and hence the C^*-algebra. We put the question "why holonomy" to Georges Skandalis and he replied that one is "forced" to consider the concept when one tries to make the graph (of the equivalence relation defined by the leaves) into a manifold. After thinking about this for some time we came up with a theorem which justifies his remark. We hasten to add that Georges Skandalis also has a theorem justifying his remark (private communication) which predates ours—the private communication tarried on the desk of an intermediary colleague for some months during which time we had begun the fomulation of our own theorem. Of course, the two theorems are similar but not identical. The main point of our theorem is to show how the holonomy groupoid fits naturally between two groupoids canonically associated with the foliation, namely, the equivalence relation and the homotopy groupoid (in Skandalis' theorem, no mention is made of the homotopy groupoid). As byproducts of this investigation

Received by the editors on March 13, 1985 and in revised form on June 20, 1985.

we show: (1) the homotopy groupoid has an essentially unique manifold structure (not necessarily Hausdorff, even when the holonomy groupoid is Hausdorff); and (2) if the foliation is given by the locally free action of the simply connected Lie group H on the connected manifold V, then the homotopy groupoid is just $H \times V$. This latter result shows that the transformation groupoid, $H \times V$, is naturally obtained from the foliation, even if it does not equal the holonomy groupoid.

In fact, the minimality of the holonomy groupoid also appears in [**6**, Theorem 1]. Moreover, the homotopy groupoid and its C^∞ structure (under the name "monodromy groupoid") were introduced in [**6**] together with a universal property which implies its maximality. With some effort, our main theorem could probably be deduced from Pradines' results [**6**, **7**]. However, at the time this paper was written we were ignorant of Pradines' work and so we took a more "nuts and bolts" approach. We hope that this approach has the advantage of clarity and accessibility to non-specialists. On the other hand, Pradine's approach does generalize to certain singular foliations [**8**].

Acknowledgements. My indebtedness to Georges Skandalis is clear and greatly appreciated. I would also like to thank the members of the Dalhousie Foliations Seminar and especially Nigel Higson and Bill (W.J.) Phillips (no relation) for many illuminating comments and conversations. Bob Paré also gets honorable mention.

I thank N.S.E.R.C. of Canada and the A.R.G.S. of Australia for supporting this research. I am especially grateful to Iain Raeburn and Gavin Brown for the invitation to visit the University of New South Wales which gave me the quiet time necessary to organise my thoughts and write down the details of this paper.

I thank the referee for pointing out the existence of Ehresmann's and Pradines' papers—it is amazing that they have remained undetected by so many people. They deserve to be more widely read.

1. Definitions and notations. Let V be a connected C^∞ n-manifold where $n = p + q$. Let $F = \{(h_i, U_i)\}_{i \in I}$ be a maximal family of charts covering V, with the property that, for each (i, j), the change of coordinates $h_j \circ h_i^{-1}$: $\mathbf{R}^p \times \mathbf{R}^q \to \mathbf{R}^p \times \mathbf{R}^q$ has the form $(h_j \circ h_i^{-1})(x, y) = (\phi(x), \psi(x, y))$. That is, the q-plane $\{x\} \times \mathbf{R}^q$ is sent to the q-plane $\{\phi(x)\} \times \mathbf{R}^q$ by the coordinate change $h_j \circ h_i^{-1}$. Such charts will be called foliation charts. If we give $\mathbf{R}^p$ the discrete topology and $\mathbf{R}^q$ the usual topology, then we can, for the moment, topologize V by letting the U_i's be basic open sets with the topology transported from $\mathbf{R}^p_{\text{disc}} \times \mathbf{R}^q$ via the h_i^{-1}'s. This is called the leaf topology on V, and the arc-components of V in this topology are called the leaves of the q-dimensional foliation (V, F). Each leaf, L is

naturally an immersed (but not usually embedded) q-dimensional submanifold of V.

If (h, U) is a foliation chart of (V, F) and L is a leaf, then the connected components of $L \cap U$ are called the plaques of L in U. If $h(U) \subseteq \mathbf{R}^p \times \mathbf{R}^q$ is the product of an open set in $\mathbf{R}^p$ with a connected open set in $\mathbf{R}^q$, then the maps $\pi_1 \circ h: U \to \mathbf{R}^p$ are a C^∞ parametrization of the plaques in U. This idea is formalized in the notion of distinguished functions.

DEFINITION. Let (V, F) be a foliated manifold of dimension $n = p + q$, with leaves of dimension q. A distinguished function is a map f from an open set O of V to $\mathbf{R}^p$ which is locally of the form $\pi_1 \circ h$ for foliation charts, (h, U). That is, for each a in O we can find a foliation chart (h, U) with $a \in U \subseteq O$ so that $f(b) = \pi_1 \circ h(b)$ for all b in U.

For $a \in V$, we denote the set of germs of distinguished functions at a by D_a and denote the germ of a distinguished function f defined in a neighbourhood of a by $[f]_a$. For these bacteriological purposes we can assume any such f has the form $\pi_1 \circ h$ for some foliation chart (h, U), and hence if $[g]_a$ is another element of D_a, we can assume $g = \phi \circ f$ for some local diffeomorphism ϕ of $\mathbf{R}^p$. Another way of saying this is, that the group of germs of local diffeomorphisms of $\mathbf{R}^p$ which fix 0 acts transitively by composition on $D_a^0 = \{[f]_a \in D_a \mid f(a) = 0\}$.

Another notion we need is that of a C^∞-groupoid. Apparently this was first introduced by C. Ehresman in [**4**].

DEFINITION. *A* (not necessarily Hausdorff) topological groupoid G is a C^∞-groupoid if it is also a C^∞-manifold with the property that the sets $G^{(0)}$ and $G^{(2)}$ are embedded submanifolds (not necessarily closed) and the maps composition $G^{(2)} \to G$, inversion $G \xrightarrow{-1} G$, range $G \xrightarrow{r} G^{(0)}$ and source $G \xrightarrow{s} G^{(0)}$ are C^∞.

