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ON THE SPACES CLASSIFYING COMPLEX VECTOR BUNDLES WITH GIVEN REAL DIMENSION

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ABSTRACT We compute the integral cohomology ring of the space classifying complex vector bundles of real geometric dimension at most n and generalise this to any complex oriented theory; also we rederive an integrality condition of Astey and Gitler for certain K-theory characteristic classes of such bundles and relate these to a "universal unit" of Ray, Switzer, Taylor.

Introduction. We will compute the integral cohomology ring of the space BU_n which classifies complex vector bundles of real geometric dimension at most n. The form of the result depends on the parity of n but in each case there is neither torsion nor non-zero odd degree cohomology. In particular our results give polynomial generators well related to the Chern classes of the canonical complex bundle over BU_n. We generalise this to an arbitrary "complex oriented" cohomology theory $E^*(\ )$ (e.g., $MU^*(\ )$, $KU^*(\ )$); the method we use for this involves calculating the E-homology of the Bott space SO/U and the construction of a dual basis in E-cohomology. Finally we consider a specific element in $KU^\circ(BU_n)$ which has been used by L. Astey and S. Gitler to derive non-sectioning results for bundles; we also explain the relation between this and a "universal unit" of **[12]**.

The results of §1 and §2 are contained in the author's 1980 Ph.D. thesis and an earlier preprint (October 1980). §4 contains results found after conversations with S. Gitler. There is some overlap with the results of **[6]**, in particular the idea of the proof of *Theorem* (*2.2*) is the same although we give our version to highlight certain details we require for later use.

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1. For any "stable subgroup" G of the infinite special orthogonal group SO we have G vector bundles and virtual bundles defined using the existence of inclusions (assumed as part of the data)

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$$G(n) \subset SO(n) \text{ and } G(n) \subset G(n+1).$$

See [7], [11], [12].

It is then an interesting question to ask whether a particular G bundle $\xi \to X$ has *real* geometric dimension at most n. This problem arises for example in connection with embeddings and immersions of manifolds. It is standard in topology to reformulate this in terms of the pullback (of fibrations) diagram

(1.1)
$$\begin{array}{ccc} BG_n^k & \xrightarrow{\chi_n^k} & BG(k) \\ {\scriptstyle \rho_n^k}\downarrow & & \downarrow{\scriptstyle Bj_k} \\ BSO(n) & \xrightarrow[Bi_n^k]{} & BSO(K) \end{array}$$

Here $1 \leqq n \leqq k \leqq \infty$.

We can actually define BG_n^k from this diagram where all maps are the obvious ones between the featured classifying spaces. Alternatively, we have as explicit models

(1.2)
$$BG_n^k = E \times_{G(k)} SO(k)/SO(n).$$

Here $SO(\infty) = SO$ and $G(\infty) = G$ and $E = ESO$ denotes a contractible free right SO, and hence $SO(r)$, space for all r; $G(k)$ then acts via the inclusion representation into $SO(k)$; finally $SO(n)$ acts on the left of $SO(k)$ in the obvious way.

Hence for a $G(k)$ bundle $\zeta \to X$ the realification $r\zeta$ has geometric dimension at most n iff its stable classifying map lifts to BG_n^∞.

Note that we can generalise all of the above in two directions:

(a) By taking a stable subgroup of the orthogonal group 0;

(b) By taking $G = \bigcup_{0<n} G(n)$ where there are compatible representations $\rho_n: G(n) \to O(n)$. The main examples are Spin, Pin, Spin^c and Pin^c. We could also use the more general notion of "X-structure" [7]. The details are left to the interested reader. From now on we concentrate on the case $k = \infty$ and will set $BG_n = BG_n^\infty$.

We will use the following notation.

(1.3) $\sigma_n \to BSO(n)$ is the canonical n-plane bundle or virtual bundle if $n = \infty$.

(1.4) $\gamma_n \to BG(n)$ is the canonical $G(n)$ n-plane bundle induced from σ_n by Bj_n; this is virtual if $n = \infty$.

(1.5) $\xi_n \to BG_n$ is the pullback of σ_n by ρ_n.

(1.6) $\zeta_n \to BG_n$ is the pullback of $\gamma \to BG$ by χ_n (this is a G-virtual bundle).

PROPOSITION 1.7. *$r\zeta_n$ is equivalent as a real virtual bundle to $\xi_n - \varepsilon_n^R$, where ε_n^R denotes the n-dimensional trivial real bundle.*

There are obvious maps

(1.8) $$q_n: BG_n \to BG_{n+1}$$

obtained using the inclusions $SO(n) \to SO(n+1)$.

PROPOSITION 1.9. *We have*

$$q_n^*\xi_{n+1} \cong \xi_n + \varepsilon_1^R$$
$$q_n^*\zeta_{n+1} \cong \zeta_n.$$

Furthermore, the map $q_n{:}BG_n \to BG_{n+1}$ is equivalent to the sphere bundle projection $S(\xi_{n+1}) \to BG_{n+1}$.

