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IMAGES AND QUOTIENTS OF SO(3, R): REMARKS ON A THEOREM OF VAN DER WAERDEN

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1. Introduction. We present an elementary proof of the following special case of a general theorem of B. L. van der Waerden: Every homomorphism from the rotation group $SO(3, \mathbf{R})$ to a compact topological group is continuous. An equivalent property of $SO(3, \mathbf{R})$, whose analogue is false for every infinite, compact, Abelian group, is this: No totally bounded topological group topology for $SO(3, \mathbf{R})$ is finer than the usual topology. The proof follows from a lemma which has an additional consequence: algebraically, the group $SO(3, \mathbf{R})$ is simple.

2. Background. In the context of (Hausdorff) topological groups, a powerful algebraic property—namely, the property that the group in question be Abelian—can have powerful topological consequences. Here are two examples of what we have in mind. (1) Every infinite Abelian group admits a totally bounded topological group topology **[14]**, **[6]**, but for non-Abelian groups the corresponding statement fails **[8]** (p. 296 ff.), **[17]** (p. 157), **[20]**, **[30]**, **[12]** (pp. 348–351). (2) Every infinite Abelian group admits a non-discrete metrizable topological group topology **[16]**, **[6]**, **[23]**, but there are non-Abelian groups which become topological groups only under the discrete topology **[24]**, **[1]** (§ 13.4).

The present paper originates with a question of much the same flavor: Can a compact topology on an infinite group be maximal among totally bounded topological group topologies? A simple argument (which we record below), based on classical cardinality constraints, shows that for Abelian groups the answer is "No". There exist, however, non-Abelian topological groups which show that the answer is "Yes". This is essentially a result, over a half-century old, due to B. L. van der Waerden **[28]**; as our Introduction suggests, our approach is via consideration of the existence of discontinuous homomorphisms into compact groups.

Our goals in this paper are to identify and describe the modern setting and vocabulary appropriate to van der Waerden's theorem, to provide an elementary proof from first principles of the result as it concerns specifically the compact group $SO(3, \mathbf{R})$, and to record some consequences of the theorem which are not mentioned explicitly by van der Waerden.

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For our purposes, a topological group is a triple $G = \langle G, \cdot, \mathscr{T} \rangle$ such that $\langle G, \cdot \rangle$ is a group, $\langle G, \mathscr{T} \rangle$ is a Hausdorff topological space, and the function $\langle a, b \rangle \to ab^{-1}$ from $G \times G$ to G is continuous. As is well known **[12]** (§ 8), the Hausdorff separation property in this context guarantees that every topological group is a Tychonoff space, i.e., a completely regular Hausdorff space.

The identity element of a (topological) group G is denoted e_G, or e when confusion is impossible; for $a \in G$, the set of neighborhoods of a is denoted $\mathscr{N}_G(a)$ or simply $\mathscr{N}(a)$.

A topological group G is said to be totally bounded if for every $U \in \mathscr{N}(e)$ there is a finite subset F of G such that $G = FU$. It is clear that a compact group, and each of its subgroups in the inherited topology, is totally bounded. Further, the product of any set of totally bounded groups is totally bounded. This may be proved directly with ease or it may be deduced from the Tychonoff product theorem and the following result of Weil **[29]**: Every totally bounded topological group G is homeomorphic with a (dense) topological subgroup of a compact group; this compact group, which is unique up to an isomorphism which fixes G pointwise, is called the Weil completion of G and is denoted $\bar{G}$.

In the following theorem, whose purpose is to illuminate our perspective on van der Waerden's theorem, we collect from the literature several results concerning the existence of totally bounded group topologies. Because the statements are not new, we include only enough details of the proof to convey to the reader the flavor of the full arguments involved.

THEOREM 2.1. *Let G be a group and let $\mathscr{B}$ be the set of topologies $\mathscr{T}$ for G such that $\langle G, \mathscr{T} \rangle$ is a (Hausdorff) totally bounded topological group.*

(a) *The set $\mathscr{B}$ is non-empty if and only if for $e \neq x \in G$ there are a compact group $K(x)$ and $h_x \in \mathrm{Hom}\,(G, K(x))$ such that $h_x(x) \neq e_{K(x)}$.*

(b) *If G is Abelian then $\mathscr{B} \neq \emptyset$;*

(c) *If $\mathscr{B} \neq \emptyset$ then there is $\mathscr{L} \in \mathscr{B}$ such that every $\mathscr{T} \in \mathscr{B}$ satisfies $\mathscr{T} \subset \mathscr{L}$;*

