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TANGENT BALL EMBEDDINGS OF SETS IN E^3

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R. H. Bing [3] and M. K. Fort, Jr. [18] asked if a 2-sphere in E^3 would be tamely embedded in E^3 if it were known to have double tangent balls on opposite sides at each of its points. A 2-sphere Σ in E^3 is said to have double tangent balls at a point p of Σ if there exist two round 3-cells B and B' such that $B \cap B' = \{p\} = \Sigma \cap (B \cup B')$. The balls B and B' are on opposite sides of Σ if they lie, except for p, in different components of $E^3 - \Sigma$. Appearing shortly after publication of Bing's 1-ULC characterization of tame surfaces [1] and about the same time as his Side Approximation Theorem [2] the question, whose answer depended upon these famous theorems, led to many related embedding facts. In this note I will summarize the evolution of "tangent ball theory" as it developed from the Bing/Fort Question. The account is historical with no proofs given, but some of the facts related here have not appeared elsewhere.

In both [3] and [18] the double tangent ball question arose in the context of piercing wild spheres with geometrically nice objects. Since Bing [3] and Fort [18] produced examples of wild 2-spheres that could be pierced with segments, it was natural to ask whether more general geometrical piercing sets such as balls and cones would be sufficient to insure the tameness of a surface. It is doubtful that either researcher was trying to produce a geometric analogue to smoothness as mildly suggested in [15] and [16]. Still it seems appropriate to give some examples to show there is no direct relation between the existence of double tangent balls and the surface having continuously differentiable defining functions. Let $f(x)$ be a function much like $x^2 \sin 1/x$ except adjusted slightly, if necessary, so that its graph in E^2 has double tangent disks at every point. The surface Σ_1, obtained by rotating this graph about the z-axis, is not continuously differentiable at the origin, yet Σ_1 has double tangent balls at each of its points. On the other hand let Σ_2 be the surface obtained by rotating the graph of $g(x) = x^{3/2}(0 \leqq x \leqq 1)$ about the z-axis. This surface has continuously differentiable coordinate functions but, with some effort, one can show there do not exist double tangent balls to Σ_2 at the origin. However a surface will have double tangent balls if it has sufficiently smooth coordinate functions [27].

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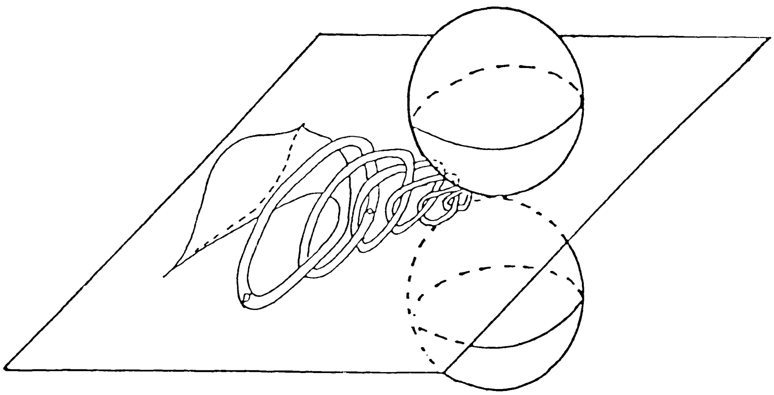
Even though double tangent ball surfaces need not be smooth, they have tangent planes at each point. Each tangent ball pair defines a unique plane tangent to the two balls at their common point. This plane is also tangent to the surface, in a loose way, so surfaces with double tangent balls also have tangent planes. However, the existence of a family of these tangent planes is not enough to insure the flatness of the surface; the Fox-Artin wild sphere can be constructed with such a family. In this connection one should mention Question 12 of the Scottish Book (see [**8**]) which is attributed to Banach. Apparently realizing in approximately 1935 that there were wild 2-spheres in E^3 with such a tangent plane at each point, he added continuity to the family of tangent planes and asked if this ruled out wildness. Burgess became aware of this question during the conference in Denton, Texas, in 1979 and answered it negatively by showing that the Fox-Artin wild 2-sphere can also be adjusted so that it has a continuous family of tangent planes [**8**]. Such an embedding is illustrated in Figure 1 where the Fox-Artin sphere is tangent to the pictured plane at its Wild point which is the point of intersection of the two round balls. By relating families of tangent planes to families of double cones, I [**24**] generalized some of Burgess' examples and gave one with continuous tangent planes where the wild set contained a simple closed curve. In such examples the wild set cannot be connected unless it is a single point [**23**]; in fact, a continuum in the wild set must be a limiting set of point components of the wild set. However, these tangent plane examples and theorems were mostly generated at least ten years after the Bing/Fort tangent ball question was resolved.

