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CHORDAL QUADRATIC SYSTEMS*

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ABSTRACT. A quadratic system is called chordal if all its singularities are on the equator of the Poincaré sphere. First, we establish necessary and sufficient conditions for a quadratic system to be chordal. Later, we determine all the phase portraits for such systems.

1. Introduction. This paper contains a study of those two-dimensional autonomous systems with quadratic polynomial right-hand sides without finite singularities. Such systems will be referred to as chordal quadratic systems (CQS, for abbreviation). The chordal systems were studied by Kaplan, see [**6**] and [**7**]. The name of chordal system is due to the fact that a such system has all its solutions starting and ending at the equator of the Poincaré sphere. For a survey on quadratic systems (QS, for abbreviation), see Coppel [**4**] and Ye Yanqian [**13**]. At the end of the paper [**4**], Coppel states that what remains to be done for quadratic systems is to determine all possible phase portraits and, ideally, to characterize them by means of algebraic inequalities on the coefficients.

This paper first establishes necessary and sufficient conditions for a QS to have all its singularities at infinity (on the equator of the Poincaré sphere), i.e., to be a CQS, and then determines all possible phase portraits for such CQS.

Our main result is the following theorem.

THEOREM. *The phase portrait of a chordal quadratic system is homeomorphic (except for perhaps the orientation) to one of the separatrix configurations shown in Figure 1. Furthermore, all the separatrix configurations of Figure 1 are realizable for the chordal quadratic systems.*

REMARK 1. The Figures 1.1 to 1.21 are realizable for properly chordal

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quadratic systems, Figures 1.22 and 1.23 for properly chordal linear systems, and Figure 1.8 for chordal constant systems, too.

Sheng Li-Ren studied in [9] the chordal quadratic systems with Reeb's components. In that work all the CQS of Figure 1 except the systems 4, 9, 13, 14, 15, 19, 21 and 23 were studied.

The unique CQS which are structurally stable are systems 1 and 8 of Figure 1, see [12].

REMARK 2. Note that it is not necessary to solve Hilbert's 16th problem

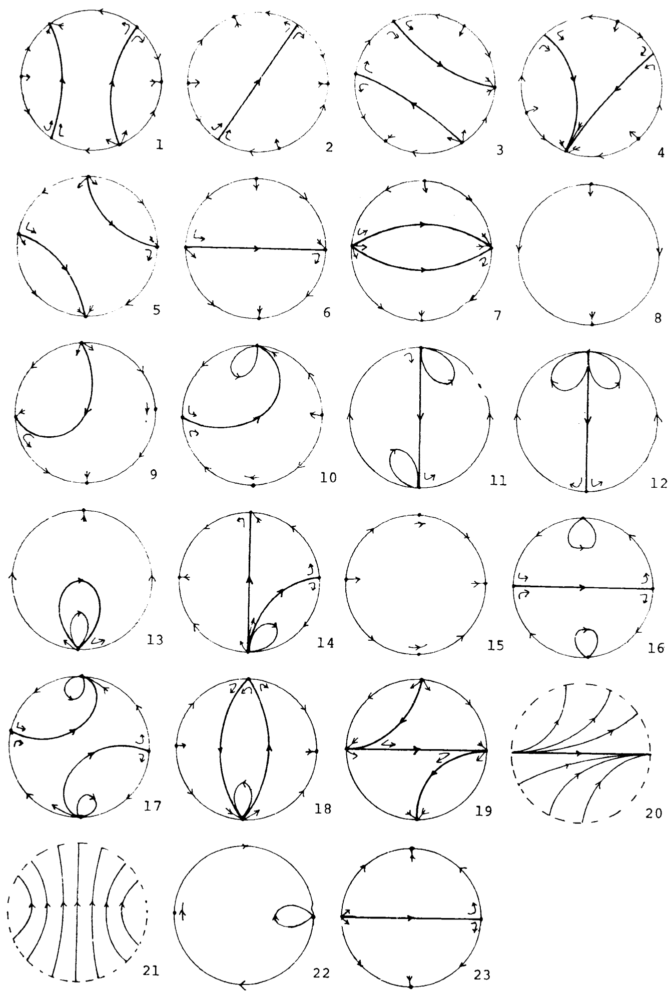

FIGURE 1. The chordal quadratic systems (except, perhaps for orientation).

in order to determine all the possible phase portraits for CQS because they do not have finite singularities.

2. Classification of CQS. First, we need a general classification of QS in which it is easy to study the finite singularities. The following lemma completes the classification of Cherkas [3] and Sheng Li-Ren [9].

LEMMA 1. *A quadratic system*

$$(1)\qquad \begin{cases} \dot{x} = \dfrac{dx}{dt} = P(x, y) \\ \dot{y} = \dfrac{dy}{dt} = Q(x, y) \end{cases}$$

is affine-equivalent, scaling the variable t if necessary, to one of the following:

$$(\mathrm{I})\ \begin{cases} \dot{x} = 1 + xy, \\ \dot{y} = Q(x, y), \end{cases} \qquad (\mathrm{VI})\ \begin{cases} \dot{x} = 1 + x^2, \\ \dot{y} = Q(x, y), \end{cases}$$

$$(\mathrm{II})\ \begin{cases} \dot{x} = xy, \\ \dot{y} = Q(x, y), \end{cases} \qquad (\mathrm{VII})\ \begin{cases} \dot{x} = x^2, \\ \dot{y} = Q(x, y), \end{cases}$$

$$(\mathrm{III})\ \begin{cases} \dot{x} = y + x^2, \\ \dot{y} = Q(x, y), \end{cases} \qquad (\mathrm{VIII})\ \begin{cases} \dot{x} = x, \\ \dot{y} = Q(x, y), \end{cases}$$

$$(\mathrm{IV})\ \begin{cases} \dot{x} = y, \\ \dot{y} = Q(x, y), \end{cases} \qquad (\mathrm{IX})\ \begin{cases} \dot{x} = 1, \\ \dot{y} = Q(x, y), \end{cases}$$

$$(\mathrm{V})\ \begin{cases} \dot{x} = -1 + x^2, \\ \dot{y} = Q(x, y), \end{cases} \qquad (\mathrm{X})\ \begin{cases} \dot{x} = 0, \\ \dot{y} = Q(x, y), \end{cases}$$

where $Q(x, y) = d + ax + by + lx^2 + mxy + ny^2$.

PROOF. We write (1) as

$$(2)\qquad \begin{cases} \dot{x} = d_1 + a_1x + b_1y + l_1x^2 + m_1xy + n_1y^2, \\ \dot{y} = d_2 + a_2x + b_2y + l_2x^2 + m_2xy + n_2y^2. \end{cases}$$

We can assume that $n_1 = 0$. Otherwise, system (2) becomes a QS without term y^2 in $P(x, y)$ if we make the change of variables $x_1 = y - rx$, $y_1 = y$ where $r \neq 0$ satisfies

$$(3)\qquad l_2 + (m_2 - l_1)r + (n_2 - m_1)r^2 - n_1r^3 = 0.$$

If $l_2 = 0$, that is, if the x^2 term does not appear in $Q(x, y)$, then it is sufficient to interchange x and y. In short, we have

$$(4)\qquad \begin{cases} \dot{x} = d_1 + a_1 x + b_1 y + l_1 x^2 + m_1 xy, \\ \dot{y} = Q(x, y). \end{cases}$$

If $m_1 \neq 0$, then we introduce the translation $x_1 \neq x + b_1 m_1^{-1}$, $y_1 = y$ and system (4) becomes

$$(5)\qquad \begin{cases} \dot{x} = d' + a'x + l'x^2 + m_1 xy, \\ \dot{y} = Q(x, y). \end{cases}$$

Now, the change $x_1 = x$, $y_1 = a' + l'x + m_1 y$ converts system (5) to

$$(6)\qquad \begin{cases} \dot{x} = d' + xy, \\ \dot{y} = Q(x, y). \end{cases}$$

We may put $x_1 = (d')^{-1}x$, $y_1 = y$ in (6) if $d' \neq 0$ to obtain (I). If $d' = 0$, then we have (II).

If $m_1 = 0$ and $b_1 \neq 0$, then the change $x_1 = x$, $y_1 = d_1 + a_1 x + b_1 y$ converts (4) to

$$(7)\qquad \begin{cases} \dot{x} = y + l_1 x^2, \\ \dot{y} = Q(x, y). \end{cases}$$

Now, if $l_1 \neq 0$, then we may put $x_1 = x$, $y_1 = l_1^{-1}y$, $t_1 = l_1 t$ in (7) to obtain (III). If $l_1 = 0$, then we have (IV).

When $m_1 = b_1 = 0$ and $l_1 \neq 0$, we put $k = a_1^2 - 4l_1 d_1$. If $k \neq 0$ the change $x_1 = 2l_1 |k|^{-1/2}(x + a_1(2l_1)^{-1})$, $y_1 = y$, $t_1 = 2^{-1}|k|^{1/2}t$ converts system (4) to (V) or (VI) according to whether k is positive or negative. If $k = 0$, then the change $x_1 = x + a_1(2l_1)^{-1}$, $y_1 = y$, $t_1 = l_1 t$ converts system (4) to (VII).

If $m_1 = b_1 = l_1 = 0$ and $a_1 \neq 0$, then the change $x_1 = x + d_1(a_1)^{-1}$, $y_1 = y$, $t_1 = a_1 t$ converts system (4) to (VIII).

Lastly, suppose that $m_1 = b_1 = l_1 = a_1 = 0$. If $d_1 \neq 0$, then the change $x_1 = x$, $y_1 = y$, $t_1 = d_1 t$ gives (IX); and if $d_1 = 0$, then we have (X).