As we shall see, both the homotopy groupoid and Winkelnkemper's holonomy groupoid are examples of C^∞-groupoids. On the other hand, if G is a C^∞-groupoid then it easily follows that the maps s and r are submersions and so G is foliated by the components of $G_a = s^{-1}(a)$ as a varies in $V := G^{(0)}$. If we assume that $s \times r : G \to V \times V$ is an immersion, then $s \times r$ (and hence r) is locally 1:1 on G_a, and so V is foliated by the components of $r(G_a)$ as a varies in V. One can then show (as Skandalis does) that a quotient of G naturally lies inside his "groupoid of all possible holonomies" and contains the holonomy groupoid. Thus, under the assumption that $s \times r : G \to G^{(0)} \times G^{(0)}$ is an immersion, we see that G is intimately connected with the holonomy groupoid of a certain natural foliation of $G^{(0)}$.

Of course, $s \times r$ need not be an immersion, in general. For example, let H be a Lie group acting on a manifold V and let G be the transforma-

tion groupoid $H \times V$, where $s(h, a) = a$ and $r(h, a) = h(a)$ and $(g, b)(h, a) = (gh, a)$ provided $h(a) = b$. In this case it is easily seen that $r \times s : H \times V \to V \times V$ defined by $(r \times s)(h, a) = (h(a), a)$ is an immersion if and only if, for each fixed $a \in V$, the map $H \to V$ given by $h \to h(a)$ is an immersion. That is, if and only if the action of H on V is "locally free". So, for example, if H acts trivially on V then $H \times V \to V \times V$ is not an immersion.

2. The homotopy groupoid of a foliation. We let $\mathscr{R} \subseteq V \times V$ be the equivalence relation defined by the leaves, i.e., $\mathscr{R} = \{(a, b) \in V \times V \mid a$ and b are in the same leaf$\}$. It is easily observed that $\mathscr{R}$ is not, generally, a manifold. For example, let V be the open Moebius band foliated by circles where the centre circle "goes around once" but all other circles "go around twice". If we consider $\mathscr{R}$ as fibred over V by the projection onto the first coordinate $\mathscr{R} \to V$, then over each point $a \in V$ sits $\{a\} \times L$ where L is the leaf containing a. In case V is the Moebius band, each L is a circle and if L_0 denotes the centre circle then $\mathscr{R}$ is the union of two 3-manifolds in $V \times V$ whose singular intersection is the 2-manifold $L_0 \times L_0$.

Of course, in the trivial case $V = \mathbf{R}^p \times \mathbf{R}^q$ with leaves of the form $\{x\} \times \mathbf{R}^q$, we see that $\mathscr{R} = \{(x, y, x, z) \in \mathbf{R}^p \times \mathbf{R}^q \times \mathbf{R}^p \times \mathbf{R}^q\}$ and the map(s) $\mathscr{R} \to V \times \mathbf{R}^q$: $(x, y, x, z) \to (x, y, z)$ (or, $\mathscr{R} \to \mathbf{R}^q \times V$: $(x, y, x, z) \to (y, x, z)$) give $\mathscr{R}$ a natural manifold structure.

In general, one would like to "enlarge" $\mathscr{R}$ to a $(2q + p)$-dimensional manifold covering $\mathscr{R}$. One concrete way of enlarging $\mathscr{R}$ is to consider $\mathrm{Hom}(V, F)$, the groupoid of all homotopy classes of piecewise C^∞ paths which lie within a leaf.

DEFINITION. Let (V, F) be a foliated manifold. Let L be a leaf of this foliation and let γ be a piecewise C^∞ path lying in L. Let $\langle\gamma\rangle$ denote the homotopy class of γ of all paths in L with the same initial and final points as γ. We define the homotopy groupoid of the foliation (V, F) to be the groupoid, $\mathrm{Hom}(V, F)$ of all such classess, $\langle\gamma\rangle$.

We topologize $\mathrm{Hom}(V, F)$ in the following way. Let γ be a path in a leaf, γ: $a \to b$. Let (h_1, U_1), (h_2, U_2) be foliation charts at a and b respectively, with $h_1(U_1) = O_p \times O_q^1$ and $h_2(U_2) = O_p \times O_q^2$ where O_p is a connected open neighbourhood of 0 in $\mathbf{R}^p$ and O_q^1 and O_q^2 are simply connected open neighbourhoods of 0 in $\mathbf{R}^q$. Furthermore, we assume that $h_1(a) = (x_0, 0) = h_2(b)$, for some $x_0 \in O_p$, and that we have a continuous family of paths γ^x each lying in a single leaf parametrized by x in O_p so that

(i) $\gamma^{x_0} = \gamma$;

(ii) $s(\gamma^x) \in U_1$ with $h_1(s(\gamma^x)) = (x, 0)$; and

(iii) $r(\gamma^x) \in U_2$ with $h_2(r(\gamma^x)) = (x, 0)$.

To see that, given γ, we can construct such data, let $\gamma = \gamma_k \cdots \gamma_2\gamma_1$, where each γ_i lies in a foliation chart (h_i, U_i). Then $(\pi_i \circ h_{i-1}) = \phi_i \circ (\pi_1 \circ h_i)$ in a neighbourhood of $r(\gamma_{i-1}) = s(\gamma_i)$, for some local diffeomorphism ϕ_i of $\mathbf{R}^p$. Thus, by successively modifying the $\mathbf{R}^p$ coordinates for $h_2, h_3, \ldots, h_k$ we can assume that $\pi_1 \circ h_{i-1} = \pi_1 \circ h_i$ in a neighbourhood of $r(\gamma_{i-1}) = s(\gamma_i)$. One can now inductively construct the continuous family of paths $\{\gamma^x\}$ with respect to (a possibly shrunken version of) the pair of charts (h_1, U_1) and (h_k, U_k). In this setting, we define a basic neighbourhood of $\langle\gamma\rangle$ by