The first progress on the tangent ball question was Griffith's [**20**] partial



FIGUGE. 1.

answer published in 1968. He proved that a 2-sphere Σ in E^3 is tamely embedded if there is a positive number δ such that, for each point p of Σ, there exist double tangent balls each of radius δ lying on opposite sides of Σ and touching only at p. Bing's Question was stated in [3] immediately after he had mentioned John Hempel's theorem on piercing a surface with a continuous family of segments. In this context the additional "uniform" or "continuous" hypothesis on the double tangent balls was a natural one. Griffith's technique was to show Σ was locally spanned in both complementary domains because Burgess [7] showed this implied Bing's 1-ULC condition. The dependence upon the 1-ULC characterization of flatness at first restricted generalization of Griffith's Theorem to $(n-1)$-spheres in E^n with $n \neq 4$; however, this dependence was later removed [15], so Griffith's Theorem is known for all n.

Working independently, H. G. Bothe [5] and I [21] resolved the Bing/Fort question by proving that a 2-sphere in E^3 must be tamely embedded when it has double tangent balls on opposite sides. Bothe's theorem was a little more general because he used double cones with sufficiently large cone angle in place of the double tangent balls. However my paper perpetuated the subject by stating several related questions in addition to the implicit one about generalizations to higher dimensions. Both proofs depended upon Bing's 1-ULC characterization of tameness.

The first question in [21] asked if a 2-sphere Σ would be tame from its exterior if, for each $p \in \Sigma$, there exists a round ball B_p such that $B_p \cap \Sigma = \{p\}$ and $B_p - \{p\} \subset \operatorname{Int} \Sigma$. (The ball B_p is called a tangent ball even though its existence does not imply that Σ has a tangent plane at p.) At about this time and perhaps during the year or two preceding my write-up, J. W. Cannon was developing his *-taming set theory [10], [11]. An easy consequence of his powerful theorems was an affirmative answer to my question and to Bing's question about the tameness of a 2-sphere that is touched at each of its points by two cones, one on each side of Σ. (See [3] and [4] for Bing's question on cones and [11] for Cannon's answer.)

It was clear that the existence of tangent balls on the interior of Σ would not make $\Sigma \cup \operatorname{Int} \Sigma$ a 3-cell, because, for example, the Fox-Artin wild sphere has these tangent balls from its wild side. Thus my second question in [21] asked if Σ would be tame from its interior if it had uniform tangent balls from its interior; that is, if there exists a $\delta > 0$ such that for each point p in Σ there exists a round ball B_p of radius δ such that $B_p - \{p\} \subset \operatorname{Int} \Sigma$ and $p \in B_p$. Although J. W. Cannon resolved three of the four questions that were asked in [21], see [11] and [12, §5], this one remained open for about ten years, until Bob Daverman and I [16] answered it in the affirmative in 1979. We felt the answer would be found once we decided whether or not the Fox-Artin sphere could be arranged to have these uniform tangent balls on its wild side. In fact our first success was to use the main result of [22] to reduce the wildness of spheres with uniform interior tangent balls to a finite set. Convinced that the Fox-Artin sphere could not have these uniform balls we then established the result through some long and detailed lemmas mostly involving plane geometry. It turned out that there was no need to require that all the uniform balls tangent to Σ be found in Int Σ [16, Theorem 2.14], so this result can be viewed as a substantial generalization of Griffith's theorem on uniform-sized double tangent balls where, instead of two tangent balls at each point, only one is required and it need not lie on a specified side of Σ.