(d) *If $\mathscr{T} \in \mathscr{B}$ then there is a proper extension of $\mathscr{T}$ in $\mathscr{B}$ if and only if there exist a compact group K and a discontinuous homomorphism from $\langle G, \mathscr{T} \rangle$ to K;*

(e) *If G is infinite and Abelian and if $\langle G, \mathscr{T} \rangle$ is compact, then the equivalent conditions of* (d) *are satisfied.*

PROOF. (a) If $\mathscr{T} \in \mathscr{B}$ then for $e \neq x \in G$ one may take for $K(x)$ the Weil completion of $\langle G, \mathscr{T} \rangle$ and for h_x the inclusion of G into $K(x)$. Conversely when $K(x)$ and h_x are defined for $e \neq x \in G$, let i be the isomorphism from G into $\prod_x K(x)$ defined by $i(p)_x = h_x(p)$. The topology $\mathcal{T}$ for G defined by the requirement that i be a homeomorphism into $\prod_x K(x)$ satisfies $\mathcal{T} \in \mathcal{B}$.

(b) A standard argument based on the fact that the circle group $\mathbf{T}$ is divisible shows that for $e \neq x \in G$ there is $h_x \in \text{Hom}(G, \mathbf{T})$ such that $h_x(x) \neq 1$. Thus (b) follows from (a).

(c) Define

$$\mathcal{A} = \{\varphi^{-1}(U)\colon \varphi \in \text{Hom}(G, K), K \text{ is a compact group}, U \in \mathcal{N}_K(e)\}.$$

It is easy to check that the family $\mathcal{A}$ satisfies

(i) for $A \in \mathcal{A}$ there is $B \in \mathcal{A}$ such that $BB^{-1} \subset A$;
(ii) for $x \in A \in \mathcal{A}$ there is $B \in \mathcal{A}$ such that $xB \subset A$;
(iii) for $A \in \mathcal{A}$, $x \in G$ there is $B \in \mathcal{A}$ such that $xBx^{-1} \subset A$; and
(iv) if $A_i \in \mathcal{A}$ for $i < n$ $(n < \omega)$ then $\bigcap_i A_i \in \mathcal{A}$.

It then follows easily, as in **[12]** (4.5), that the family $\{xA\colon x \in G, A \in \mathcal{A}\}$ is a base for a topological group topology $\mathcal{L}$ on G. It is clear that $\mathcal{L}$ is as required.

(d) It is clear from the proof of (c) that for $\mathcal{T} \in \mathcal{B}$ we have $\mathcal{T} \neq \mathcal{L}$ if and only if $\mathcal{N}_{\langle G, \mathcal{T}\rangle}(e) \neq \mathcal{A}$.

(e) It is a special case of the famous duality theorem of Pontrjagin **[21]**, **[22]** and van Kampen **[15]** that the (discrete) dual group $\hat{G}$ of $\langle G, \mathcal{T}\rangle$ has a dual $\hat{\hat{G}}$ which is topologically isomorphic to $\langle G, \mathcal{T}\rangle$. In particular, $|G| = |\hat{\hat{G}}|$. Since $|\text{Hom}(A, \mathbf{T})| = 2^{|A|}$ for every infinite Abelian group A **[14]** we have $|G| = 2^{|\hat{G}|}$. Thus if every homomorphism from $\langle G, \mathcal{T}\rangle$ to $\mathbf{T}$ were continuous—that is, if $\text{Hom}(G, \mathbf{T}) = \hat{G}$—then we have the contradiction

$$|G| = |\hat{\hat{G}}| = 2^{|\hat{G}|} = 2^{|\text{Hom}(G,\mathbf{T})|} = 2^{2^{|G|}}.$$

REMARK 2.2. The portions of 2.1 which concern infinite Abelian groups combine to yield the statement that such a group admits a largest totally bounded group topology $\mathcal{L}$ and $\langle G, \mathcal{L}\rangle$ is not compact. More is known: $\langle G, \mathcal{L}\rangle$ is not pseudocompact **[7]**. In related work **[5]** we have shown that on an Abelian group no compact group topology of uncountable weight is maximal among pseudocompact group topologies. We do not know whether every pseudocompact group topology of uncountable weight on an Abelian group extends properly to a larger such topology.