$$\begin{aligned} W &= W(\gamma, (h_1, U_1), (h_2, U_2), \{\gamma^x\}) \\ &= \{\langle\gamma'\rangle \in \mathrm{Hom}(V, F) \mid s(\gamma') \in U_1,\ r(\gamma') \in U_2, \\ &\qquad \text{and } \gamma' \text{ is homotopic to } (s(\gamma') \to s(\gamma^x) \xrightarrow{\gamma^x} r(\gamma^x) \to r(\gamma'))\}, \end{aligned}$$

where the arcs $s(\gamma') \to s(\gamma^x)$ and $r(\gamma^x) \to r(\gamma')$ lie in $h_1^{-1}(\{x\} \times O_q^1)$, $h_2^{-1}(\{x\} \times O_q^2)$, repectively. As O_q^1 and O_q^2 are simply connected, the neighbourhood W is in bijective correspondence with $O_p \times O_q^1 \times O_q^2$.

PROPOSITION. *The set of such neighbourhoods,* $W = W(\gamma, (h_1, U_1), (h_2, U_2), \{\gamma^x\})$ *form a basis for a topology on* $\mathrm{Hom}(V, F)$. *With the obvious coordinate maps* $W \to O_p \times O_q^1 \times O_q^2$, $\mathrm{Hom}(V, F)$ *becomes a* C^∞*-groupoid of dimension* $(2q + p)$ *with* $\mathrm{Hom}(V, F)^{(0)} = V$.

PROOF. First we observe that if $\langle\gamma'\rangle \in W = W(\gamma, (h_1, U_1), (h_2, U_2), \{\gamma^x\})$ so that (without loss of generality) $\gamma' = (s(\gamma') \to s(\gamma^x) \xrightarrow{\gamma^x} r(\gamma^x) \to r(\gamma'))$, then we can shrink U_1 and U_2 about $s(\gamma')$ and $r(\gamma')$, respectively, and modify h_1, h_2 and the family $\{\gamma^x\}$ so that

$$\langle\gamma'\rangle \in W(\gamma', (h_1', U_1'), (h_2', U_2'), \{\gamma'^x\}) \subseteq W.$$

Now, if $\langle\gamma'\rangle$ is in the intersection of two basic neighbourhoods, then by the previous observation we can assume they are of the form

$$W' = W(\gamma', (h_1', U_1'), (h_2', U_2'), \{\gamma'^x\})$$

and

$$W'' = W(\gamma'', (h_1'', U_1''), (h_2'', U_2''), \{\gamma''^x\}),$$

where $\langle\gamma'\rangle = \langle\gamma''\rangle$. Now, by shrinking the U_i''s and the U_i'''s and modifying $\{\gamma''^x\}$ and h_2'' we can assume

$$W'' = W(\gamma'', (h_1', U_1'), (h_2'', U_2''), \{\gamma''^x\})$$

so that, in particular, $s(\gamma'^x) = s(\gamma''^x)$. At the moment, we do not yet know that we have $r(\gamma'^x)$ and $r(\gamma''^x)$ in the same plaque of, say, U_2'. However,

as γ' is homotopic to γ'', we can cover the homotopy by a finite family of foliation charts and can then find a finite family $\gamma_1 = \gamma', \gamma_2, \ldots, \gamma_k = \gamma''$ of paths in this homotopy such that any pair γ_j, γ_{j+1} are both covered by the same finite sequence of foliation charts. From this, we now see that, by shrinking the U's and modifying $\{\gamma''^x\}$ we can assume $r(\gamma'^x) = r(\gamma''^x)$ and also that γ'^x is homotopic to γ''^x. Hence, we can arrange, by shrinking, a basic neighbourhood

$$W''' = W(\gamma'', (h_1''', U_1'''), (h_2''', U_2'''), \{\gamma''^x\}) \subseteq W' \cap W''.$$

Thus we have a well-defined topology on $\mathrm{Hom}(V, F)$. It is not hard to show that the coordinate maps $W \to O_p \times O_q^1 \times O_q^2$ are local homeomorphisms. For example, if $W' \subseteq W$ are basic neighbourhoods and $\langle\gamma\rangle \in W'$, then the argument above shows that we can find another basic neighbourhood W'' containing $\langle\gamma\rangle$ with $W'' \subseteq W' = W' \cap W$ so that the data defining W'' are merely restrictions of the data defining W. The image of W'' in $O_p \times O_q^1 \times O_q^2$ is then clearly open. Thus, the coordinate maps are open and similar sorts of considerations show that they are continuous.

It is fairly evident that the coordinate changes are C^∞ and so $\mathrm{Hom}(V, F)$ is a C^∞-manifold of dimension $(2q + p)$. That $\mathrm{Hom}(V, F)$ is actually a C^∞-groupoid is also straightforward.

REMARK. As in the case of the holonomy groupoid, $\mathrm{Hom}(V, F)$ is not Hausdorff, in general. However, the two phenomenon are somewhat distinct. For instance, in Winkelnkemper's example where the holonomy groupoid is not Hausdorff, V is the punctured plane, $\mathbf{C} \setminus \{0\}$ foliated in the following way.

Here the lack of Hausdorff-ness is caused by a sequence of loops with

trivial holonomy converging to a loop with nontrivial holonomy. However, by the next proposition, $\mathrm{Hom}(V, F) \cong \mathbf{R} \times V$ which is certainly Hausdorff. In order to get an example where $\mathrm{Hom}(V, F)$ is not Hausdorff we need a sequence of paths which are homotopically trivial converging to a path which is not trivial. This is easy to arrange. Let V be punctured 3-space, $\mathbf{R}^3\backslash\{0\}$, foliated by the horizontal planes and let γ be a loop around the origin in the plane $z = 0$ so that γ is not homotopically trivial. By translating γ parallel to the z axis we can get a sequence $\{\gamma_n\}$ of loops each of which is contractible in its own leaf and $\gamma_n \to \gamma$. If we let τ denote the trivial loop with the same basepoint as γ, then we can easily show that $\langle\gamma_n\rangle \to \langle\gamma\rangle$ and $\langle\gamma_n\rangle \to \langle\tau\rangle \neq \langle\gamma\rangle$ in $\mathrm{Hom}(V, F)$. In this example, however, all loops are holonomically trivial and so the holonomy groupoid is Hausdorff!