Very recently I showed that there are not many regular solids that can substitute for the round ball in the Daverman-Loveland theorem. Perhaps next to the solid sphere in symmetry and beauty are the five convex regular polyhedral solids, the ones Plato regarded as symbolizing the four elements earth, fire, air, and water together with the dodecahedron which to Plato symbolized in some way the entire universe [13, p. 149]. None of the first four of these Platonic solids tames a crumpled cube in the same sense as the solid round sphere. To see this one first embeds the Fox-Artin arc on the pages of a 3-page book as described in [22], then, after rotating the pages to be exactly 120° apart (see Figure 2), one "indents" the book along the arc with the tip of a pencil to convert the arc to the Fox-Artin 2-sphere. This indenting or etching process is illustrated in Figure 3. It turns out that the resulting wild sphere can be touched at each of its points

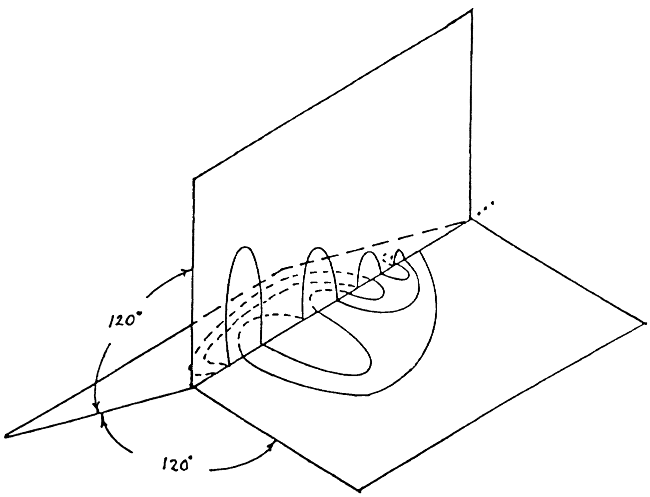


FIGURE. 2.

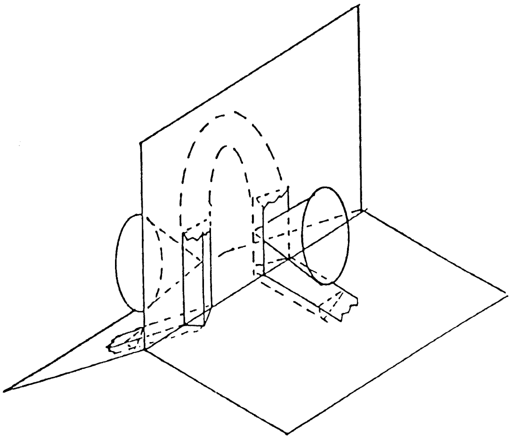


FIGURE 3.

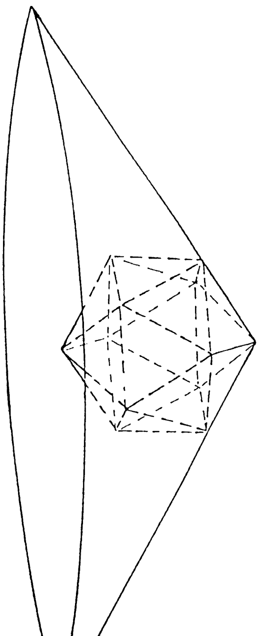


FIGURE 4.

by the vertex of a cone (the pencil) that lies, except for its vertex, in the exterior of the sphere. The construction can be carried out as long as the cone angle is less than 120°. By calculating angles in the five Platonic solids one finds that each of the first four, the tetrahedron, cube, octahedron, and the icosahedron, lies in a cone whose cone angle is less than 120° with a vertex of the solid at the vertex of the cone. The icosahedron is pictured in such a cone in Figure 4. Thus the Fox-Artin 2-sphere described in this way, can be touched at each of its points by the vertex of a Platonic solid from a family of congruent solids each of which is either a cube, tetrahedron, octahedron, or an icosahedron. The dodecahedron is naturally more troublesome since it "envelops the whole universe" **[13]**. It cannot be constructed to lie in a cone with cone angle 120° (it requires almost 139°), in fact, one finds the angle between one edge and its non-adjacent face, the one with which the edge shares a vertex, is more than 121°. This prevents the same use of the dodecahedron as the uniform etching or indenting tool in place of the cone or pencil in the reconstruction of the Fox-Artin sphere from the 120° three-page book. These examples are meant to point out the sharpness of the Daverman-Loveland theorem.