The present elementary treatment of $SO(3, \mathbf{R})$ does not require even the rudiments of the theory of Lie groups—indeed we omit even the definition, referring the interested reader to such authoritative texts as **[19]**, **[11]** and **[13]**. In any event we recall this characterization: A topological group is a Lie group if and only if its topology is locally Euclidean **[19]** (p. 70). A Lie group is semisimple if and only if $\{e\}$ is its only connected normal, Abelian subgroup [11] (p. 121); a compact, connected, Lie group is semisimple if and only if its center is finite (cf. [13] (XIII.1.3) and [11] (pp. 254 and 268)).

The expression "G is a simple Lie group" is usually defined or described in terms of the Lie algebra of G. For our purposes the following characterization may be taken as a definition. A semisimple Lie group G is a simple Lie group if G is locally indecomposable in this sense: there are no infinite connected Lie groups E and F such that G is locally isomorphic to $E \times F$. Our point here is to alert the reader to the fact that our use of the term "simple Lie group" is in consonance with the convention favored by experts in the field: A simple Lie group need not be algebraically simple; indeed a simple Lie group may contain a non-trivial discrete normal subgroup, and there are no obvious reasons for eliminating *a priori* the possibility that a simple Lie group might contain a proper dense normal subgroup.

The theorem of van der Waerden [28] referred to in our title may be regarded as a technical lemma which has two major results as consequences. In their full generality, these read as follows.

Let G be a connected, semisimple Lie group with center $Z(G)$. Then

(a) Every local homomorphism from G to a compact group is continuous; and

(b) G is a simple Lie group if and only if $Z(G)$ contains every proper normal subgroup of G.

For our purposes it is convenient to emphasize the following specializations.

Let G be a compact, connected, semisimple Lie group with center $Z(G)$. Then

(a′) Every homomorphism from G to a compact group is continuous; and

(b′) G is a simple Lie group if and only if the group $G/Z(G)$ is algebraically simple.

For H a group and $a \in H$ we denote by $M_H(a)$, or by $M(a)$ if confusion is impossible, the set

$$M_H(a) = \{cbab^{-1}a^{-1}c^{-1}: b, c \in H\}.$$

(For the general treatment given in [28], where Lie groups are coordinatized by canonical coordinates of the second kind, van der Waerden finds it convenient for n-dimensional Lie groups H and $a \in H$ to define and analyze the set

$$H(a) = \{\textstyle\prod_{i=1}^{n} h_i: \text{each } h_i \in M_H(a)\}.$$

For our purposes the sets $M(a)$ are adequate and the sets $H(a)$ may be ignored.)

The foregoing remarks concerning Lie groups and the results of **[28]** are included in the interest of scientific and historical perspective. We emphasize that our treatment of $SO(3, \mathbf{R})$, though based almost entirely on ideas in **[28]**, is completely self-contained.

3. The Theorem of van der Waerden. We proceed as follows. First, an elementary lemma (3.1) from **[28]**; second, notation, terminology and definitions (3.2); third, verification that two naturally defined topologies for $SO(3, \mathbf{R})$ coincide (3.4) and make $SO(3, \mathbf{R})$ into a compact group (3.3);and finally, an extended list of properties enjoyed by $SO(3, \mathbf{R})$ (3.5, 3.6), from which follow the principal results (3.7) and some corollaries (3.8,3.9).

LEMMA 3.1. *Let K be a compact group and let a_λ be a net in K such that $a_\lambda \to e_K$. Then for every $V \in \mathscr{N}(e)$ there is λ such that $M(a_\lambda) \subset V$.*

PROOF. If the theorem fails then for each λ there are $b_\lambda, c_\lambda, x_\lambda \in K$ such that $x_\lambda = c_\lambda b_\lambda a_\lambda b_\lambda^{-1} a_\lambda^{-1} c_\lambda^{-1} \in K \backslash V$. Passing to subnets if necessary we assume without loss of generality that there are $b, c \in K$ such that $b_\lambda \to b$ and $c_\lambda \to c$. From continuity we have $x_\lambda \to cbeb^{-1}e^{-1}c^{-1} = e$, and since $K\backslash V$ is compact we have $e = \lim_\lambda x_\lambda \in K\backslash V$, a contradiction.