In order to study the singularities at infinity of the ten systems of Lemma 1, we need the Poincaré compactification [5], [11]. Consider the sphere $S^2 = \{y \in \mathbf{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$, let $q = (0, 0, 1)$ be the north pole of S^2, and T_qS^2 be the plane $\{y \in \mathbf{R}^3 : y_3 = 1\}$. Let $p^+ : T_qS^2 \to S^2$ and $p^- : T_qS^2 \to S^2$ be the central projections, i.e., $p^+(y)$ (resp. $p^-(y)$) is the intersection of the line joining y to the origin with the northern (resp. southern) hemisphere of S^2. Let X be a polynomial vector field of degree d on the plane and let $f: S^2 \to \mathbf{R}$ be defined by $f(y) = y_3^{d-1}$. Then the vector fields $f \cdot (p^+)_* X = f \cdot Dp^+(X \circ (p^+)^{-1})$ and $f \cdot (p^-)_* X$, extend X to an analytic vector field $p(X)$, on S^2. The equator is invariant under the flow of $p(X)$ and a neighborhood of the equator corresponds to a neighborhood of infinity in $\mathbf{R}^2$.

To study $p(X)$ we use the following coordinate systems on S^2. Let $U_i = \{y \in S^2: y_i > 0\}$ and $V_i = \{y \in S^2: y_i < 0\}$. Let $F_i: U_i \to \mathbf{R}^2$ be given by $F_i(y) = (y_j y_i^{-1}, y_k y_i^{-1})$, for $j < k$ and $j, k \neq i$. We define $G_i: V_i \to \mathbf{R}^2$ by the same expression. Consider the vector field $p(X)$, where $X(x, y) = (P(x, y), Q(x, y))$ is such that P and Q are polynomials of degree at most d and at least one of them has degree d. Then $(F_1)_*(p(X))$ is given by

$$\text{(8)}\qquad \begin{cases} \dot{z}_1 = \Delta(z)^{1-d} z_2^d(-z_1 P(z_2^{-1}, z_1 z_2^{-1}) + Q(z_2^{-1}, z_1 z_2^{-1})), \\ \dot{z}_2 = -\Delta(z)^{1-d} z_2^{d+1} P(z_2^{-1}, z_1 z_2^{-1}), \end{cases}$$

where $\Delta(z) = (1 + z_1^2 + z_2^2)^{1/2}$, $z_1 = y_2 y_1^{-1} = yx^{-1}$, $z_2 = y_3 y_1^{-1} = x^{-1}$. Here the points of the equator are represented by $z_2 = 0$, the points of the northern hemisphere by $z_2 > 0$ and the point $(0, 0)$ corresponds to the point $(1, 0, 0) \in S^2$. $(F_2)_*(p(X))$ is given by

$$\text{(9)}\qquad \begin{cases} \dot{z}_1 = \Delta(z)^{1-d} z_2^d(P(z_1 z_2^{-1}, z_2^{-1}) - z_1 Q(z_1 z_2^{-1}, z_2^{-1})), \\ \dot{z}_2 = -\Delta(z)^{1-d} z_2^{d+1} Q(z_1 z_2^{-1}, z_2^{-1}). \end{cases}$$

As before the northern hemisphere corresponds to $z_2 > 0$ and $(0, 0) = F_2(0, 1, 0)$. The vector fields $(G_i)_*(p(X))$ have the same expressions as $(F_i)_*(p(X))$, multiplied by $(-1)^{d-1}$, but in this case the northern hemisphere corresponds to $z_2 < 0$.

In short, systems (8) and (9) will be sufficient in order to study the singularities at infinity of system (1).

From now on we shall denote by (y, z) the coordinates $(z_1, z_2) = F_1(y_1, y_2, y_3)$, where $(y_1, y_2, y_3) \in U_1$, and by (x, z) the coordinates $(z_1, z_2) = F_2(y_1, y_2, y_3)$ where $(y_1, y_2, y_3) \in U_2$.

We shall say that $X(x, y) = (P(x, y), Q(x, y))$ is degenerate at infinity or that the CQS is degenerate, if all the points of the equator of S^2 are singularities of $p(X)$ and X is properly a quadratic system. If X is nondegenerate at infinity, then from (8) and (9), and the fact that each singularity of $p(X)$ at infinity completely determines its antipodal singularity, it follows that there are only one, two or three singularities at infinity to be considered.

Let $X(x, y) = (ax + by + F(x, y), cx + dy + G(x, y))$ be a vector field such that F and G are analytic in a neighborhood of the origin and have expansions that begin with second degree terms in x and y, and the origin $(0, 0)$ is an isolated singularity. Then, we say that $(0, 0)$ is a singularity of type:

E if $(0, 0)$ is a nondegenerate singularity;

S if the linear part $DX(0, 0)$ has an unique eigenvalue equal to zero;

H if the linear part $DX(0, 0)$ has the two eigenvalues equal to zero and $DX(0, 0)$ is not zero; and

T if the linear part is zero.

Now, we shall study the singularities at infinity for the system (I). For this system, from (8) and (9), we have

$$\begin{cases} \dot{y} = l + my + az + (n-1)y^2 + byz + dz^2 - yz^2, \\ \dot{z} = -yz - z^3, \end{cases} \tag{10}$$

$$\begin{cases} \dot{x} = (1-n)x - mx^2 - bxz + z^2 - lx^3 - ax^2z - dxz^2, \\ \dot{z} = -nz - mxz - bz^2 - lx^2z - axz^2 - dz^3, \end{cases} \tag{11}$$

where we omit the positive function $\Delta = (1 + y^2 + z^2)^{1/2}$ in (10), and $\Delta = (1 + x^2 + z^2)^{1/2}$ in (11). So, the singularities at infinity of the system (I) will be the singularities $(y, 0)$ of (10) and $(x, 0)$ of (11).

The point $(y, 0) \in F_1(U_1)$ will be a singularity of (10) if and only if y satisfies $(n-1)y^2 + my + l = 0$. The matrix of the linear part $DX(y, 0)$ of the vector field (10) at the singularity $(y, 0)$ is given by

$$\begin{bmatrix} m + 2(n-1)y & a + by \\ 0 & -y \end{bmatrix} \tag{12}$$

The unique singularity $(x, 0) \in F_2(U_2)$ of (11) which does not belong to $F_1(U_1)$ is the singularity $(0, 0)$. The matrix of the linear part $DX(0, 0)$ of the vector field (11) at the singularity $(0, 0)$ is equal to

$$\begin{bmatrix} 1-n & 0 \\ 0 & -n \end{bmatrix} \tag{13}$$

Since system (I) does not have finite singularities, the polynomial $lx^4 + ax^3 + (d-m)x^2 - bx + n$ has no real roots different from zero.

From (10)–(13) and this last condition it is easy to classify the singularities at infinity for system (I), the result is given in Table 1.

Similarly, we can classify the singularities at infinity for systems (II)-(X) when they have no finite singularities, the results are given in Tables 2–10. Note that to study system (X) it is necessary to consider the imaginary conics.

Table 11 summarizes the classification of the singularities at infinity.

3. Phase portraits for degenerate, linear and constant CQS.

LEMMA 2. *For a degenerate, linear or constant* CQS (1), *there exists an affine transformation and a scaling of the variable t which reduces it to one of the systems:*

TABLE 1. The singularities at infinity for system (I).

System (I) has at infinity	3	singularities of type	(E,E,E)	if	$n \neq 0, 1$	$l \neq 0$	$m^2 - 4(n-1)l > 0$	(I.1)	with	(A)
	1		(E)				$m^2 - 4(n-1)l < 0$	(I.2)		
	2		(E,S)				$m^2 - 4(n-1)l = 0$	(I.3)		
	3		(E,E,S)			$l = 0$	$m \neq 0$	(I.4)		(B)
	2		(E,T)				$m = 0$	(I.5)		
	3		(E,E,S)		$n = 0$	$l \neq 0$	$m^2 + 4l > 0$	(I.6)		(C)
	1		(S)				$m^2 + 4l < 0$	(I.7)		
	2		(S,S)				$m^2 + 4l = 0$	(I.8)		
	3		(E,S,S)			$l = 0$	$m \neq 0$	(I.9)		(B)
	2		(S,H)				$m = 0$, $a \neq 0$	(I.10)		
	2		(S,T)				$m = 0$, $a = 0$	(I.11)		
	2		(E,S)		$n = 1$	$l \neq 0$	$m \neq 0$	(I.12)		(A)
	1		(S)				$m = 0$	(I.13)		
	2		(S,S)			$l = 0$	$m \neq 0$	(I.14)		(B)
	∞		degen.				$m = 0$	(I.15)		

In Table 1, we have:

(A) The polynomial $lx^4 + ax^3 + (d-m)x^2 - bx + n$ has no real roots different from zero.

(B) Either $a = 0$, $d - m \neq 0$ and $b^2 - 4n(d-m) < 0$,
or $a = b = n = 0$ and $d - m \neq 0$, or $a = d - m = b = 0$ and $n \neq 0$,
or $a = d - m = n = 0$ and $b \neq 0$, or $a \neq 0$, $n = 0$ and $(d-m)^2 + 4ab < 0$,
or $a \neq 0$ and $n = b = d - m = 0$,

(C) Either $b = 0$ and $a^2 - 4l(d-m) < 0$, or $b = a = d - m = 0$.

TABLE 2. The singularities at infinity for system (II).

System (II) has at infinity	3	singularities of type	(E,E,E)	if	$n \neq 0, 1$	$l \neq 0$	$m^2 - 4(n-1)l > 0$	(II.1)	with	(A)
	1		(E)				$m^2 - 4(n-1)l < 0$	(II.2)		
	2		(E,S)				$m^2 - 4(n-1)l = 0$	(II.3)		
	3		(E,E,S)			$l = 0$	$m \neq 0$	(II.4)		(B)
	2		(E,T)				$m = 0$	(II.5)		
	3		(E,E,S)		$n = 0$	$l \neq 0$	$m^2 + 4l > 0$	(II.6)		(C)
	1		(S)				$m^2 + 4l < 0$	(II.7)		
	2		(S,S)				$m^2 + 4l = 0$	(II.8)		
	3		(E,S,S)			$l = 0$	$m \neq 0$	(II.9)		(D)
	2		(S,T)				$m = 0$	(II.10)		
	2		(E,S)		$n = 1$	$l \neq 0$	$m \neq 0$	(II.11)		(A)
	1		(S)				$m = 0$	(II.12)		
	2		(S,S)			$l = 0$	$m \neq 0$	(II.13)		(B)
	∞		degen.				$m = 0$	(II.14)		

In Table 2, we have:

(A) $b^2 - 4nd < 0$ and $a^2 - 4\,ld < 0$,

(B) $a = 0$ and $b^2 - 4nd < 0$,

(C) $b = 0$ and $a^2 - 4\,ld < 0$,

(D) $a = b = 0$ and $d \neq 0$.