In the case of the Reeb foliation of S^3 [5], we see that $\mathrm{Hom}(V, F)$ is the holonomy groupoid. In fact, there is a sequence $\gamma_n \to \gamma$ where all the γ_n are homotopically trivial, but γ is not even holonomically trivial so that the class of γ cannot be separated from the class of the trivial path.

REMARK. The definition of this topology on $\mathrm{Hom}(V, F)$ is, of course, modelled on the usual definition of the topology on the homotopy groupoid of a manifold (without foliation). Viewed in this light, the machinations required to define the topology are very easy to understand. We note that we have not seen this topology explicitly constructed before, but it must be known to J. Pradines and others in the field. When we come to the main theorem however, we will not use the full force of this construction which is tantamount to the construction of holonomy, but we will use a trivial special case. As a corollary to the main theorem we will show that the C^∞-groupoid structure on $\mathrm{Hom}(V, F)$ is essentially unique.

Concerning the homotopy groupoid, we have two positive results which deserve to be recorded.

PROPOSITION. *Let H be a connected Lie group acting locally freely on the connected manifold V, and let $\tilde{H} \to H$ be the simply connected cover of H. Let F be the foliation of V induced by this action. Then* $\mathrm{Hom}(V, F) \cong \tilde{H} \times V$ *as C^∞-groupoids.*

PROOF. Since H acts locally freely if and only if $\tilde{H}$ acts locally freely, we can assume that H is simply connected. Let $\langle\gamma\rangle \in \mathrm{Hom}(V, F)$ and let $a = s(\gamma)$. Now the map $H \to H \cdot a$ has kernel $H_a = \{h \in H \mid h \cdot a = a\}$ and since H_a is discrete, $H/H_a \to H \circ a$ is a diffeomorphism and $H \to H \cdot a$ is a covering map. Since γ is a path in $H \cdot a$, starting at $e \cdot a = a$, there is a unique lifting to a path $\tilde{\gamma}$ in H with initial point e and so that $\gamma(t) =$

$\tilde{\gamma}(t) \cdot a$ for all t in [0, 1]. Thus, we map $\langle\gamma\rangle$ in Hom(V, F) to $(\tilde{\gamma}(1), s(\gamma)) \in H \times V$. This clearly depends only on the class $\langle\gamma\rangle$ and not on γ itself.

On the other hand, if $(h, a) \in H \times V$, then there is a unique path $\tilde{\gamma}$ (up to homotopy) starting at e and ending at h. Let $\gamma(t) = \tilde{\gamma}(t) \cdot a$. Then, $(h, a) \to \langle\gamma\rangle$ is a well-defined inverse of the previous map, and so Hom(V, F) $\to H \times V$ is a bijection. It is clearly an isomorphism of groupoids.

To see that it is a diffeomorphism we take a point $\langle\gamma\rangle$ in Hom(V, F) and construct a neighbourhood W of $\langle\gamma\rangle$ and a neighbourhood N of its image (g, a) inside the image of W and give coordinates for N and W so that the diagram

$$\begin{array}{ccc} N & \longrightarrow & W \\ & \searrow \quad \swarrow & \\ & \mathbf{R}^{2q+p} & \end{array}$$

commutes.

To this end let $\tilde{\gamma}$ be the unique path in H with initial point e such that $\tilde{\gamma}(t) \cdot a = \gamma(t)$ and let $g = \tilde{\gamma}(1)$ so that (g, a) is the image of $\langle\gamma\rangle$ in Hom(V, F). Let (h_1, U_1) be a foliation chart at a with $h_1(a) = (x_0, 0) \in \mathbf{R}^p \times \mathbf{R}^q$. Then $(h_2, U_2) = (h_1 \circ g^{-1}, g \cdot U_1)$ is a foliation chart at $b = r(\gamma)$ and $(h_1 \circ g^{-1})\ (b) = (h_1 \circ g^{-1})\ (\gamma(1)) = (h_1 \circ g^{-1})\ (g \cdot a) = h_1(a) = (x_0, 0)$ as required in the definition of the topology on Hom(V, F). Now, we suppose $h_1(U_1) = O_p \times O_q$, where O_p is open in $\mathbf{R}^p$, O_q is open and simply connected in $\mathbf{R}^q$, and $h_2(U_2) = O_p \times O_q$ as well. For $x \in O_p$, let $a^x = h_1^{-1}(x, 0)$ and define $\gamma^x(t) = \tilde{\gamma}(t) \cdot a^x$ so that $\gamma^{x_0} = \gamma$. Now $W(\gamma, (h_1, U_1), (h_2, U_2), \{\gamma^x\})$ is a basic open neighbourhood of $\langle\gamma\rangle$ in Hom(V, F). We wish to shrink U_1 and U_2 and modify h_1, h_2 (without affecting the $\mathbf{R}^p$-coordinates) to obtain the W we require. Now, let O be a simply connected neighbourhood of e in H such that $O \cdot T \subseteq O^2 \cdot T \subseteq U_1$ where $T = h_1^{-1}(O_p \times \{0\})$. Then $O \cdot T$ and $O^2 \cdot T$ are open neighbourhoods of T. Shrinking O, if necessary, we can assume that $O^2 \times T \to O^2 \cdot T$ is a diffeomorphism, so that we get coordinates for $O^2 \cdot T$ (and hence $O \cdot T$) by taking coordinates for O^2 and T separately which will not affect the $\mathbf{R}^p$-coordinates of $T \subseteq U_1$ (nor its $\mathbf{R}^q$-coordinate, namely zero, of course). Let h_1' be the modified version of h_1, and let $W = W(\gamma, (h_1', O \cdot T), (h_1' \circ g^{-1}, gO^2 \cdot T), \{\gamma^x\})$, a neighbourhood of $\langle\gamma\rangle$. Let $N = (g \cdot O) \times (O \cdot T)$ which is a neighbourhood of (g, a) in $H \times V$. To see that N gets mapped into W, let $(g', a') \in N = (g \cdot O) \times (O \cdot T)$ so that $a' = g''a^x$ and $g' = g \cdot g'''$, for some $g'', g''' \in O$ and some x in O_p. Let $\tilde{\gamma}''$ denote the path