3.2. For an integer $n > 0$ we denote by $M(n, \mathbf{R})$ the set of $n \times n$ real matrices. This (and its subsets) are topologized as subsets of $\mathbf{R}^{n^2}$; thus for $A^{(m)} = (a_{ij}^{(m)}) \in M(n, \mathbf{R})$ and $A = (a_{ij}) \in M(n, \mathbf{R})$ we have $A^{(m)} \to A$ if and only if $a_{ij}^{(m)} \to a_{ij}$ whenever $i, j \in \{1, 2, \ldots, n\}$.

For $A \in M(n, \mathbf{R})$ we denote by A' the transpose of A defined as usual: A' is that $n \times n$ matrix $B = (b_{ij})$ such that $b_{ij} = a_{ji}$. Using the usual inner product $\langle , \rangle$ on $\mathbf{R}^n$ given by $\langle u, v\rangle = \sum_{i=1}^n u_i v_i$, we note that $\langle A'(v), w\rangle = \langle v, A(w)\rangle$ for all $A \in M(n, \mathbf{R})$ and $v, w \in \mathbf{R}^n$; in particular $\langle A'A(v), v\rangle = \langle Av, Av\rangle$ for such A and v.

We denote by $E(n)$, or simply by E when confusion is impossible, that $n \times n$ matrix $E(n) = (e_{ij})$ such that

$$\begin{aligned} e_{ij} &= 1 \text{ if } i = j, \\ &= 0 \text{ if } i \neq j. \end{aligned}$$

Those matrices $A \in M(n, \mathbf{R})$ for which the determinant $\det A$ satisfies $\det A \neq 0$ have an inverse; this we denote by A^{-1}.

A matrix $A \in M(n, \mathbf{R})$ is a special orthogonal matrix if $\det A = 1$ and $A^{-1} = A'$. Since $\det(AB) = (\det A)\cdot(\det B)$ and $(AB)' = B'A'$ for all $A, B \in M(n, \mathbf{R})$, the set $SO(n, \mathbf{R})$ of (real) special orthogonal matrices is a group.

LEMMA 3.3. *$SO(n, \mathbf{R})$ is a compact topological group.*

PROOF. Continuity of the functions $A \to A^{-1}$ and $\langle A, B\rangle \to AB$ follows from the fact that $\mathbf{R}$ is a topological field; thus $SO(n, \mathbf{R})$ is a topological group. The functions $A \to \det A$, $A \to A^{-1}$ and $A \to A'$ being continuous, $SO(n, \mathbf{R})$ is (homeomorphic to a set which is) closed in $\mathbf{R}^{n^2}$. With $(b_{ij}) = B = A^{-1}$ we have

$$1 = \sum_j a_{ij} b_{ji} = \sum_j a_{ij} a_{ij}$$

for $1 \leqq i \leqq n$, so each $|a_{ij}| \leqq 1$ and $SO(n, \mathbf{R})$ is bounded in $\mathbf{R}^{n^2}$.

It is convenient now to specialize to three dimensions. We write $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$.

For $v = \langle v_1, v_2, v_3 \rangle \in \mathbf{R}^3$ we set

$$\|v\| = |\langle v, v \rangle|^{1/2} = (v_1^2 + v_2^2 + v_3^2)^{1/2}.$$

We set $S^2 = \{v \in \mathbf{R}^3 : \|v\| = 1\}$, and for $A \in M(3, \mathbf{R})$ we define

$$\|A\| = \sup \{\|A(v)\| : v \in S^2\}.$$

We note that if $A \in SO(3, \mathbf{R})$ then

$$\|A(v)\| = |\langle A(v), A(v) \rangle|^{1/2} = |\langle v, v \rangle|^{1/2} = \|v\|$$

for all $v \in \mathbf{R}^3$. It follows that $\|A\| = 1$ and $\|AX\| = \|XA\| = \|X\|$ whenever $A \in SO(3, \mathbf{R})$ and $X \in M(3, \mathbf{R})$.

It is clear that the function $\langle A, B \rangle \to \|A - B\|$ from $M(n, \mathbf{R}) \times M(n, \mathbf{R})$ to $\mathbf{R}$ satisfies the formal requirements of a metric on the set $M(n, \mathbf{R})$. From this, or directly, follows $\|A + B\| \leqq \|A\| + \|B\|$ for $A, B \in M(n, \mathbf{R})$.