TABLE 3. The singularities at infinity for system (III).

System (III) has at infinity	3	singularities of type	(E,E,E)	if	$n \neq 0$	$(m-1)^2 - 4nl > 0$	(III.1)	with	(A)
	1		(E)			$(m-1)^2 - 4nl < 0$	(III.2)		
	2		$(E,\dot{S})$			$(m-1)^2 - 4nl = 0$	(III.3)		
	2		(E,H)		$n = 0$		(III.4)		(B)

In Table 3, we have:

(A) the polynomial $nx^4 - mx^3 + (l-b)x^2 + ax + d$ has no real roots,

(B) either $m = 0$, $l - b \neq 0$ and $a^2 - 4d(l-b) < 0$, or $m = l - b = a = 0$ and $d \neq 0$.

TABLE 4. The singularities at infinity for system (IV).

System (IV) has at infinity	3	singularities of type	(E,S,S)	if	$n \neq 0$	$m^2 - 4nl > 0$	(IV.1)
	1		(E)			$m^2 - 4nl < 0$	(IV.2)
	2		(E,H)			$m^2 - 4nl = 0$, $4an^2 - 2bmn - m^2 \neq 0$	(IV.3)
	2		(E,T)			$m^2 - 4nl = 0$, $4an^2 - 2bmn - m^2 = 0$	(IV.4)
	2		(S,H)		$n = 0$	$m \neq 0$	(IV.5)
	1		(H)			$m = 0$, $l \neq 0$	(IV.6)
System (IV) is linear when						$n = m = l = 0$	(IV.7)

In Table 4, we have always either $l \neq 0$ and $a^2 - 4ld < 0$, or $l = a = 0$ and $d \neq 0$.

TABLE 5. The singularities at infinity for system (V).

System (V) has at infinity	3	singularities of type	(E,E,E)	if	$n \neq 0$	$(m-1)^2 - 4nl > 0$	(V.1)	with	(A)
	1		(E)			$(m-1)^2 - 4nl < 0$	(V.2)		
	2		(E,S)			$(m-1)^2 - 4nl = 0$	(V.3)		
	2		(E,T)		$n = 0$		(V.4)		(B)

In Table 5 we have:

(A) $(m+b)^2 - 4n(d+a+l) < 0$ and $(m-b)^2 - 4n(d-a+l) < 0$,

(B) $m = b = 0$ and $d + l \neq \pm a$.

TABLE 6. The singularities at infinity for system (VI).

System (VI) has at infinity	3	singularities of type	(E,E,E)	if	$n \neq 0$	$(m-1)^2 - 4nl > 0$	(VI.1)
	1		(E)			$(m-1)^2 - 4nl < 0$	(VI.2)
	2		(E,S)			$(m-1)^2 - 4nl = 0$	(VI.3)
	2		(E,T)		$n = 0$	$m \neq 1$	(VI.4)
	1		(T)			$m = 1,\ l \neq 0$	(VI.5)
	∞		degen.			$m = 1,\ l = 0$	(VI.6)

TABLE 7. The singularities at infinity for system (VII).

System (VII) has at infinity	3	singularities of type	(E,E,E)	if	$n \neq 0$	$(m-1)^2 - 4nl > 0$	(VII.1)	with	$b^2 - 4nd < 0$
	1		(E)			$(m-1)^2 - 4nl < 0$	(VII.2)		
	2		(E,S)			$(m-1)^2 - 4nl = 0$	(VII.3)		
	2		(E,T)		$n = 0$	$m \neq 1$	(VII.4)		$b = 0,\ d \neq 0$
	1		(T)			$m = 1,\ l \neq 0$	(VII.5)		
	∞		degen.			$m = 1,\ l = 0$	(VII.6)		

TABLE 8. The singularities at infinity for system (VIII).

System (VIII) has at infinity	3	singularities of type	(E,S,S)	if	$n \neq 0$	$m^2 - 4nl > 0$	(VIII.1)	with	$b^2 - 4nd < 0$
	1		(E)			$m^2 - 4nl < 0$	(VIII.2)		
	2		(E,H)			$m^2 - 4nl = 0,\ 2an - (b-1)m \neq 0$	(VIII.3)		
	2		(E,T)			$m^2 - 4nl = 0,\ 2an - (b-1)m = 0$	(VIII.4)		
	2		(S,T)		$n = 0$	$m \neq 0$	(VIII.5)		$b = 0,\ d \neq 0$
	1		(T)			$m = 0,\ l \neq 0$	(VIII.6)		
System (VIII) is linear when						$n = m = l = 0$	(VIII.7)		

TABLE 9. The singularities at infinity for system (IX).

System (IX) has at infinity	3	singularities of type	(E,S,S)	if	$n \neq 0$	$m^2 - 4nl > 0$	(IX.1)
	1		(E)			$m^2 - 4nl < 0$	(IX.2)
	2		(E,H)			$m^2 - 4nl = 0,\ 2an - bm \neq 0$	(IX.3)
	2		(E,T)			$m^2 - 4nl = 0,\ 2an - bm = 0$	(IX.4)
	2		(S,T)		$n = 0$	$m \neq 0$	(IX.5)
	1		(T)			$m = 0,\ l \neq 0$	(IX.6)
System (IX) is linear or of degree zero when $n = m = l = 0$							(IX.7)

TABLE 10. The singularities at infinity for system (X).

System (X) has at infinity	1	singularities of type	(E)	if	$n \neq 0$	$m^2 - 4nl < 0$ and either $lD > 0$ or $D = 0$	(X.1)
	2		(E,T)			$m^2 - 4nl = 2an - bm = D = 0,\ b^2 - 4nd < 0$	(X.2)
	1		(T)		$n = 0$	$m = b = 0,\ l \neq 0$ and $a^2 - 4ld < 0$	(X.3)
System (X) is of degree zero when					$n = l = m = a = b = 0,\ d \neq 0$		(X.4)

In Table 10 the equation $d + ax + by + lx^2 + mxy + ny^2 = 0$ has no real solutions and

$$D = \begin{vmatrix} l & m/2 & a/2 \\ m/2 & n & b/2 \\ a/2 & b/2 & d \end{vmatrix}$$

TABLE 11. The classification of the singularities at infinity for properly chordal quadratic systems.

(E,E,E)	(I.1), (II.1), (III.1), (V.1), (VI.1), (VII.1).
(E,E,S)	(I.4), (I.6), (II.4), (II.6).
(E,S,S)	(I.9), (II.9), (IV.1), (VIII.1), (IX.1).
(E,S)	(I.3), (I.12), (II.3), (II.11), (III.3), (V.3), (VI.3), (VII.3).
(S,S)	(I.8), (I.14), (II.8), (II.13).
(E)	(I.2), (II.2), (III.2), (IV.2), (V.2), (VI.2), (VII.2), (VIII.2), (IX.2), (X.1).
(S)	(I.7), (I.13), (II.7), (II.12).
(E,H)	(III.4), (IV.3), (VIII.3), (IX.3).
(E,T)	(I.5), (II.5), (IV.4), (V.4), (VI.4), (VII.4), (VIII.4), (IX.4), (X.2).
(S,H)	(I.10), (IV.5).
(S,T)	(I.11), (II.10), (VIII.5), (IX.5).
(H)	(IV.6).
(T)	(VI.5), (VII.5), (VIII.6), (IX.6), (X.3).
Degenerate	(I.15), (II.14), (VI.6), (VII.6)

$$(\mathrm{D.1})\begin{cases}\dot{x} = 1 + xy,\\ \dot{y} = y^2,\end{cases} \qquad (\mathrm{D.4})\begin{cases}\dot{x} = 1,\\ \dot{y} = y,\end{cases}$$

$$(\mathrm{D.2})\begin{cases}\dot{x} = xy,\\ \dot{y} = 1 + by + y^2,\end{cases} \qquad (\mathrm{D.5})\begin{cases}\dot{x} = 1,\\ \dot{y} = 0,\end{cases}$$

$$(\mathrm{D.3})\begin{cases}\dot{x} = y,\\ \dot{y} = 1,\end{cases}$$

where $|b| < 2$.

PROOF. System (I.15) with $d = 0$ is (D.1). If $d \neq 0$, then $d > 0$ (because $b^2 - 4d < 0$). Therefore, the change $x_1 = d^{1/2}x$, $y_1 = d^{-1/2}y$, $t_1 = d^{1/2}t$ converts the system to the form

$$\begin{cases}\dot{x} = 1 + xy,\\ \dot{y} = 1 + b'y + y^2,\end{cases} \tag{14}$$

and after the change $x_1 = x - b' - y$, $y_1 = y$ the system becomes (D.2).

System (II.14) becomes (D.2) in a similar way to system (I.15) with $d \neq 0$.

System (IV.7) becomes (D.3) if $b = 0$. If $b \neq 0$ the change $x_1 = bd^{-1}(y - bx)$, $y_1 = d + by$, $t_1 = bt$ transforms the system to (D.4).