$$\underbrace{g'' \longrightarrow e}_{\text{in } O} \xrightarrow{\tilde{\gamma}} \underbrace{g \longrightarrow gg''' \ g''}_{\text{in } gO^2}$$

in H and let $\tilde{\gamma}'(t) = \tilde{\gamma}''(t)(g'')^{-1}$ so that $\tilde{\gamma}' : e \to gg''' = g'$ in H. If we define

$\gamma'(t) = \tilde{\gamma}'(t)a'$, then $\gamma' : a' \to g'a'$. Since $\gamma'(t) = \tilde{\gamma}''(t)(g'')^{-1}a' = \tilde{\gamma}''(t) \cdot a^x$ we see that γ' is given by

$$a' = \underbrace{g'' \cdot a^x \longrightarrow e \cdot a^x}_{\text{in } O \cdot T} \xrightarrow{\tilde{\gamma} a^x} \underbrace{g \cdot a^x \longrightarrow gg''' g'' a^x}_{\text{in } gO^2 \cdot T} = g'a'.$$

This is just

$$\underbrace{a' \longrightarrow a^x}_{\text{in } O \cdot T} \xrightarrow{\gamma^x} \underbrace{r(\gamma^x) \longrightarrow g'a'}_{\text{in } gO^2 \cdot T}$$

so that $\langle \gamma' \rangle \in W$ as desired. If we have chosen coordinates for O^2 (that is, $\mathbf{R}^q$-coordinates) as above, then we have $\mathbf{R}^q$ coordinates for 0, $g \cdot O^2$, $g \cdot O$, $O \cdot T$, O^2T and $gO^2 \cdot T$. Then, clearly, we have coordinates for N so that

$$\begin{array}{ccc} N & \longrightarrow & W \\ & \searrow \quad \swarrow & \\ & \mathbf{R}^{2q+p} & \end{array}$$

commutes.

EXAMPLE. Bundles with discrete structure group [5]. Let X and F be connected C^∞ manifolds and let $p : \tilde{X} \to X$ be the universal cover of X. Let $K = \pi_1(X)$ so that $X = \tilde{X}/K$ and suppose K acts as diffeomorphisms of F. Then K acts on $F \times \tilde{X}$ discretely, and $V = F \times \tilde{X}/K$ is a C^∞-manifold. The product foliation of $F \times \tilde{X}$ with leaves $\{f\} \times \tilde{X}$ passes to a foliation of V since the action of K maps leaves to leaves. Moreover, the projection $F \times \tilde{X} \to X$ induces a map $q : V \to X$ which makes V into a fibre bundle over X with fibre F and structure group K. The fibres are transverse to the foliation and the restriction of q to a leaf is a covering of X.

In this example, one can show that the homotopy groupoid is diffeomorphic to $(F \times \tilde{X} \times \tilde{X})/K$ which is a Hausdorff manifold. This example was pointed out to me by Bill Phillips. In case X is a Lie group, this foliation is induced by the locally free action of $\tilde{X}$ on $V = (F \times \tilde{X})/K : g \cdot (f, h)K = (f, gh)K$ where $g, h \in \tilde{X}$ and $f \in F$. Thus, the homotopy group oid is also diffeomorphic to $V \times \tilde{X} = (F \times X)/(K \times \tilde{X})$. To realize the diffeomorphism $(F \times \tilde{X} \times \tilde{X})/K = (F \times \tilde{X})/K \times \tilde{X}$ concretely, we just map $(f, g, h)K \to ((f, g)K, g^{-1}h)$ for $f \in F$ and $g, h \in \tilde{X}$.

REMARK. It is curious that $\mathrm{Hom}(V, F)$ can be non-Hausdorff when the foliation is trivial (i.e., given by a submersion) but that it is always Hausdorff in the two (most?) important classes of examples: locally free actions of Lie groups and bundles with discrete structure group.

3. The Main Theorem. Before embarking on the main theorem we outline the proof of a result which is implict, perhaps, in [9] but which deserves more attention as it clarifies the notion of holonomy and shows that homotopy need not be included in the definition of holonomy. Recall that if $a \in V$, a foliated manifold, then D_a denotes the set of germs of distinguished functions at a.

PROPOSITION. *There is a unique family of maps $H_\gamma: D_a \to D_b$, one for each piecewise C^∞ path $\gamma: a \to b$ lying within a leaf, with the following two properties:*

1. $H_{\gamma_2\gamma_1} = H_{\gamma_2} \circ H_{\gamma_1}$ *if* $r(\gamma_1) = s(\gamma_2)$,
2. $H_\gamma([f]_a) = [f]_b$ *if* $\gamma \subseteq \operatorname{dom} f$. *This family automatically satisfies:*
3. $H_\gamma = H_{\gamma'}$ *if γ and γ' are homotopic*;
4. $H_\gamma([\phi \circ f]_a) = [\phi]_{f(a)} H_\gamma([f]_a)$ *if ϕ is a local diffeomorphism of $\mathbf{R}^p$ in a neighbourhood of $f(a)$.*

PROOF. We show uniqueness by giving a procedure to calculate $H_\gamma([f]_a)$. Let $f_1 = f$ and let $\gamma = \gamma_k \cdots \gamma_2\gamma_1$, where each γ_i lies in $\operatorname{dom} f_i$ for some distinguished functions f_i. Then, $f_{i-1} = \phi_i \circ f_i$ in a neighbourhood of $r(\gamma_{i-1}) = s(\gamma_i)$, for some local diffeomorphism ϕ_i of $\mathbf{R}^p$. Repeated applications of 1 and 2 yield $H_\gamma([f]_a) = [\phi_2 \circ \phi_3 \circ \cdots \circ \phi_k \circ f_k]_b$, and so the family is unique.