LEMMA 3.4. (a) *For* $A \in M(3, \mathbf{R})$ *the function* $v \to A(v)$ *from* $\mathbf{R}^3$ *to* $\mathbf{R}^3$ *is continuous*;

(b) *the function* $A \to \|A\|$ *from* $M(3, \mathbf{R})$ *to* $\mathbf{R}$ *is continuous*;

(c) *the topology of the topological group* $SO(3, \mathbf{R})$ *is equal to the topology induced by the function* $\| \ \|$.

PROOF. (a) With $A = (a_{ij})$ we have $A(v)_i = \sum_j a_{ij} v_j$, so (a) is obvious.

(b) Since $\|A + B\| \leqq \|A\| + \|B\|$ for $A, B \in M(3, \mathbf{R})$, it is enough to show that if $A^{(m)} = a_{ij}^{(m)} \in M(3, \mathbf{R})$ and $A = (a_{ij}) \in M(3, \mathbf{R})$ and $A^{(m)} \to A$, then $\|A^{(m)} - A\| \to 0$. For $\varepsilon > 0$ there is N such that if $m > N$ then $|a_{ij}^{(m)} - a_{ij}| < \varepsilon/(3\sqrt{3})$ whenever $i, j \in \{1, 2, 3\}$. For $v = \langle v_1, v_2, v_3 \rangle \in S^2$ we have $|v_j| \leqq 1$ for $1 \leqq j \leqq 3$, so that

$$\left|\sum_{j=1}^3 (a_{ij}^{(m)} - a_{ij}) \cdot v_j\right| < \varepsilon/\sqrt{3}$$

and hence

$$\|(A^{(m)} - A)(v)\| = \left(\sum_{i=1}^3 \left[\sum_{j=1}^3 (a_{ij}^{(m)} - a_{ij}) \cdot v_j\right]^2\right)^{1/2} < \varepsilon.$$

It follows that $\|A^{(m)} - A\| \leqq \varepsilon$ for $m > N$.

(c) According to (b), the topology given on G by $\| \quad \|$ is contained in the topological group topology (inherited from $\mathbf{R}^9$). Thus the identity function $A \to A$ is a one-to-one continuous function from the compact group G to G with the (Hausdorff) topology given by $\| \quad \|$. Statement (c) is then immediate.

In the following theorem we assemble several facts—most of them well-known and each of them susceptible to an elementary proof—concerning the group $SO(3, \mathbf{R})$.

For $\theta \in [-\pi, \pi]$ we define $E_\theta \in SO(3, \mathbf{R})$ by

$$E_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

For notational simplicity in what follows we write $G = SO(3, \mathbf{R})$.

THEOREM 3.5. *The topological group* $G = SO(3, \mathbf{R})$ *has the following properties.*

(a) *G acts transitively on S^2, that is, for all u, $v \in S^2$ there is $A \in G$ such that $A(u) = v$.*

(b) *If $C \in G$ and $C(e_3) = e_3$ then there is $\theta \in [-\pi, \pi]$ such that $C = E_\theta$.*

(c) *If $D = (d_{ij})$ is a real 3×3 diagonal matrix, then* $\|D\| = \max\{|d_{ii}| : 1 \leqq i \leqq 3\}$.

(d) *For every $A \in G$ there is a unique $\theta \in [0, \pi]$ such that $\|A - E\| = (2 - 2\cos(\theta))^{1/2}$.*

(e) *For every $A \in G$ there is an eigenvector w with eigenvalue* 1 *such that $\|w\| = 1$; i.e., there is $w \in S^2$ such that $A(w) = w$.*

(f) *If $A \in G$ and $\|A - E\| = (2 - 2\cos(\varphi))^{1/2}$ with $\varphi \in [0, \pi]$ then there is $B \in G$ such that $BAB^{-1} = E_\varphi$.*

(g) *The group G is arc-wise connected.*

(h) *If A, $X \in G$ and $\|A - E\| = \|X - E\|$ then there is $C \in G$ such that $X = CAC^{-1}$.*

(i) *The conjugacy classes in G are precisely the sets $S(r) = \{X \in G: \|X - E\| = r\}$* for $0 \leqq r \leqq 2$, and *each $S(r)$ is non-empty.*

(j) *G has trivial center.*

PROOF. (a) It is enough to note that for every $v = \langle v_1, v_2, v_3 \rangle \in S^2$ there are θ and φ such that

$$v = \langle (\sin\varphi)(\cos\theta), (\sin\varphi)(\sin\theta), \cos\varphi \rangle,$$

and that

$$\begin{pmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

(b) From $C(e_3) = e_3$ follows $c_{13} = c_{23} = 0$ and $c_{33} = 1$, and then from $\sum_{j=1}^{3} a_{ij}^2 = 1$ for $1 \leqq i \leqq 3$ it follows that $c_{31} = c_{32} = 0$ and that the "upper left corner" of C is an element of $SO(2, \mathbf{R})$.