The translation $x_1 = x + b$, $y_1 = y + a$ converts system (VI.6) to

$$\begin{cases} \dot{x} = x^2 - 2bx + b^2 + 1, \\ \dot{y} = d' + xy. \end{cases} \tag{15}$$

If $d' \neq 0$, then using the change $x_1 = (b^2 + 1)^{1/2}(d')^{-1}y$, $y_1 = (b^2 + 1)^{-1/2}x$, $t_1 = (b^2 + 1)^{1/2}t$ system (15) becomes (14). If $d' = 0$, then the change $x_1 = y$, $y_1 = (b^2 + 1)^{-1/2}x$, $t_1 = (b^2 + 1)^{1/2}t$ converts (15) to (D.2).

System (VII.6) is equivalent to system (D.1) using the change $x_1 = d^{-1}(y + a)$, $y_1 = x$.

System (VIII.7) becomes(D.4) if $a = 0$. If $a \neq 0$ the change $x_1 = d^{-1}(y - ax)$, $y_1 = ax$ converts the system to (D.4), too.

System (IX.7) becomes (D.4), (D.3) or (D.5) according to $b \neq 0$, $b = 0$ and $a \neq 0$, or $a = b = 0$, respectively. Lastly, system (X.4) is equivalent to (D.5).

THEOREM 3. *The phase portrait of a degenerate, linear or constant* CQS *is homeomorphic (except, perhaps for the orientation) to one of the separatrix configurations shown in* 20, 21, 22, 23 *and* 8 *of* Figure 1. *Furthermore, systems* (D.1)–(D.5) *realize all of these configurations.*

PROOF. It will be sufficient to describe the flow of the systems given in Lemma 2. System (D.1) has the following orbits $x = -(2y)^{-1} - ky$ and $y = 0$; so, its phase portrait is given by 20 of Figure 1.

In U_1 system (D.2) becomes

$$\begin{cases} \dot{y} = (by + z)z, \\ \dot{z} = -yz. \end{cases} \tag{16}$$

If we omit the factor z we can draw the orbits of (16), so the phase portrait of (D.2) is given in Figure 1.21. Lastly, the orbits of systems (D.3), (D.4) and (D.5) are $x = y^2 2^{-1} + k$, $y = k \exp(x)$ and $x = t$, $y = k$; respectively. Hence, 22, 23 and 8 of Figure 1 follow.

4. Phase portraits for CQS with all the singularities of type E and S. Let X be a polynomial vector field. Since the equator of S^2 is invariant by the flow of $p(X)$ (see §2), a singularity at infinity of type E will be a saddle or a node. Similarly, a singularity at infinity of type S will be a saddle, a node or a saddle-node (see Theorems E and S of the appendix).

From Theorems E and S, the stable and unstable separatrices of a saddle p of type E or S form an angle into the point p. So, the equator separates the hyperbolic sectors as in Figure 2. From Theorem S, the equator separates a saddle-node p as in Figure 3. We note that the matrix of the linear part, $Dp(X)(p)$, will be either

$$\begin{bmatrix} \neq 0 & * \\ 0 & 0 \end{bmatrix}, \text{ or } \begin{bmatrix} 0 & * \\ 0 & \neq 0 \end{bmatrix}$$

according to whether p is of type S_1 or S_2, respectively.

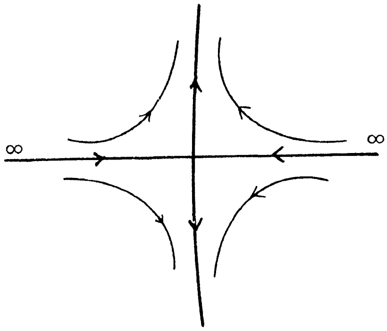

FIGURE 2. A saddle of type E or S on the equator of S^2 (We can reverse the orientation of the orbits).

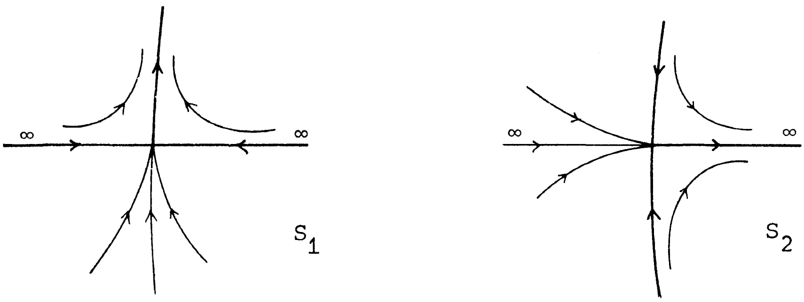

FIGURE 3. The saddle-nodes of type S_1 or S_2 of $p(X)$ on the equator of S^2 (We can reverse the orientation of the orbits).

The following lemma generalizes Lemma 6 of [**9**].

LEMMA 4. *Let* $X(x, y) = (P(x, y), Q(x, y))$ *be a* QS. *Suppose that* X *has at the equator a singularity of type* S_2, H *or* T. *Then*

(1) *there exist at most two singularities at the equator*

(2) *if there is another singularity at the equator it can not be of type* S_2, H *or* T.

PROOF. Since there is a singularity at infinity of type S_2, H or T, the infinity of X is nondegenerate. Without loss of generality we can choose a coordinate system (x, y) for X such that all the singularities at the equator are contained in U_1.

We write X in the form of equation (2). From (8), X becomes

$$
(17) \quad \begin{cases} \dot{y} = l_2 + (m_2 - l_1)y + a_2 z + (n_2 - m_1)y^2 + (b_2 - a_1)yz + d_2 z^2 \\ \qquad - n_1 y^3 - b_1 y^2 z - d_1 z^2 y, \\ \dot{z} = - l_1 z - a_1 z^2 - m_1 yz - n_1 zy^2 - b_1 yz^2 - d_1 z^3. \end{cases}
$$

If $(y, 0)$ is a singularity of (17), then it satisfies the equation

$$
(18) \qquad n_1 y^3 - (n_2 - m_1)y^2 - (m_2 - l_1)y - l_2 = 0
$$

The matrix of the linear part, $Dp(X)\,(y, 0)$, is given by

$$
\begin{bmatrix} (m_2 - l_1) + 2(n_2 - m_1)y - 3n_1 y^2 & a_2 + (b_2 - a_1)y - b_1 y^2 \\ 0 & - l_1 - m_1 y - n_1 y^2 \end{bmatrix}.
$$

Since $(y, 0)$ is of type S_2, H or T we have that $(m_2 - l_1) + 2(n_2 - m_1)y - 3n_1 y^2 = 0$. Therefore y is a multiple root of (18). Hence (1) follows.

Let $(y_1, 0)$ and $(y_2, 0)$ be two singularities at infinity of type S_2, H or T with $y_1 \neq y_2$. Then, we have $n_1 = n_2 - m_1 = m_2 - l_1 = l_2 = 0$, and this is a contradiction because the equator would be degenerate.

THEOREM 5 (*Poincaré's index theorem, see* [8]). *The index of a surface relative to any vector field X with at most a finite number of singularities,*

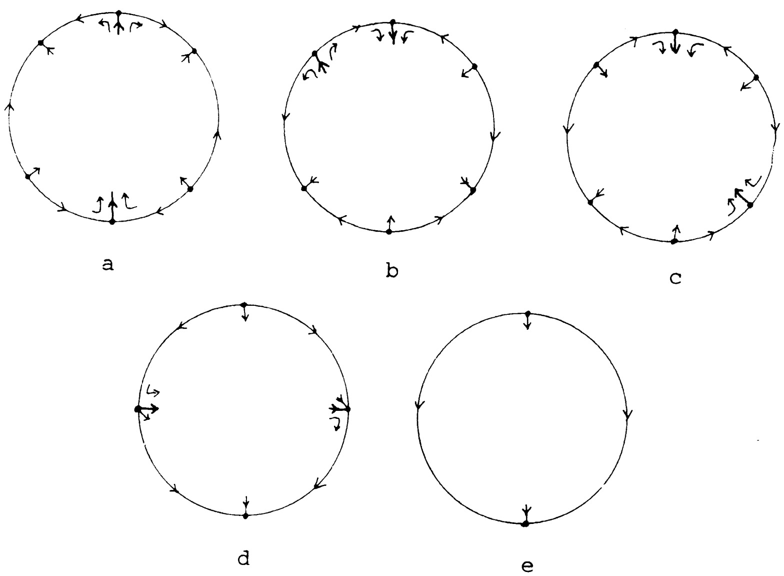

FIGURE 4. (a) Singularities of type (E, E, E), (E, E, S) or (E, S, S) with indices $(+1, -1, +1)$

(b) and (c) Singularities of type (E, S, S) with indices $(+1, 0, 0)$

(d) Singularities of type (E, S) or (S, S) with indices $(+1, 0)$

(e) Singularities of type (E) or (S) with index $(+1)$.

is independent of the vector field and equal to the Euler-Poincaré characteristic of the surface.

LEMMA 6. *Let X be a* CQS *with all the singularities of type E or S. Then the behaviour of X near the equator of the Poincaré sphere is shown in Figure* 4.

PROOF. From Theorem E, the index of a singularity of type E is $+1$ or -1. While, by Theorem S, the index will be $+1$, -1 or 0 for a singularity of type S.

We note that, by Theorems E and S, and Figure 2 we have that a singularity of type E or S with the same index has the same behavior with respect to the equator.

Suppose that X has three singularities at the equator. From Theorem 5, their indices are $(+1, -1, +1)$ or $(+1, 0, 0)$. If the indices are $(+1, -1, +1)$, Figure 4.a follows. Now, assume that the indices are $(+1, 0, 0)$. By Lemma 4, the two saddle-nodes are of type S_1. Hence, Figures 4.b and 4.c follow.

If X has two singularities at the equator, then the indices are $(+1, 0)$ and Figure 4.d follows. Lastly, Figure 4.e shows the equator with a unique singularity.

Markus **[10]** has shown that in the plane two C^1 systems with isolated singularities and no limit separatrices are equivalent if and only if their separatrix configuration are equivalent. Thus, if a QDS X is such that $p(X)$ has only a finite number of singularities, it suffices to determine all possible separatrix configurations in order to determine all possible phase portraits.