In order to see that this procedure is well-defined, a standard sort of refinement argument allows us to restrict to two cases: (a) subdividing γ at one new point while retaining the same distinguished function for the two new segments of γ; and (b) leaving the subdivision of γ alone, but changing one distinguished function, other than $f_1 = f$, of course. It is easy to verify that the calculated value $H_\gamma([f]_a)$ does not change by an application of (a) or (b) and so the family H_γ is well-defined by this procedure.

To see property 4, we apply H_γ to $\phi \circ f$ and get

$$\begin{aligned} H_\gamma([\phi \circ f]_a) &= [(\phi \circ \phi_2) \circ \phi_3 \circ \cdots \circ \phi_k \circ f_k]_b \\ &= [\phi]_{f(a)}[\phi_2 \circ \cdots \circ \phi_k \circ f_k]_b = [\phi]_{f(a)} H_\gamma([f]_a), \end{aligned}$$

as required.

To see homotopy invariance, it suffices to see that $H_\gamma = H_{\gamma'}$, if γ and γ' are homotopic and very close. In this case, we can cover both γ and γ' by the domains of the same sequence of distinguished functions $\{f_1, \ldots, f_k\}$, and so $H_\gamma([f]_a) = [\phi_2 \circ \cdots \circ \phi_k \circ f_k]_b = H_{\gamma'}([f]_a)$.

DEFINITION. Let (V, F) be a foliation. We define $\operatorname{Hol}(V, F)$ to be the groupoid of equivalence classes of paths in $\operatorname{Hom}(V, F)$, where $\gamma_1 \sim \gamma_2$ if and only if $H_{\gamma_1} = H_{\gamma_2}$. Clearly, $\operatorname{Hom}(V, F) \twoheadrightarrow \operatorname{Hol}(V, F)$ is a surjective

groupoid morphism. A neighbourhood of $[\gamma] \in \mathrm{Hol}(V, F)$ is obtained by fixing a distinguished function f in a neighbourhood U of $a = s(\gamma)$ and a second distinguished function f' in a neighbourhood U' of $b = r(\gamma)$ and such that $H_\gamma([f]_a) = [f']_b$. We then define $W(f, f') = \{[\gamma'] \mid s(\gamma') \in \mathrm{dom}\, f, r(\gamma') \in \mathrm{dom}\, f'$ and $H_{\gamma'}([f]_{s(\gamma')}) = [f']_{r(\gamma')}\}$. If $f = \pi_1 \circ h$ and $f' = \pi_1 \circ h'$ for foliation charts (h, U), (h', U'), then

$$\begin{aligned}\Phi([\gamma']) &= (f(s(\gamma')), \pi_2 \circ h(s(\gamma')), \pi_2 \circ h'(r(\gamma'))) \\ &= (f'(r(\gamma')), \pi_2 \circ h(s(\gamma')), \pi_2 \circ h'(r(\gamma')))\end{aligned}$$

defines $\mathbf{R}^{2q+p}$-coordinates in a neighbourhood of $[\gamma]$. With these coordinates, $\mathrm{Hol}(V, F)$ becomes a C^∞-groupoid and $\mathrm{Hom}(V, F) \twoheadrightarrow \mathrm{Hol}(V, F)$ is a C^∞ map.

DEFINITION. If (h, U) is a foliation chart of V with $h(U) = O_p \times O_q$, where O_q is simply connected in $\mathbf{R}^q$, then the set $N = \{\langle\gamma\rangle \in \mathrm{Hom}(V, F) \mid \gamma \subseteq U\}$ is called a trivial neighbourhood in $\mathrm{Hom}(V, F)$. Clearly, $N \cong O_p \times O_q \times O_q$, and such neighbourhoods form an open cover of $V \to \mathrm{Hom}(V, F)$.

THEOREM. *Let F be a codimension p foliation of the connected $(p + q)$-dimensional C^∞-manifold, V, and let $\mathscr{R} \subseteq V \times V$ denote the equivalence relation determined by the leaves. Let G be a C^∞-groupoid of dimension $(2q + p)$ such that $G^{(0)} = V$ and $s \times r : G \twoheadrightarrow \mathscr{R} \subseteq V \times V$ is surjective and C^∞. If there is a groupoid morphism $\mathrm{Hom}(V, F) \twoheadrightarrow G$ which is C^∞ on trivial neighbourhoods and so that*

$$\begin{array}{ccc} \mathrm{Hom}(V, F) & \twoheadrightarrow & G \\ & \searrow \quad \swarrow & \\ & \mathscr{R} & \end{array}$$

commutes, then there is a unique morphism $G \twoheadrightarrow \mathrm{Hol}(V, F)$ so that

$$\begin{array}{ccc} \mathrm{Hom}(V, F) & \twoheadrightarrow & G \\ & \searrow \quad \swarrow & \\ & \mathrm{Hol}(V, F) & \end{array}$$

commutes. Moreover,

$$\begin{array}{ccc} G & \twoheadrightarrow & \mathrm{Hol}(V, F) \\ & \searrow \quad \swarrow & \\ & \mathscr{R} & \end{array}$$

commutes also, and if $G \twoheadrightarrow \mathscr{R} \subseteq V \times V$ is an immersion, then $G \twoheadrightarrow \mathrm{Hol}(V, F)$ is a local diffeomorphism.

PROOF. We take the notationally convenient point of view that elements of G are just paths lying in a leaf modulo some equivalence relation. Our job is to show that this relation is at least as fine as holonomy.