The proof is easy and follows from the definitions and a well-known analogous result for the map $BSO(n) \to BSO(n+1)$.

PROPOSITION (1.10) *There is an equivalence $h{:}SO/G \simeq BG_1$ with the properties that $\chi \simeq \chi_1.h$, where $\chi{:}SO/G \to BG$ includes the fibre of $BG \to BSO$. Hence*

$$h^*\zeta_1 \cong \chi^*\gamma$$
$$\xi_1 \cong \varepsilon_1^R.$$

The proof is again easily seen from the definitions and a careful comparison of the various bundles involved.

2. We now take $G = U$, the unitary group, with

$$G(2k) = G(2k+1) = U(k) \subset SO(2k+1).$$

We will compute the integral cohomology of the spaces BU_N, and as a corollary in Section 3 will deduce results on their cohomology with respect to any "complex oriented" theory.

Our results will require induction on n to calculate the cohomologies of BU_{2n} and BU_{2n+1}, starting with knowledge of the case BU_1, provided by

LEMMA 2.1. $H^*(BU_1) = \mathbf{Z}[y_1, y_3, \ldots, y_{2k+1}, \ldots]$ *where* $y_{2k+1} = (1/2)\,\chi_1 c_{2k+1}(\zeta_1)$.

In the above $c_k(\ )$denotes the k-th Chern class. The proof uses (1.10) and the fact that SO/U is a Bott space with $SO/U \simeq \Omega_0^2\, B\mathrm{Spin}$, and from [4]

$$H^*SO/U = \mathbf{Z}[y_1', y_3', \ldots, y_{2k+1}', \ldots]$$

where $y_{2k+1}' = (1/2)\, c_{2k+1}(\chi^*\gamma)$.

Recall that the projection $q_k : BU_k \to BU_{k+1}$ has an interpretation as the projection map of the sphere bundle $\xi_{k+1} \to BU_{k+1}$. Hence we can use the Gysin sequence to calculate $H^*(BU_{k+1})$ form $H^*(BU_k)$ since the bundle ξ_{k+1} is orientable.

THEOREM 2.2. *$H^*(BU_{2n}) = \mathbf{Z}[c_1, c_2, \ldots, c_n, w_n, y_{n+1}, y_{n+2}, \ldots, y_{2n-2}, y_{2n-1}, y_{2n+1}, y_{2n+3}, \ldots, y_{2r+1}, \ldots]$ for $r \geqq n$, and $H^*(BU_{2n+1}) = \mathbf{Z}[c_1, c_2, \ldots, c_n, y_{n+1}, y_{n+2}, \ldots, y_{2n}, y_{2n+1}, y_{2n+3}, y_{2n+5}, \ldots]$ for $r \geqq n$, where*

$$c_k = c_k(\zeta_N)$$

$$y_k = \frac{1}{2}\, c_k(\zeta_N)$$

$$w_n = \frac{1}{2}\, [c_n(\zeta_{2n}) - e(\xi_{2n})].$$

Here we use $e(\xi_{2n})$ to denote the Euler class of the bundle ξ_{2n}.

The proof will follow by induction on the integer n in the statement of the Theorem with the aid of (2.1) and the following Lemmas.

LEMMA 2.3. *If*

$$H^*(BU_{2n-1}) = \mathbf{Z}[c_1, c_2, \ldots, c_{n-1}, y_n, y_{n+1}, \ldots, y_{2n-2}, y_{2n-1}, y_{2n+1}, \ldots, y_{2r+1}, \ldots],$$

then

$$H^*(BU_{2n}) = \mathbf{Z}[c_1, c_2, \ldots, c_n, w_n, y_{n+1}, \ldots, y_{2n-2}, y_{2n+1}, y_{2n+1}, y_{2n+3}, \ldots, y_{2r+1}, \ldots].$$

Here all notation agrees with that in (2.2).

LEMMA 2.4. *If*

$$H^*(BU_{2n}) = \mathbf{Z}[c_1, c_2, \ldots, c_n, w_n, y_{n+1}, y_{n+2}, \ldots, y_{2n-2}, y_{2n-1}, y_{2n+1}, y_{2n+3}, \ldots, y_{2r+1}, \ldots],$$ *then*

$$H^*(BU_{2n+1}) = \mathbf{Z}[c_1, c_2, \ldots, c_n, y_{n+1}, y_{n+2}, \ldots, y_{2n}, y_{2n+1}, y_{2n+3}, y_{2n+5}, \ldots, y_{2r+1}, \ldots].$$

Here all notation agrees with that in (2.2).

We thus only need to prove (2.3) and (2.4). We will only sketch the main points.