THEOREM 7. *The phase portrait of* CQS *with all the singularities of type E and S is homeomorphic (except, perhaps for orientation) to one of the separatrix configurations shown in* 1, 2, 3, 4, 5, 6, 7 *and* 8 *of* Figure 1.

The proof follows easily from Lemma 6. We note that 1 and 2 of Figure 1 have indices $(+1, -1, +1)$, and 3 and 4 have indices $(+1, 0, 0)$.

TABLE 12.

	1		(II.1) with $n > 1$
	2	of	(II.1) with $n \in (0, 1)$
	3	Figure 1	$\dot{x} = y, \dot{y} = 1 - y + xy + y^2$
The	4	is	$\dot{x} = y, \dot{y} = -1 - y + xy + y^2$
configuration	5	realizable	(II.3), (II.8), (V.3), (VII.3)
	6	by the	(II.11), (II.13)
	8	systems	all the systems of type (E) or (S) of Table 11

It is not difficult to verify Table 12. So, the eight possible configurations given by Theorem 7 except, perhaps, configuration 7 are realizable for CQS with all the singularities of type E and S. The configuration 7 will be realized later on.

5. Phase portrait for CQS with some singularity of type *H*. From Table 11 a singularity of type H at infinity for a CQS appears in the cases (E, H), (S, H) and (H).

We shall need the following lemmas.

LEMMA 8. *For $n \neq 0$ the system*

$$\begin{cases} \dot{x} = P(x, y), \\ \dot{y} = d + ax + by + lx^2 + mxy + ny^2, \end{cases}$$

with the change of variables $x_1 = x$, $y_1 = y + m(2n)^{-1}x$ becomes

$$\begin{cases} \dot{x} = P(x, y - m(2n)^{-1}x), \\ \dot{y} = d + (2an - mb)(2n)^{-1}x + by + (4nl - m^2)(4n)^{-1}x^2 \\ \qquad + ny^2 + m(2n)^{-1}P(x, y - m(2n)^{-1}x). \end{cases}$$

The proof follows easily.

LEMMA 9. *For $n \neq 0$ the system*

$$\begin{cases} \dot{x} = P(x, y), \\ \dot{y} = d + ax + by + ny^2, \end{cases}$$

with $a = 0$ can be written as

$$\begin{cases} \dot{x} = x^2 + 1, \\ \dot{y} = Q(x, y), \end{cases} \qquad \begin{cases} \dot{x} = x^2, \\ \dot{y} = Q(x, y), \end{cases} \qquad \begin{cases} \dot{x} = x^2 - 1, \\ \dot{y} = Q(x, y), \end{cases}$$

according to whether $k = b^2 - 4nd$ is negative, zero or positive. And if $a \neq 0$ it can be written as

$$\begin{cases} \dot{x} = y + x^2, \\ \dot{y} = Q(x,y). \end{cases}$$

PROOF. This follows easily with the changes of variables:

$$x_1 = 2n|k|^{-1/2}(y + b(2n)^{-1}), \qquad y_1 = 4nak^{-1}x - k\,|k|^{-1},$$
$$t_1 = 2^{-1}\,|k|^{1/2}\,t \text{ if } k \neq 0,\ a \neq 0;$$

$$x_1 = 2n|k|^{-1/2}(y + b(2n)^{-1}), \qquad y_1 = x, \qquad t_1 = 2^{-1}\,|k|^{1/2}\,t \text{ if } k \neq 0, a = 0;$$

$$x_1 = y + b(2n)^{-1}, \qquad y_1 = an^{-1}x, \qquad t_1 = nt \text{ if } k = 0,\ a \neq 0;$$

and

$$x_1 = y + b(2n)^{-1}, \qquad y_1 = x, \qquad t_1 = nt \text{ if } k = a = 0.$$

LEMMA 10. *Let $X(x, y) = (P(x, y), Q(x, y))$ be a* CQS *with singularities at the equator of type (E, H). Then the local phase portrait of the singularity of type H is given by* Figure 5.a.

PROOF. From Theorems E, H and 5, the indices are $+1$ for the singularity of type E and 0 for the one of type H. Again, from Theorem H we have that the singularity of type H is a saddle-node.

Since the parabolic sector of a saddle-node of type H reaches the singularity in a unique direction given by the one of the three separatrices (see Theorem H) and the infinity of a QS is invariant, we have that a saddle-node of type H at the equator for a QS must be as in Figure 5.

Since the singularity of type E is a node, the orbits on the equator are as in Figure 6. So, the configurations for the saddle-node shown in Figures 5.c and 5.d are not possible.

By Lemma 8 and 9 systems (IV.3), (VIII.3) and (IX.3) can be transformed into system (III.4). This system has the singularity of type H at the point (0, 0) of the local chart (U_2, F_2). In this local chart the equation becomes

$$\text{(19)} \qquad \begin{cases} \dot{x} = z + x^2 - bxz - lx^3 - ax^2z - dxz^2, \\ \dot{z} = -bz^2 - lx^2z - axz^2 - dz^3. \end{cases}$$

We apply to this system two successive changes of variables $x = x$, $z = w_1x$ and $x = x$, $w_1 = wx$. Therefore, system (19) is equivalent (after omitting a common factor x) to

$$\begin{cases} \dot{x} = x + wx - lx^2 - bx^2w - ax^3w - dx^4w^2, \\ \dot{w} = -2w - 2w^2 + lxw + bxw^2 + ax^2w^2 + dx^3w^3. \end{cases}$$

This system has exactly two singularities on the w-axis, from Theorems E and S; they are a saddle at (0, 0) and a saddle-node at $(0, -1)$, and the saddle-node has the two hyperbolic sectors either to the right or to the left of the invariant w-axis. So, from Figure 7, the lemma follows. (For more details, see pp. 335–336 of [**2**]).

The next theorem follows immediately from Lemma 10.

THEOREM 11. *The phase portrait of a* CQS *with singularities at the equator of type (E, H) is homeomorphic (except, perhaps for orientation) to the configuration shown in* Figure 1.9.

Now we shall study the case (S, H). This case appears in systems (I.10) and (IV.5). System (I.10) with the change of variables $x_1 = y$, $y_1 = d + ax + by$ becomes a system contained in (IV.5). By Theorem H we have that the singularity of type H of system (IV.5) is the union of a hyperbolic and an elliptic sector (because $\alpha = 3$, $\beta = 1$, $a = -m^2$, $b =$

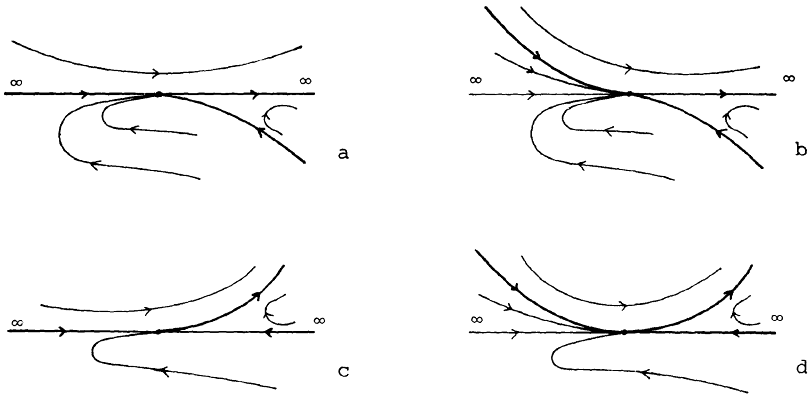

FIGURE 5. The possible saddle-nodes of type H on the equator of S^2. In fact, only configuration a is possible (We can reverse the orientation of the orbits).

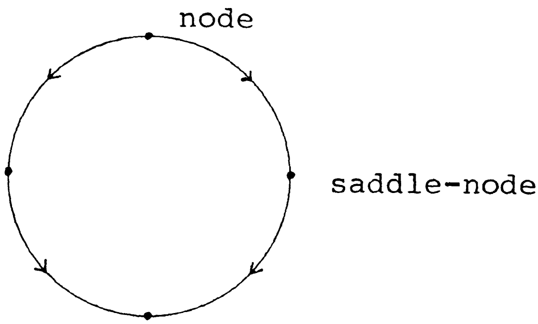

FIGURE 6.

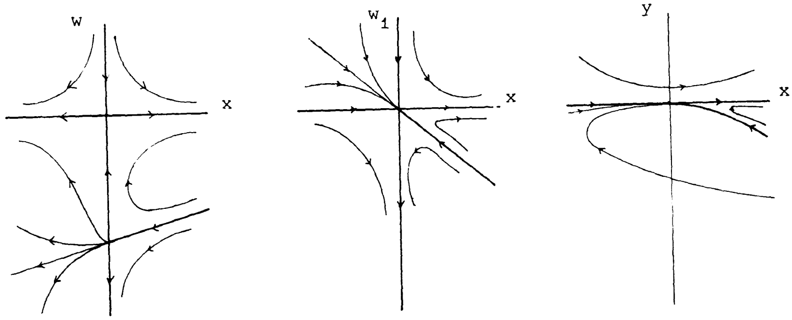

FIGURE 7.

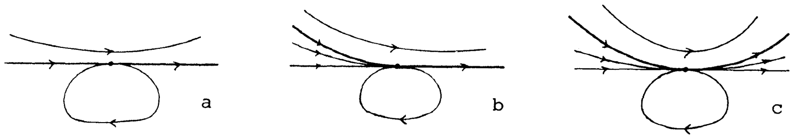

FIGURE 8. The possible configurations for a singularity of type H at infinity which is the union of a hyperbolic and elliptic sector (We can reverse the orientation of the orbits).

$-3m$). Then, by Lemma 4, the singularity of type S is a saddle-node of type S_1.

From Theorem H, the orbits which reach or leave the singularity of type H do so in a unique direction. So, we have that the local behaviour at the singularity H must be as shown in Figure 8. The following lemma tells us that, for (IV.5), the unique possibility for a singularity of type H is given by Figure 8.a.