Before we begin the main part of the proof, we recall that the maps r and $s : G \to V$ are submersions; this follows easily from the axioms for a C^∞-groupoid. Now, let γ be in G and let (h, U) be a foliation chart in a neighbourhood of $a = s(\gamma)$. Since r is a submersion, we can find an open neighbourhood O of γ in G so that $r : O \to r(O) \subseteq V$ is a product projection; that is, O is diffeomorphic to $N \times r(O)$ and

$$\begin{array}{ccc} O & \cong & N \times r(O) \\ & {\scriptstyle r}\searrow \quad \swarrow & \\ & r(O) & \end{array}$$

commutes. By shrinking, if necessary, we can assume that O (and hence $r(O)$ and N) are connected and that $s(O) = U$. Now, let $\rho: r(O) \to O$ be any C^∞ section for r so that $\pi_1 \circ h \circ s \circ \rho: r(O) \to \mathbf{R}^p$ is C^∞ and well-defined. To see that this map does not depend on ρ, we observe that, for $b' \in r(O)$, $r^{-1}(b')$ is connected. Then $s(r^{-1}(b'))$ is connected and lies in a single leaf and so in a plaque for (h, U). Thus, $\pi_1 \circ h \circ s(r^{-1}(b'))$ is a singleton and $\pi_1 \circ h \circ s \circ \rho$ does not depend on the choice of ρ. Another way of saying this is that the following diagram commutes:

$$\begin{array}{ccc} 0 & \xrightarrow{\pi_1 \circ h \circ s} & \mathbf{R}^p \\ \downarrow & \nearrow_{\pi_1 \circ h \circ s \circ \rho} & \\ r(0) & & \end{array}$$

Since r and $\pi_1 \circ h \circ s$ are submersions, we see that $\pi_1 \circ h \circ s \circ \rho$ is also a submersion. To see that $\pi_1 \circ h \circ s \circ \rho$ is a distinguished function, we only need show that it is constant on plaques. But, if P is a plaque in $r(O)$, then $r^{-1}(P)$ is connected in O, and $s(r^{-1}(P))$ is connected and lies in a single leaf and so in a plaque for (h, U). Thus, $\pi_1 \circ h \circ s(r^{-1}(P))$ is a singleton, as required. In summary, we have found a neighbourhood O of γ in G so that, for $\gamma' \in O$, the map $r(\gamma') \to \pi_1 \circ h \circ s(\gamma') : r(O) \to \mathbf{R}^p$ is a distinguished function in a neighbourhood of $b = r(\gamma)$ in V. By its very form, the germ of this distinguished function at b depends only on γ and the germ of $\pi_1 \circ h$ at a. We suggestively denote the germ of this distinguished function at b by $H'_\gamma([\pi_1 \circ h]_a)$. To see that $H'_{\gamma_2\gamma_1} = H'_{\gamma_2} \circ H'_{\gamma_1}$, let $\gamma = \gamma_2\gamma_1$: $a \to b$, where γ_1: $a \to b_1$ and $\gamma_2 : b_1 \to b$. Let O, O_1, O_2 be neighbourhoods of γ, γ_1, γ_2 as in the construction above and assume that $O_2 \cdot O_1 \subseteq O$. Now, $b_1 \in r(O_1) \cap s(O_2)$ is open in V, and so, by shrinking O_1 and O_2, we can assume that $r(O_1) = s(O_2)$. Then, $r(O_2 \cdot O_1) = r(O_2)$ is an open neighbourhood of b in V. Now, let f_1 be any distinguished function in a neighbourhood of a and let f_2 be the distinguished function in a neighbourhood of

b_1 given by $r(\gamma_1') \to f_1(s(\gamma_1'))$; that is, $f_2(r(\gamma_1')) = f_1(s(\gamma_1'))$, and so $H'_{\gamma_1}([f_1]_a) = [f_2]_{b_1}$. Then, let f_3 be the distinguished function in a neighbourhood of b given by $r(\gamma_2') \to f_2(s(\gamma_2'))$, that is, $f_3(r(\gamma_2')) = f_2(s(\gamma_2'))$, and so $H'_{r_2}([f_2]_{b_1}) = [f_3]_b$. Now, for $\gamma' = \gamma_2'\gamma_1' \in O_2 \cdot O_1 \subseteq O$, we have

$$\begin{aligned} f_3(r(\gamma')) &= f_3(r(\gamma_2')) = f_2(s(\gamma_2')) = f_2(r(\gamma_1')) \\ &= f_1(s(\gamma_1')) = f_1\,(s(\gamma')), \end{aligned}$$

and since $r(0_2 \cdot 0_1)$ is an open neighbourhood of b, we see that

$$H'_\gamma([f_1]_a) = [f_3]_b = H'_{\gamma_2}([f_2]_{b_1}) = H'_{\gamma_2}(H'_{\gamma_1}([f_1]_a))$$

as required.

Now, if γ lies in a foliation chart for V, (h, U) with $h(U) = O_p \times O_q$, where O_q is simply connected in $\mathbf{R}^q$, then $\langle\gamma\rangle$ lies in a trivial neighborhood N of $\mathrm{Hom}(V, F)$. Since $N \to \mathscr{R}$ is a diffeomorphism onto a submanifold of $V \times V$, we see that $\tilde{N}$, the image of N in G, must also be diffeomorphic to this submanifold of $V \times V$ since

$$\begin{array}{ccc} N & \longrightarrow & \tilde{N} \\ & \searrow \quad \swarrow & \\ & \mathscr{R} & \end{array}$$

commutes. Also, since $N \to \tilde{N}$ is bijective, C^∞ and full rank, $\tilde{N}$ is open in G. Hence, to compute $H'_\gamma([\pi_1 \circ h]_a)$ using the open set $\tilde{N}$, we could equally use N as

$$\begin{array}{ccc} N & \longrightarrow & \tilde{N} \\ & \searrow \quad \swarrow & \\ & \mathscr{R} & \end{array}$$

commutes. But then $f_2(r(\gamma')) = \pi_1 \circ h(s(\gamma')) = \pi_1 \circ h(r(\gamma'))$, and so $H'_\gamma([\pi_1 \circ h]_a) = [\pi_1 \circ h]_b$.