(c) There is $k \in \{1, 2, 3\}$ such that the indicated maximal value is equal to $|d_{kk}|$. Then for $v = \langle v_1, v_2, v_3 \rangle \in S^2$ we have

$$\|D(v)\|^2 = \textstyle\sum_{i=1}^{3}(d_{ii}v_i)^2 \leqq d_{kk}^2 \sum_{i=1}^{3} v_i^2 = d_{kk}^2 \|v\|^2 = d_{kk}^2$$

and hence $\|D(v)\| \leqq |d_{kk}|$. For the reverse inequality we note from $e_k \in S^2$ that

$$\|D\| \geqq \|D(e_k)\| = |\langle D(e_k), D(e_k)\rangle|^{1/2} = |d_{kk}|.$$

(d) This is immediate from the inequalities

$$0 \leqq \|A - E\| \leqq \|A\| + \|E\| = 1 + 1 = 2.$$

(e) [This is obvious geometrically since A in effect is a rotation of $\mathbf{R}^3$ about a line through the origin; one may take for w either of the points on this line which satisfy $\|w\| = 1$. We will argue algebraically.] Let

$$\det(A - \lambda I) = f(\lambda) = k_3\lambda^3 + k_2\lambda^2 + k_1\lambda + k_0$$

be the characteristic polynomial of A, and note from

$$A^{-1} - \lambda I = A' - \lambda I = (A - \lambda I)'$$

that $f(\lambda)$ is also the characteristic polynomial of A^{-1}. Clearly $k_3 = -1$ and $k_0 = \det A = 1$. From the celebrated Cayley-Hamilton theorem we have, denoting by 0 the 3×3 matrix whose entries are all 0,

$$(*) \quad -A^3 + k_2A^2 + k_1A + I = 0, \text{ and}$$
$$(**) \quad -A^{-3} + k_2A^{-2} + k_1A^{-1} + I = 0.$$

Multiplying (*) by A^{-1} and (**) by A^2 and adding we have $(k_1 + k_2)(A + I) = 0$. Since $\det A = 1$ the relation $A = -I$ is false; hence $k_1 + k_2 = 0$ and

$$\det(A - I) = f(1) = 0.$$

It is well-known (see for example [18] (IX, § 5)) that there is $u \in \mathbf{R}^3$ such that $u \neq 0$ and $A(u) = u$. Then $w = u/\|u\|$ is as required.

(f) Use (e) to choose $w \in S^2$ such that $A(w) = w$, and use (a) to choose $B \in G$ such that $B^{-1}(e_3) = w$. Writing $C = BAB^{-1}$ we have $C(e_3) = e_3$, so by (b) there is $\theta \in [-\pi, \pi]$ such that $C = E_\theta$. Denoting by F the diagonal matrix with entries $(1, -1, -1)$, we note that $FBA\,(FB)^{-1}(e_3) = e_3$

and $FCF^{-1} = E_{-\theta}$. Thus we may assume without loss of generality, replacing if necessary B by FB(and hence θ by $-\theta$), that $\theta \in [0, \pi]$. It remains to show $\theta = \varphi$, i.e., that $\|A - E\| = (2 - 2\cos(\theta))^{1/2}$.

A routine computation shows, writing $T_\theta = E_\theta - E$ and denoting by D the diagonal matrix with diagonal entries $((2-2\cos(\theta))^{1/2}, (2-2\cos(\theta))^{1/2}, 0)$, that $T'_\theta T_\theta = D^2 = D'D$. Now for $v \in \mathbf{R}^3$ we have

$$\begin{aligned}\|T_\theta(v)\|^2 &= \langle T_\theta(v), T_\theta(v)\rangle = \langle T'_\theta T_\theta(v), v\rangle \\ &= \langle D'D(v), v\rangle = \langle D(v), D(v)\rangle = \|D(v)\|^2,\end{aligned}$$

and hence $\|T_\theta\| = \|D\|$. It then follows that

$$\begin{aligned}\|A - E\| &= \|BAB^{-1} - BEB^{-1}\| = \|E_\theta - E\| = \|T_\theta\| = \|D\| \\ &= (2 - 2\cos(\theta))^{1/2}\end{aligned}$$

as required.