LEMMA 12. *If $m \neq 0$ the semiaxes $x > 0$ and $x < 0$ are separatrices of the origin for the system*

$$\text{(20)} \qquad \begin{cases} \dot{x} = z - mx^2 - bxz - lx^3 - ax^2z - dxz^2, \\ \dot{z} = -mxz - bz^2 - lx^2z - axz^2 - dz^3. \end{cases}$$

PROOF. We apply to system (20) two successive changes of variables $x = x$, $z = w_1x$ and $x = x$, $w_1 = wx$. Therefore (20) is equivalent (after omitting a common factor x) to

$$\begin{cases} \dot{x} = x(-m - lx + w - bxw - ax^2w - dx^3w^2), \\ \dot{w} = w(m + lx - 2w + bxw + ax^2w + dx^3w). \end{cases}$$

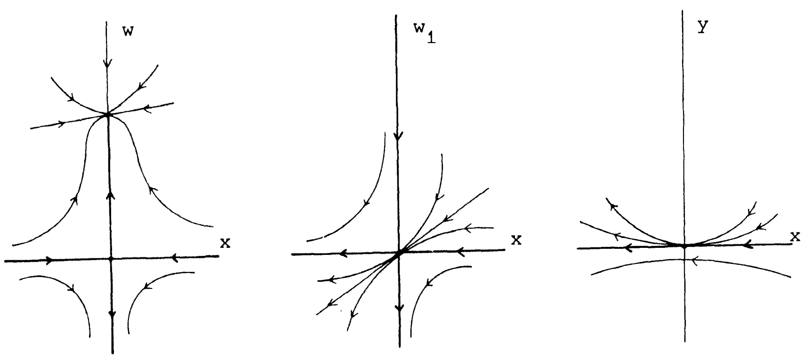

FIGURE 9.

This system has exactly two singularities on the w-axis, a saddle at (0, 0) and a node at $(0, 2^{-1}m)$. So, from Figure 9, the lemma follows.

In short, since the singularity of type H of system (IV.5) in the local chart U_2 is the origin of system (20), we have the following theorem.

THEOREM 13. *The phase portrait of a* CQS *with singularities at infinity of type* (S, H) *is homeomorphic* (*except, perhaps for orientation*) *to the configuration shown in* Figure 1.10.

Lastly, the case (H) only appears in system (IV.6). By Poincaré's index theorem and Theorem H we have that the singularity of type H is either a topological node or the union of a hyperbolic and an elliptic sector (see Figure 8). The second case implies that the behaviour of the flow on the equator of the Poincaré sphere must be like that in Figure 10 and this fact is impossible for the quadratic systems. So the phase portrait of system (IV.6) is like 8 of Figure 1.

6. Phase portraits for CQS with some singularity of type *T*. From Table 11 we must consider the systems whose singularities at equator are of type (T), (S, T) and (E, T).

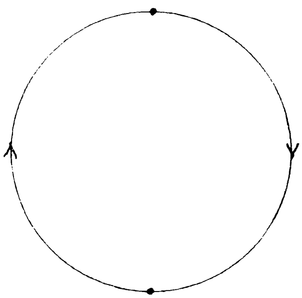

FIGURE 10.

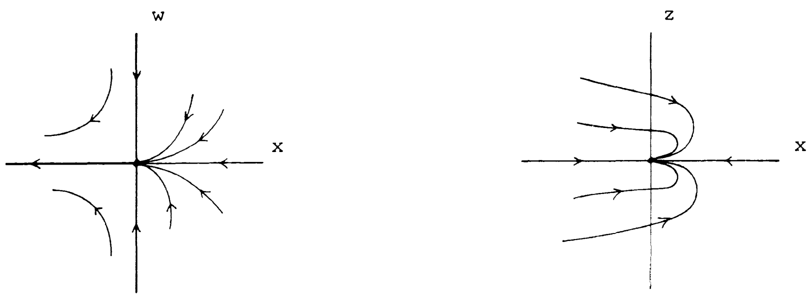

FIGURE 11.

Case (T). In this case we have the following systems (VI.5), (VII.5), (VIII.6), (IX.6) and (X.3).

The system (VI.5) has the equations

$$
\begin{cases} \dot{x} = 1 + x^2, \\ \dot{y} = d' + by + x^2 + xy, \end{cases} \tag{20}
$$

after the change of variables $x_1 = x$, $y_1 = (y + a)l^{-1}$. It has, at the equator, a singularity of type (T) at the point (0, 0) of the local chart (U_2, F_2). In this local chart the system becomes

$$
\begin{cases} \dot{x} = -bxz + z^2 - x^3 - d'xz^2, \\ \dot{z} = -xz - bz^2 - x^2z - d'z^3. \end{cases} \tag{21}
$$

So, the zeros of the equation $z(x^2 + z^2) = 0$ give the directions to reach the singularity of type (T). Then, the unique possible direction to reach this singularity is given by the equator, $z = 0$.

If we make the change of variables $x = x$, $z = wx$, (21) becomes (after omitting a common factor x)

$$
\begin{cases} \dot{x} = -bxw + xw^2 - x^2 - d'x^2w^2, \\ \dot{w} = -w - w^3. \end{cases}
$$

The unique singularity of this system is (0, 0) and, by Theorem S, it is a saddle-node. Since the unique direction to reach the origin of (21) is given by the equator, Figure 11 follows. So Figure 1.8 gives us the phase portrait of (20).

System (VII.5) is given by

$$
\begin{cases} \dot{x} = x^2, \\ \dot{y} = d + ax + lx^2 + xy, \end{cases} \tag{22}
$$

with $d \neq 0$ and $l \neq 0$. We introduce the change $x_1 = l|ld|^{-1/2}x$, $y_1 = |ld|^{-1/2}(a + lx + y)$, $t_1 = l^{-1}|dl|^{1/2}t$. The equations (22) becomes

$$
\begin{cases} \dot{x} = x^2, \\ \dot{y} = \pm 1 + x^2 + xy. \end{cases}
$$

This system has the line $x = 0$ invariant and the other solutions are given by $y = \mp(2x)^{-1} + x\log|x| + kx$. Drawing these solutions for the minus sign we obtain again a phase portrait like that in Figure 1.8. But for the plus sign the phase portrait is shown in 11 of the same Figure.

System (VIII.6) is given by

$$
\begin{cases} \dot{x} = x, \\ \dot{y} = d + ax + lx^2. \end{cases} \tag{23}
$$

with $d \neq 0$ and $l \neq 0$. The transformation $x_1 = |ld^{-1}|^{1/2}x$, $y_1 = l^{-1}|ld^{-1}|$ $(y - ax)$ converts (23) to one of the two forms

$$\begin{cases} \dot{x} = x, \\ \dot{y} = \pm 1 + x^2. \end{cases}$$

This system has the solutions $x(t) = k \exp(t)$, $y(t) = \pm t + k^2 2^{-1} \exp(2t)$. When we choose the plus sign we obtain 8 in Figure 1. But, for the minus sign, the phase portrait is shown in Figure 1.12.

System (IX.6) has the equations

$$\begin{cases} \dot{x} = 1, \\ \dot{y} = d + ax + by + lx^2, \end{cases} \tag{24}$$

with $l \neq 0$. If $b = 0$ the transformation $x_1 = x + a(2l)^{-1}$, $y_1 = l^{-1}(y - (d - a^2(4l)^{-1})(x + a(2l)^{-1}))$ converts (24) to the form

$$\begin{cases} \dot{x} = 1, \\ \dot{y} = x^2. \end{cases} \tag{25}$$

Since the solutions of (25) are $y = 3^{-1}x^3 + k$, its phase portrait is like 8 in Figure 1.

When $b \neq 0$ in (24) the change of variables $x_1 = b(x + a(2l)^{-1})$, $y_1 = b^3 l^{-1}(y + b^{-1}(d - a^2(4l)^{-1}))$, $t_1 = bt$ converts system (24) to the form

$$\begin{cases} \dot{x} = 1, \\ \dot{y} = y + x^2. \end{cases} \tag{26}$$

Since the solutions of (26) are $y = k \exp(x) - x^2 - 2x - 2$ its phase portrait is given in 13 of Figure 1.

Lastly, system (X.3) looks like 8 in Figure 1.

Case (S, T). From Table 11 we must study the systems (I.11), (II.10), (VIII.5) and (IX.5).

System (I.11) can be transformed into systems (VIII.5) or (IX.5) accordingly as $b \neq 0$ or $d \neq 0$ with the changes of variables $x_1 = b^{-1}y$, $y_1 = bx$, $t_1 = bt$ or $x_1 = y$, $y_1 = x$, $t_1 = dt$, respectively. In a similar way, system (II.10) is transformed into system (IX.5) with the change of variables $x_1 = y$, $y_1 = x$, $t_1 = dt$.

Now we consider the system (VIII.5) which has the equations

$$\begin{cases} \dot{x} = x, \\ \dot{y} = d + ax + lx^2 + mxy, \end{cases}$$

with $d \neq 0$ and $m \neq 0$. We introduce the variables $x_1 = mx$, $y_1 = (md)^{-1}$ $(a + lx + my + lm^{-1})$. In the new variables system (VIII.5) becomes

$$\begin{cases} \dot{x} = x, \\ \dot{y} = 1 + xy. \end{cases} \tag{27}$$

Since the solutions of (27) are given by $x(t) = c \exp(t)$, $y(t) = \exp(c \exp(t)) (\int_0^t \exp(-c \exp(t))\, dt + k)$ its phase portrait is shown in Figure 1.14.