Therefore, by the previous proposition, $H'_\gamma = H_\gamma$, for all γ, and so the map $\gamma \to \langle\gamma\rangle : G \twoheadrightarrow \mathrm{Hol}(V, F)$ is well-defined. It is clearly a groupoid morphism and

$$\begin{array}{ccc} G & \twoheadrightarrow & \mathrm{Hol}(V, F) \\ & \searrow \quad \swarrow & \\ & \mathscr{R} & \end{array}$$

commutes. Moreover, it is clear that

$$\begin{array}{ccc} \mathrm{Hom}(V, F) & \twoheadrightarrow & G \\ & \searrow \quad \swarrow & \\ & \mathrm{Hol}(V, F) & \end{array}$$

commutes and so we must have that $G \twoheadrightarrow \mathrm{Hol}(V, F)$ is unique.

If $G \twoheadrightarrow \mathscr{R} \subseteq V \times V$ is an immersion, then given γ in G we can find a neighbourhood O of γ in G so that O', the image of O in $V \times V$, is a submanifold of $V \times V$ and $O \to O'$ is a diffeomorphism. By shrinking, if necessary, we can also assume that $\tilde{O}$, a neighbourhood of $\langle\gamma\rangle$ in $\mathrm{Hol}(V, F)$, is also diffeomorphic to O' via $\tilde{O} \to O'$. Since $O \to \tilde{O}$ is bijective and

$$\begin{array}{ccc} O & \longrightarrow & \tilde{O} \\ & \searrow \quad \swarrow & \\ & O' & \end{array}$$

commutes, we see that $O \to \tilde{O}$ is a diffeomorphism.

COROLLARY. *There is a unique C^∞-manifold structure on* $\mathrm{Hom}(V, F)$ *making it into a C^∞-groupoid over V which has the obvious structure on trivial neighborhoods and so that* $\mathrm{Hom}(V, F) \twoheadrightarrow \mathscr{R} \subseteq V \times V$ *is an immersion.*

PROOF. If we had a second such structure on $\mathrm{Hom}(V, F)$, then, letting $G = \mathrm{Hom}(V, F)$ with this structure, we would get

$$\begin{array}{ccc} \mathrm{Hom}(V, F) & \xrightarrow{\mathrm{id}} & \mathrm{Hom}(V, F) \\ & \searrow \quad \swarrow & \\ & \mathrm{Hol}(V, F) & \end{array}$$

commuting, so the two vertical maps would be the same and both would be local diffeomorphisms. Thus, the identity map on $\mathrm{Hom}(V, F)$ would be a diffeomorphism between the two structures.

COROLLARY. *Suppose the connected Lie group H acts locally freely on the connected manifold V and let (V, F) be the induced foliation. Then there is a local diffeomorphism $H \times V \twoheadrightarrow \mathrm{Hol}(V, F)$ which is a diffeomorphism if H acts freely.*

PROOF. Since $\mathrm{Hom}(V, F) \cong \tilde{H} \times V$, where $\tilde{H}$ is the simply connected cover of H, we have that

$$\begin{array}{ccc} \tilde{H} \times V & \to & H \times V \\ & \searrow \quad \swarrow & \\ & \mathscr{R} & \end{array}$$

commutes. Since $H \times V \twoheadrightarrow \mathscr{R} \subseteq V \times V$ is an immersion, the theorem applies. If H acts freely, then $H \times V \twoheadrightarrow \mathscr{R}$ is a bijection, and so $H \times V \twoheadrightarrow \mathrm{Hol}(V, F)$ is also a bijection.

REMARK. Just as in the case of the holonomy groupoid [1, 2], we can associate a normed *-algebra with $\mathrm{Hom}(V, F)$ which we can complete

in either the universal C^*-norm or the "reduced" C^*-norm. In case the foliation is given by a locally free action of a simply connected Lie group H on V we obtain the crossed product $C_0(V) \times H$ and the reduced crossed product $C_0(V) \times_{\text{red}} H$ respectively. Hence, both these algebras are anonically determined by the foliation itself.

Concluding Remarks. The main theorem gives two justifications for the term "holonomic imperative". The first and most obvious explanation is that the holonomy groupoid is the "smallest manifold" which removes the singularities of the equivalence relation. The second, less obvious, explanation is that the homotopy groupoid, which is certainly a natural object to study, has a unique C^∞-groupoid structure and the construction of this differentiable structure uses holonomy in an essential way.

References

1. A Connes, *Sur la théorie non-commutative de l'intégration*, Lecture Notes in Math. **725**, Springer-Verlag, Berlin and New York, 19–143.

2. ———, *A survey of foliations and operator algebras*, Proceedings of the A.M.S. Symposium on Operator Algebras and their Applications, vol. **38**, part I, 1980.

3. C. Ehresmann, *Structures feuilletées*, Proc. 5th Canadian Math. Congress, (1961), 109–172, Univ. of Toronto Press, 1963.

4. ———, *Sur les catégories différentiables*, Atti Conv. Int. Geom. Diff., Bologna (1967), 31–40.

5. A. Haefliger, *Variétés feuilletées*, Ann. Scuola Norm. Sup. Pisa, (3) **16** (1962), 367–397.

6. J. Pradines, *Théorie de Lie pour les groupoides différentiables*, Relations entre propriétés locales et globales, C.R.A.S., Paris **263** série A (1966), 907–910.

7. ———, *Feuilletages: Holonomie et graphes locaux*, C.R.A.S., Paris, **298** série I (13) (1984), 297–300.

8. ———, and B. Bigonnet, *Graphe d'un feuilletage singulier*, C.R.A.S., Paris **300** série I (13) (1985), 439–442.

9. H. E. Winkelnkemper, *The graph of a foliation*, preprint, to appear.

Department of Mathematics Dalhousie University, Halifax, N.S. B3H 4H8 CANADA