Extending vivid descriptions of an earlier theorem given by Benny Rushing, I like to call this result the Rattle Theorem. In this sense the crumpled cube is the baby's rattle and the rattler inside is a marble or ball. If the rattler touches every point of the boundary of the rattle when it is shaken, then the rattle is a 3-cell. Except for rattlers which themselves are touched from their interiors by marbles, the marble is the only rattler I know that tames the rattle. A natural guess is that a sufficiently fine polyhedral approximation to a round ball would work just as well in taming a rattle. Working in this direction I **[26]** proved that a 2-sphere must be locally tame except possibly at finite number of points if it is touched from its interior at each of its points by the vertex of a cone from a family of congruent cones with sufficiently large cone angle. I suspect the same conclusion is true when the dodecahedron is used as the rattler. Of course Cannon's work **[11]** implies the rattle would be a tame 3-cell if, at each of its points, its boundary is touched from each side by a dodecahedron or by any polyhedral 3-cell.

The higher dimensional analogue of the Bing/Fort Question led to results that were surprising to me. In 1978 Daverman and I **[15]** constructed, for each $n \geqq 4$, an $(n-1)$-sphere Σ in E^n such that Σ was locally polyhedral modulo a Cantor set X, Σ was wild at each point of X, and Σ had uniform-sized double tangent balls on opposite sides of Σ at each point of X. Later David Wright and I **[27]** showed how to construct a similar example where $\Sigma - X$ was "smoothed out" enough so that Σ had double tangent balls (but certainly not uniform-sized ones) on opposite

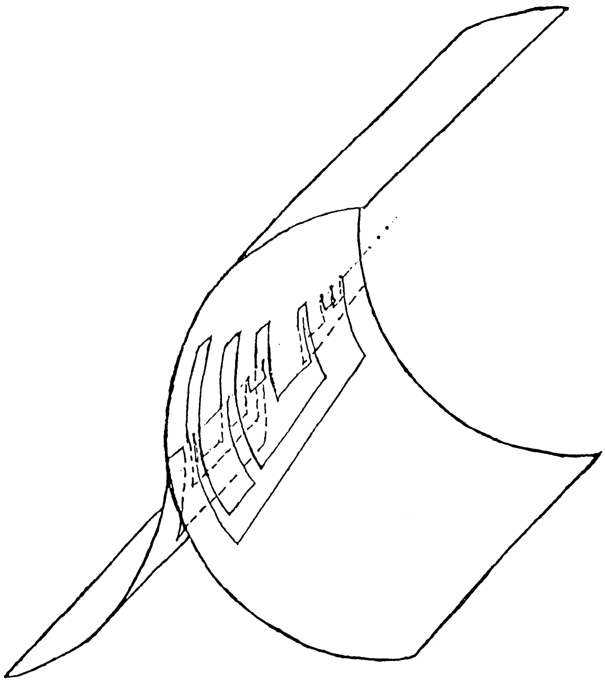


FIGURE 5

sides at each of its points. Thus the Bing/Fort Question has an affirmative answer in E^3 and a negative answer in E^n for $n \geqq 4$.

Using the inflation technque developed by Daverman [**14**], he and I [**16**] constructed an example of a 3-sphere Σ in E^4 such that Σ was wildly embedded in E^4, yet Σ had uniform-sized exterior tangent balls at every point. However the wildness of Σ was from Int Σ so the question of generalizing the Daverman-Loveland Rattle Theorem to higher dimensions remains unanswered.