System (IX.5) is given by

$$\begin{cases} \dot{x} = 1, \\ \dot{y} = d + ax + by + lx^2 + mxy, \end{cases}$$

with $m \neq 0$. We consider the variables $x_1 = |m|^{1/2}(x + bm^{-1})$, $y_1 = a - 2blm^{-1} + l(x + bm^{-1}) + my$, $t_1 = |m|^{1/2}t$. In the new variables system (IX.5) has the following equations

$$\begin{cases} \dot{x} = 1, \\ \dot{y} = d' \mp xy. \end{cases} \tag{28}$$

Note that we can assume that $d' = 0$ or $d' = 1$. The solutions of (28) are $x(t) = t$, $y(t) = \exp(\mp 2^{-1}t^2)$ $(d' \int_0^t \exp(\pm 2^{-1}t^2)dt + k)$. So the phase portraits of (28) for $d' = 0$, minus sign; $d' = 0$, plus sign; $d' = 1$, minus sign; $d' = 1$, plus sign are given by 15, 16, 15 and 17 of Figure 1, respectively.

Case (E, T). We must study the cases (I.5), (II.5), (IV.4), (V.4), (VI.4), (VII.4), (VIII.4), (IX.4) and (X.2).

By Lemmas 8 and 9, the cases (I.5), (II.5), (IV.4), (VIII.4) and (IX.4) are contained in the cases (V.4), (VI.4) and (VII.4). It is clear that system (X.2) has a phase portrait like Figure 1.15. So we must study these last three cases.

System (V.4) is given by

$$\begin{cases} \dot{x} = x^2 - 1, \\ \dot{y} = d + ax + lx^2, \end{cases}$$

with $d + l \neq \pm a$. If $a = 0$, then the change of variables $x_1 = x$, $y_1 = (d + l)^{-1} (y - lx)$ converts system (V.4) to the form

$$\begin{cases} \dot{x} = x^2 - 1, \\ \dot{y} = 1. \end{cases}$$

The solutions of this system are $x(t) = (1 + k \exp(2t)) (1 - k \exp(2t))^{-1}$, $y(t) = t$. So, its phase portrait is shown in Figure 1.7. Note that this phase portrait was the one not realized in §4.

When $a \neq 0$ we introduce the coordinates $x_1 = x$, $y_1 = a^{-1}(y - lx)$ and system (V.4) has the equations

$$\begin{cases} \dot{x} = x^2 - 1, \\ \dot{y} = x + k, \end{cases} \tag{29}$$

where $k = (d + l)a^{-1}$. Note that we can assume that $k \geqq 0$, $k \neq 1$. This system is solvable. If $k > 1$, then the configuration of (29) is shown in Figure 1.7. If $0 \leqq k < 1$, then its phase portrait is like 18 in Figure 1.

System (VI.4) is given by the equations

$$\begin{cases} \dot{x} = x^2 + 1, \\ \dot{y} = d + ax + by + lx^2 + mxy, \end{cases} \tag{30}$$

where $m \neq 1$. The singularity of type (T) for (30) is the point (0, 0) of the local chart (U_2, F_2) and the system in this chart has the equation

$$\begin{cases} \dot{x} = (1 - m)x^2 - bxz + z^2 - lx^3 - ax^2z - dxz^2, \\ \dot{z} = -mxz - bz^2 - lx^2z - axz^2 - dz^3. \end{cases} \tag{31}$$

So the unique direction to reach this singularity is the direction given by $z = 0$.

Assume $m < 1$. Then the singularity of type (E) is a node. By making the change of variables $x = x$, $z = wx$ in (31), we obtain (after omitting a common factor x)

$$\begin{cases} \dot{x} = (1 - m)x - bxw + xw^2 - lx^2 - ax^2w - dx^2w^2, \\ \dot{w} = -w - w^3. \end{cases} \tag{32}$$

Since the unique singularity of (32) on the w-axis is a saddle and the unique direction to the the origin of (31) is $z = 0$, we obtain Figure 12. So the phase portrait of (30) with $m < 1$ is given by Figure 1.15.

Assume $m > 1$. In this case the origin of (32) is a topological node, and since $z = 0$ is the unique direction to reach the origin of (31), we have that the behaviour of (31) near $z = 0$ is given by Figure 13 but without knowing if there is some elliptic sector. The singularity of type (E) for (30) is the point $(l(1 - m)^{-1}, 0)$ of the local chart (U_1, F_1) and it is a saddle. By making the inner product of the vector field (30) with the vector$(l(1 - m)^{-1}, -1)$ on the line $r = \{(x, l(1 - m)^{-1}x - am^{-1} - blm^{-1}(1 - m)^{-1}) : x \in \mathbf{R}\}$ we obtain $c = l(1 - m)^{-1} - d + bam^{-1} + lb^2(1 - m)^{-1}m^{-1}$. So if $c = 0$, the line r is an invariant straight line connecting the two opposite saddle points, and, by the Poincaré-Bendixson theory on the sphere, the phase portrait of (30) is given in Figure 1.16. If $c \neq 0$, r is a line without contact and so the separatrices of the saddles cannot connect. Hence, from Figure 13, and, again, by the Poincaré-Bendixson theory, the phase portrait of (30) must be like in Figure 1.17.

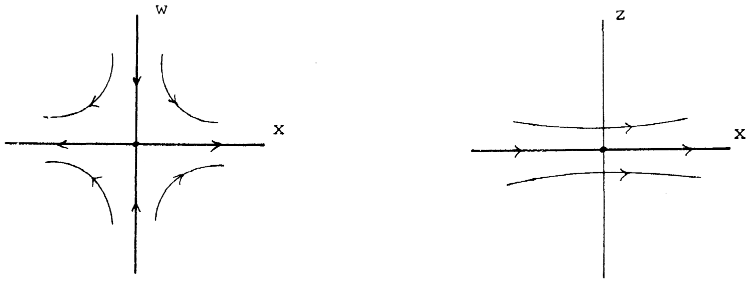

FIGURE 12.

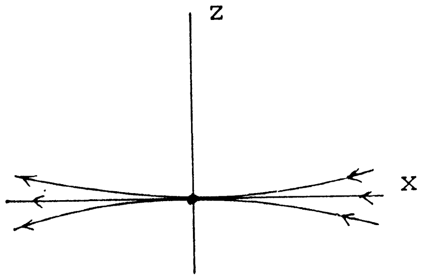

FIGURE 13.

System (VII.4) is given by

$$\begin{cases} \dot{x} = x^2, \\ \dot{y} = d + ax + lx^2 + mxy, \end{cases}$$

with $d \neq 0$ and $m \neq 1$. We introduce the variables $x_1 = x$, $y_1 = y + l(m - 1)^{-1}x$. In the new variables system (VII.4) becomes

$$\begin{cases} \dot{x} = x^2, \\ \dot{y} = d + ax + mxy. \end{cases} \tag{33}$$

If $m \neq 0$ the transformation $x_1 = (md)^{-1}(a + my)$, $y_1 = mx$ converts (33) to

$$\begin{cases} \dot{x} = 1 + xy, \\ \dot{y} = m^{-1}y^2. \end{cases} \tag{34}$$

Since $m \neq 1$ the point (0, 0) in the local chart (U_2, F_2) is a node if $m < 1$ and a saddle if $m > 1$. By the symmetries of system (34) it suffices to study the half-plane $-ym^{-1} > 0$. The solutions are $x = k(-ym^{-1})^m - m((m + 1)y)^{-1}$ for $m \neq -1$ and $x = -y^{-1}(\log|y| + k)$ for $m = -1$.

Drawing these curves we obtain 6, 17 and 19 in Figure 1 for $m \in [-1, 0) \cup (0, 1)$, $m > 1$ and $m < -1$, respectively.

If $m = 0$ and $a \neq 0$, then system (33) becomes

$$\begin{cases} \dot{x} = x^2, \\ \dot{y} = 1 + x, \end{cases} \tag{35}$$

using the transformation $x_1 = ad^{-1}x$, $y_1 = a^{-1}y$, $t_1 = da^{-1}t$.

Lastly, if $m = a = 0$, then system (33) is equivalent to the system

$$\begin{cases} \dot{x} = x^2, \\ \dot{y} = 1. \end{cases} \tag{36}$$

Systems (35) and (36) have the solutions $x(t) = -t^{-1}$, $y(t) = t - \log|t| + k$ and $x(t) = -t^{-1}$, $y(t) = t + k$, respectively. Hence, they have a phase portrait like 6 in Figure 1.

Appendix

This appendix contains the theorems which we use in this paper concerning the local behaviour near a singularity of type E, S or H.

THEOREM E. (see [**2**]). *Let* $(0, 0)$ *be an isolated singularity of the vector field* $X(x, y) = (ax + by + F(x, y), cx + dy + G(x, y))$, *where* F *and* G *are analytic in a neighborhood of the origin and have expansions that begin with second degree terms in* x *and* y. *We say that* $(0, 0)$ *is a nondegenerate singularity if* $ad - bc \neq 0$. *Let* λ_1 *and* λ_2 *be the eigenvalues of* $DX(0, 0)$. *Then the following hold.*

(1) *If* λ_1, λ_2 *are real and* $\lambda_1 \lambda_2 < 0$, *then* $(0, 0)$ *is a saddle* (Figure 14.a) *whose separatrices tend to* $(0, 0)$ *in the directions given by the eigenvectors associated with* λ_1 *and* λ_2.

(2) *If* λ_1, λ_2 *are real and* $\lambda_1 \lambda_2 > 0$, *then* $(0,0)$ *is a node* (Figure 14.b). *If* $\lambda_1 > 0$ (*resp.* < 0) *then it is a source* (*resp. sink*).

(3) *If* $\lambda_1 = \alpha + \beta i$ and $\lambda_2 = \alpha - \beta i$ *with* $\alpha, \beta \neq 0$, *then* $(0,0)$ *is a focus* (Figure 14.c). *If* $\alpha > 0$ (*resp.* $\alpha < 0$) *then it is repellor* (*resp. attractor*).

(4) *If* $\lambda_1 = \beta i$ *and* $\lambda_2 = -\beta i$, *then* $(0, 0)$ *is a linear center, topologically a focus or a center* (Figure 14.d)

The corresponding indices are -1, $+1$, $+1$, $+1$.