(g) To connect E to an (arbitrary) $A \in G$, use (d) and (f) to find $\varphi \in [0, \pi]$ and $B \in G$ such that $A = B^{-1}E_\varphi B$; then define f: $[0, 1] \to G$ by the rule $f(t) = B^{-1}E_{t\varphi}B$. Then f is continuous, and $f(0) = E, f(1) = A$.

(h) From (d) and (f) there are $\theta \in [0, \pi]$ and $B, D \in G$ such that $E_\theta = BAB^{-1} = DXD^{-1}$; take $C = D^{-1}B$.

(i) Use (d) to show that $G = \bigcup \{S(r): 0 \leqq r \leqq 2\}$ and use the computation $\|E_\theta - E\|^2 = 2 - 2\cos\theta$ in the proof of (f) to show that $S(r)$ is nonempty when $0 \leqq r \leqq 2$. For $A, X \in S(r)$, use (h) to show that A and X are conjugate. Conversely if $A \in S(r)$ and $X = CAC^{-1}$ then we have

$$\|X - E\| = \|C(A - E)C^{-1}\| = \|A - E\| = r$$

and hence $X \in S(r)$, as required.

(j) If A is central in G then from (f) there is $\varphi \in [0, \pi]$ such that $A = E_\varphi$. Then $A(e_3) = e_3$ and for every $B \in G$ we have $A(B(e_3)) = B(A(e_3)) = B(e_3)$. It then follows from (a) that $A(v) = v$ for all $v \in \mathbf{R}^3$, so that $A = E$.

In the following technical lemma we retain the notation

$$M_G(A) = M(A) = \{CBAB^{-1}A^{-1}C^{-1}: B, C \in G\}$$

for $A \in G$.

Lemma 3.6. *If* $E \neq A \in G = SO(3, \mathbf{R})$ *then there is* $U \in \mathcal{N}_G(E)$ *such that* $U \subset M(A)$.

Proof. From 3.5(j) there is $B \in G$ such that $AB \neq BA$. Define $\varepsilon = \|BAB^{-1}A^{-1} - E\|$ and choose a continuous f: $[0, 1] \to G$ such that $f(0) = E$ and $f(1) = B$. Now for $t \in [0, 1]$ write $B_t = f(t)$ and $A_t = B_tAB_t^{-1}A^{-1}$ and note that the function $t \to \|A_t - E\|$ satisfies $0 \to 0$ and $1 \to \varepsilon$. By 3.4(b) this function is continuous, so for $0 \leqq r \leqq \varepsilon$ there is $t(r) \in [0, 1]$ such that $\|A_{t(r)} - E\| = r$. Writing $S(r) = \{X \in G: \|X - E\| = r\}$ for

$0 \leqq r \leqq \varepsilon$ and setting $U = \bigcup\{S(r): 0 \leqq r < \varepsilon\}$, we have from 3.5(h) that

$$S(r) \subset \{CA_{t(r)}C^{-1}: C \in G\} \subset M(A)$$

and from 3.4(c) that U, since it is open in the topology given by $\| \quad \|$, is in fact open in G that is, $U \in \mathscr{N}_G(e)$.

The properties of $SO(3, \mathbf{R})$ established by van der Waerden **[28]** are now readily accessible.

THEOREM 3.7. (a) *Every homomorphism from* $SO(3, \mathbf{R})$ *to a compact group is continuous.*

(b) *The group* $SO(3, \mathbf{R})$ *is algebraically simple, that is, every normal subgroup* N *of* $SO(3, \mathbf{R})$ *satisfies either* $N = \{e\}$ *or* $N = SO(3, \mathbf{R})$.

(c) *Every element of* $SO(3, \mathbf{R})$ *is a commutator, that is,* $SO(3, \mathbf{R}) = \{XYX^{-1}Y^{-1}: X, Y \in SO(3, \mathbf{R})\}$.

PROOF. Again we write $G = SO(3, \mathbf{R})$.

(a) Let $h \in \mathrm{Hom}(G, K)$ with K compact and let $V \in \mathscr{N}_K(e)$. We will find $U \in \mathscr{N}_G(e)$ such that $h[U] \subset V$. We assume without loss of generality, replacing if necessary K by $G \times K$, and h by $h' \in \mathrm{Hom}(G, G \times K)$ defined by $h'(x) = \langle x, h(x)\rangle$, that h is a one-to-one function. (We note concerning these replacements that h is continuous if and only if h' is continuous.) It then follows that $h(e_G) = e_K$ is not an isolated point of $h[G]$, for if e_K were isolated then $h[G]$ is a discrete subgroup of the compact group K and hence finite.