With the Bing/Fort Question and the one on uniform one-sided tangent balls settled, I extended my focus from 2-spheres to arbitrary subsets of E^3. The Fox-Artin arc can be embedded so as to have uniform (single) tangent balls at each of its points (see Figure 5) or to have double (nonuniform) tangent balls everywhere [**23**], so the natural condition to impose on arbitrary subsets to make them tamely embedded was the generalization of Griffith's uniform double tangent balls. In [**23**] I showed that an arbitrary subset X of E^3 must locally lie on a flat 2-sphere if there is a positive number δ such that for each $p \in X$ there are two three-dimensional balls B and B', each with radius δ, such that $\{p\} = B \cap B' = (B \cup$

$B') \cap X$. Notice that when X is taken to be a 2-sphere this slightly generalizes Griffith's Theorem [**20**] because the double tangent balls are not required to lie on opposite sides of the 2-sphere. Burgess and I [**9**] studied 2-spheres having these "indiscrete" (but not uniform) double tangent balls, giving wild examples and finally proving that the wild set W of a 2-sphere Σ having indiscrete double tangent balls over W cannot be connected unless it is a single point. The proof of this theorem [**9**, Corollary 3.4] depended upon Theorem 3.1 of [**22**]. Strictly interpreted (in a way that was not intended), the question credited to Bing in 1965 [**4**, Question 3c] has a negative answer because there was no stated requirement that the tangent balls be on opposite sides of the sphere. In a sense the Burgess-Loveland paper [**9**] gave limits on the examples supporting a negative answer to this question.

Tangent ball theorems are closely related to questions about ξ-boundaries. If X is a subset of a metric space (Y, d) and $\xi > 0$, the ξ-boundary, $\partial(\xi, X)$, of X in Y is $\{y \in Y \mid d(X, y) = \xi\}$. Many authors, including Brown [**6**] and Ferry [**17**], have studied ξ-boundaries. Ferry showed [**17**] that the ξ-boundary of a subset X of E^3 is almost always a 2-manifold. When $\partial(\xi, X)$ is a 2-manifold one can easily see that it would have one-sided tangent balls of radius ξ, so it would be tamely embedded in E^3 [**16**, Theorem 4.1]. This answered a question raised by Weill [**28**]. Curious about an arc when it is realized as the ξ-boundary of some subset of E^3, I showed that a 1-dimensional subset of E^3 must also locally lie on a flat 2-sphere when it is the ξ-boundary of another set [**25**]. As far as I know, the analogous ξ-boundary results in dimensions greater than 3 have not been established.

In summary, uniform-sized double tangent balls locally tame all subsets of E^3 and flatten $(n - 1)$-spheres in E^n for each n. Double tangent balls (with no size restriction) tame a 2-sphere Σ in E^3 provided the ball pairs lie on opposite sides of Σ, but such pairs of tangent balls do not locally tame arcs in E^3 or flatten $(n - 1)$-spheres in E^n when $n > 3$. Single, uniform-sized, tangent balls tame a 2-sphere in E^3, even with no requirement that they lie on the same side of Σ, but they fail to locally tame arbitrary subsets of E^3 and fail to flatten $(n - 1)$-spheres in E^n when $n > 3$. As a corollary it follows that ξ-boundaries in E^3 are locally tame wherever they are locally 2-manifolds. More recently, 1-dimensional subsets of E^3 were proven locally tame when they can be realized as ξ-boundaries.

On the other hand there are some interesting questions in this area that are not yet answered.

1. Is there a Rattle Theorem for $n > 3$?

2. Are ξ-boundaries in $E^n (n > 3)$ locally flat from one side wherever they are locally known to be manifolds?

3. For $n > 3$ do uniform double tangent balls tame subsets other than $(n - 1)$-manifolds? Wright [29] has proven that a compactum X in E^n is tamely embedded if each point of X can be touched by the tip of a cone from $E^n - X$ and if X has dimension $n - 3$ or less.

4. Is there a wild 2-sphere in E^3 that is touched at each point from its interior by the vertex of a cone from a family of congruent cones with cone angles 120°? What is the greatest lower bound of the cone angles in such examples?

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