THEOREM S. (see Theorem 65 of [**2**]). Let $(0, 0)$ *be an isolated singularity of the system*

$$\begin{cases} \dot{x} = X(x, y), \\ \dot{y} = y + Y(x, y). \end{cases}$$

where X *and* Y *are analytic in a neighborhood of the origin and have ex-*

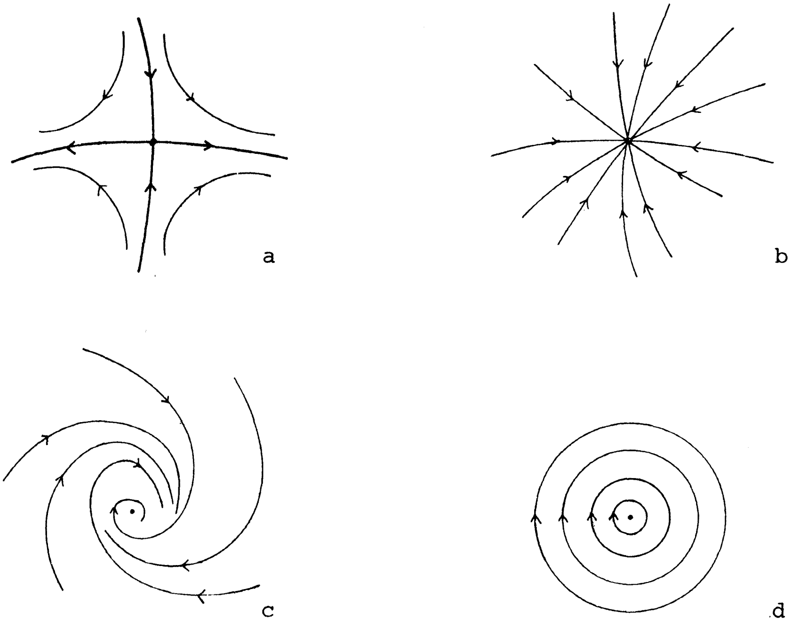

FIGURE 14. The local behaviour near a singularity of type E
(We can reverse the orientation of the orbits).

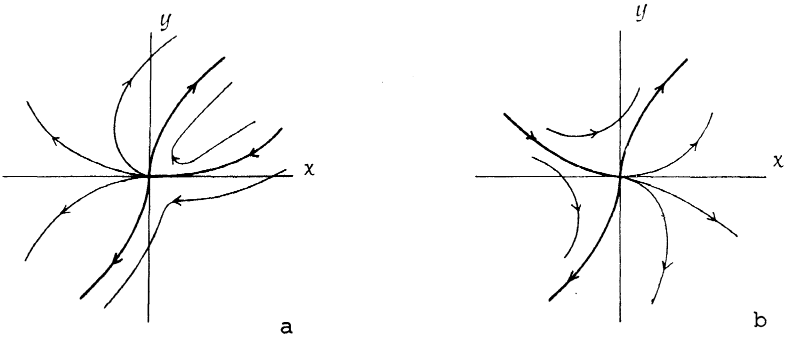

FIGURE 15. The saddle-nodes of type S
(We can reverse the orientation of the orbits).

pansions that begin with second degree terms in x and y. Let $y = f(x)$ be the solution of the equation $y + Y(x, y) = 0$ in the neighborhood of $(0, 0)$, *and assume that the series expansion of the function $g(x) = X(x, f(x))$ has the form $g(x) = a_m x^m + \ldots$, where $m \geqq 2$, $a_m \neq 0$. Then the following are true.*

(1) *If m is odd and* $a_m > 0$, *then* $(0, 0)$ *is a topological node.*

(2) *If m is odd and* $a_m < 0$, *then* $(0, 0)$ *is a topological saddle, two of whose separatrices tend to* $(0, 0)$ *in the directions* 0 *and* π, *the other two in the directions* $\pi/2$ *and* $3\pi/2$.

(3) *If m is even, then* $(0, 0)$ *is a saddle-node, i.e., a singularity whose neighborhood is the union of one parabolic and two hyperbolic sectors, two of whose separatrices tend to* $(0, 0)$ *in the directions* $\pi/2$ *and* $3\pi/2$ *and the other in the direction* 0 *or* π *according to* $a_m < 0$ (Figure 15.a) *or* $a_m > 0$ (Figure 15.b).

The corresponding indices are $+1, -1, 0$ *so they may serve to distinguish the three types.*

THEOREM H. (see **[1]**). *Let* $(0, 0)$ *be an isolated singularity of the system*

$$\begin{cases} \dot{x} = y + X(x, y) \\ \dot{y} = Y(x, y) \end{cases},$$

where X and Y are analytic in a neighborhood of the origin and have expansions that begin with second degree terms in x and y. Let $y = F(x) = a_2x^2 + a_3x^3 + \cdots$ *be a solution of theequa tion* $y + X(x, y) = 0$ *in the neighborhood of* $(0, 0)$, *and assume that we have the following series expansions for the functions* $f(x) = Y(x, F(x)) = ax^{\alpha}(1 + \cdots)$ *and* $\Phi(x) = (\partial X/\partial x + \partial Y/\partial y)(x, F(x)) = bx^{\beta}(1 + \cdots)$ *where* $\alpha \neq 0$, $a \geqq 2$ *and* $\beta \geqq 1$. *Then*

(1) *If* α *is even, and*

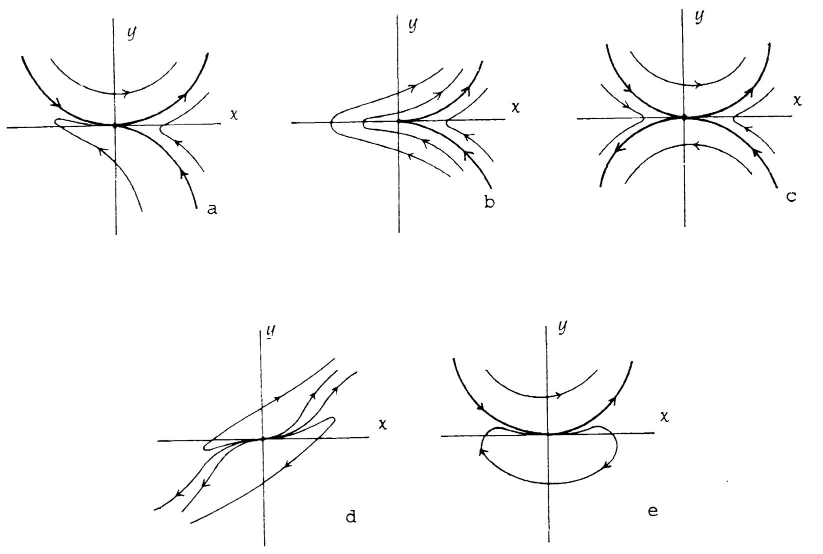

FIGURE 16. The local behaviour near a singularity of type H (We can reverse the orientation of the orbits).

(1.a) *$\alpha > 2\beta + 1$, then the origin is a saddle-node (index 0), see* Figure 16.a.

(1.b) *either $\alpha < 2\beta + 1$ or $\Phi(x) \equiv 0$, then the origin is a singularity whose neighborhood is the union of two hyperbolic sectors (index 0), see* Figure 16.b.

(2) *If α is odd and $a > 0$, then the origin is a saddle (index -1), see* Figure 16.c.

(3) *If α is odd, $a < 0$, and*

(3.a) *either $\alpha > 2\beta + 1$ and β even, or $\alpha = 2\beta + 1$, β even and $b^2 + 4a(\beta + 1) \geqq 0$, then the origin is a node (index $+1$); see* Figure 16.d. *The node is stable if $b < 0$, or unstable if $b > 0$.*

(3.b) *either $\alpha > 2\beta + 1$ and β odd, or $\alpha = 2\beta + 1$, β odd and $b^2 + 4a(\beta + 1) \geqq 0$, then the origin is the union of a hyperbolic and an elliptic sector (index $+1$), see* Figure 16.e.

(3.c) *either $\alpha = 2\beta + 1$ and $b^2 + 4a(\beta + 1) < 0$, or $\alpha < 2\beta + 1$* (or $\Phi(x) \equiv 0$), *then the origin is either a focus, or a center, respectively (index $+1$).*

REFERENCES

1. A.F. Andreev, *Investigation of the behaviour of the integral curves of a system of two differential equations in the neighborhood of a singular point*, Translation of A.M.S. **8** (1958), 183–207.

2. A.A. Andronov, E.A. Leontovich, I.I. Gordon, and A.L. Maier, *Qualitative theory of second-order dynamic systems*, John Wiley & Sons, New York (1973).

3. L.A. Cherkas, *On the conditions for a center for certain equations of the form $yy' = P(x) + Q(x)y + R(x)y^2$*, Diff. Eq. **8** (1972), 1104–1107.

4. W.A. Coppel, *A survey of quadratic systems*, J. Differential Equations **2** (1966), 293–304.

5. E.A.V. Gonzales, *Generic properties of polynomials vector fields at infinity*, Trans. of A.M.S., **143** (1969), 201–222.

6. W. Kaplan, *Regular curve-families filling the plane*, I, Duke Math. J. **7** (1940), 154–185.

7. ———, *Regular curve-families filling the plane*, II, Duke Math. J. **8** (1941), 11–46.

8. S. Lefschetz, *Differential Equations: Geometric Theory*, Interscience, New York (1962).

9. Sheng Li-Ren, *Systèmes quadratiques et composantes de Reeb*, Preprint, Université de Strasbourg (1984).

10. L. Markus, *Quadratic differential equations and non-associative algebras*, Ann. Math. Studies **45** (1960), 185–123.

11. J. Sotomayor, *Curvas definidas por equações diferenciais no plano*, Instituto de Matemática Pura e Aplicada, Rio de Janeiro (1981).

12. G. Tavares dos Santos, *Classification of generic quadratic vector fields with no limit cycles*, Lect. Notes in Math. **597** (1977), 605–640.

13. Ye Yanqian, *Some problems in the qualitative theory of ordinary differential equations*, J. Differential Equations **46** (1982), 153–164.

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