It follows that there is a net A_λ in G such that $h(A_\lambda) \to e_K$ and $h(A_\lambda) \neq e_K$; hence by 3.1 (applied to the net $h(A_\lambda)$) there is λ such that $M_K(h(A_\lambda)) \subset V$. Let A be such an element A_λ of G and (using $A \neq E = e_G$ and 3.6) choose $U \in \mathscr{N}_G(E)$ such that $U \subset M_G(A)$. Since h is a homomorphism we have

$$h[U] \subset h[M_G(A)] \subset M_K(h(A)) \subset V,$$

as required.

(b) If there is $A \neq E$ such that $A \in N$ then by 3.6 there is $U \in \mathscr{N}(E)$ such that $U \subset M(A) \subset N$. It follows that N is an open, hence closed, subgroup of G. Since G is connected (3.4(g)) we have $N = G$, as required.

(c) It follows from 3.5(d) and 3.5(f) that every $A \in G$ has the form $A = B^{-1}E_\varphi B$ with $B \in G$ and $\varphi \in [0, \pi]$. Since the conjugate of a commutator is a commutator, it is therefore enough to show that E_φ is a commutator. For this we take $\theta = -\varphi/2$ and we note, denoting as in 3.5(f) by F the diagonal matrix with diagonal entries $(1, -1, -1)$, that $E_\varphi = FE_\theta F^{-1}E_\theta^{-1}$.

For a proof (in geometric language) that the groups $SO(n, \mathbf{R})$ for n odd are simple, the interested reader might consult Artin **[2]** (5.3).

We conclude with two consequences of the theorem just proved. The second asserts in effect that the (continuous) homomorphisms we have been considering are in fact homeomorphisms.

COROLLARY 3.8. *The only totally bounded topological group topology for the group* $SO(3, \mathbf{R})$ *is the usual* (compact) *topology.*

PROOF. Let $\langle G, \mathscr{T} \rangle$ be $SO(3, \mathbf{R})$ with a totally bounded topological group topology, and let K be the Weil completion of G (described above in the seventh paragraph of § 2). The function i: $SO(3, \mathbf{R}) \to K$ given by $i(x) = x$ is a continuous, one-to-one function from the compact space $SO(3, \mathbf{R})$ onto the Hausdorff space $\langle G, \mathscr{T} \rangle$. The function i is then a homeomorphism.

COROLLARY 3.9. *Let h be a homomorphism from* $SO(3, \mathbf{R})$ *into a compact group. Then either* $h(A) = e$ *for all* $A \in SO(3, \mathbf{R})$ *or h is a topological isomorphism* (*i.e., an isomorphism and a homeomorphism*) *onto its range.*

PROOF. If ker $h \neq SO(3, \mathbf{R})$ then h is a one-to-one continuous fucntion with compact domain and Hausdorff range, hence is a homeomorphism.

REMARKS 3.10. Here we note briefly the existence of some other papers related to van der Waerden's theorem.

(a) We have specialized the 1933 results of van der Waerden to the compact case. This specialization had already been established by Cartan [**4**] in 1930, with a much more difficult proof. Cartan's primary concern was isomorphisms (rather than homomorphisms); he apparently deduces continuity by establishing the continuity of various restrictions to three-dimensional subgroups. This paper has an excellent elementary treatment of $SO(3, \mathbf{R})$.

(b) Freudenthal [**10**] proved interesting generalizations to a class of non-compact semisimple Lie groups. Using a theorem now available in [**19**] (4.6, page 175), his results can be viewed as asserting the uniqueness of a topology within a specified class of topologies. Specifically, the Lie groups which Freudenthal characterizes have a unique connected locally compact topology. Tits [**27**] established generalizations, including an analysis of Lie groups with proper connected normal Abelian subgroups.

(c) Related results were obtained by Stewart [**25**] for compact connected groups, by van Est [**9**] for Lie groups, and by Borel and Tits [**3**] and Tits [**26**]. The latter two papers analyze groups related to matrix groups over an arbitrary field; the theorems have significant topological content when the field is locally compact and non-discrete.

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