PROOF OF (2.3): We have a Gysin sequence

$$\cdots \xrightarrow{\psi} H^{*-2n}(BU_{2n}) \xrightarrow{e} H^*(BU_{2n}) \xrightarrow{q} H^*(BU_{2n-1}) \xrightarrow{\psi} \cdots$$

Here e denotes multiplication by the Euler class $e(\xi_{2n})$ (raising degree by $2n$) $q = q_{2n-1}^*$ and ψ is the Gysin boundary map which lowers degree by $2n - 1$. However it is easily seen that this splits into a short exact sequence with ψ identically 0 and $H^*(BU_{2n})$ having trivial torsion and odd degree groups by the induction hypothesis. Now setting w_n to be any element of $H^{2n}(BU_{2n})$ such that $q(w_n) = y_n$ we obtain a relation of the form

$$c_n(\zeta_{2n}) = ae(\xi_{2n}) + bw_n + \theta$$

where a, $b \in \mathbf{Z}$ and θ is decomposable. We have on applying q that $c_n(\zeta_{2n-1}) = by_n + q(\theta)$, hence $b = 2$ and $\theta = 0$ by the induction hypothesis and decomposability. But now we can see that a is odd by, for example, considering the bundle $\gamma_n \to BU(n)$ which has a lift to BU_{2n}. After redefining w_n (if necessary) by adding an even multiple of $e(\xi_{2n})$ we can finally deduce that

$$c_n(\xi_{2n}) = e(\xi_{2n}) + 2w_n.$$

The rest of the proof is routine.

PROOF OF (2.4): Once again our Gysin sequence

$$\cdots \xrightarrow{\psi} H^*(BU_{2n+1}) \xrightarrow{e} H^{*+2n+1}(BU_{2n+1}) \xrightarrow{q} H^{*+2n+1}(BU_{2n}) \xrightarrow{\psi} \cdots$$

splits into a short exact sequence since $e(\xi_{2n+1}) = 0$; this follows by considering the coefficient sequence induced by the multiplication by 2 map on $\mathbf{Z}$ wherein we have $e(\xi_{2n+1}) = \delta w_{2n}(\xi_{2n+1})$, which is 0 since $w_{2n}(\xi_{2n+1}) = \rho_2 c_n(\zeta_{2n+1})$; here δ and ρ_2 are the boundary and reduction maps respectively. We can therefore deduce that there is a class $z \in H^{2n}(BU_{2n})$ such that as an $H^*(BU_{2n+1})$-module

$$H^*(BU_{2n}) = H^*(BU_{2n+1}) \{1, z\}.$$

We also have a single multiplicative relation

$$z^2 = \beta . z + \alpha . 1$$

for some α, $\beta \in H^*(BU_{2n+1})$. Now observe that in fact we can take $\pm w_n$ for z since this class is characterised by the property $\psi(z) = 1$ (see [8]) and $\pm w_n$ also satisfies this by an analysis of im q making use of the induction hypothesis. We then obtain (with careful checking of the effect of the sign chosen)

(2.5) $$w_n^2 = \beta . w_n + \alpha . 1,$$

(2.6) $$\psi(w_n^2) = \beta,$$

by [8] (9.1).

We can compute α and β by the following line of argument. We have three identities:

$$e(\xi_{2n})^2 = p_n(\xi_{2n}) = q[p_n(\xi_{2n+1})]; \tag{2.7}$$

$$e(\xi_{2n}) = c_n(\zeta_{2n}) - 2w_n; \tag{2.8}$$

$$p_n(\xi_{2n}) = (-1)^n[2c_{2n}(\zeta_{2n}) - 2c_1(\zeta_{2n})c_{2n-1}(\zeta_{2n}) + \cdots + (-1)^n c_n(\zeta_{2n})^2]. \tag{2.9}$$

The first of these is a basic relation between Euler and Pontrjagin classes, the second a part of our inductive assumption, and the third uses the definition of the Pontrjagin class as

$$(-1)^n c_{2n}(\xi_{2n} \otimes \mathbf{C})$$

together with the identification

$$\xi_{2n} \otimes \mathbf{C} \cong \zeta_{2n} + \zeta_{2n}^*$$

and the Cartan formula.

Combining these yields

$$\begin{aligned} -4c_n w_n + 4w_n^2 = (-1)^n[2c_{2n}(\zeta_{2n}) &- 2c_1(\zeta_{2n})c_{2n-1}(\zeta_{2n}) \\ &+ \cdots + (-1)^{n-1}2c_{n-1}(\zeta_{2n})c_{n+1}(\zeta_{2n})]. \end{aligned} \tag{2.10}$$

We can now apply ϕ to (2.10), and by appealing to [8] §3, Lemma 1 use

$$\phi(c_n w_n) = c_n \phi(w_n) \tag{2.11}$$

to deduce

$$\beta = c_n. \tag{2.12}$$

This last step requires the facts that im q = ker ϕ and that $H^*(BU_{2n})$ is torsion free. Now (2.5), and (2.12) imply

$$\alpha.1 = w_n^2 - c_n w_n \tag{2.13}$$

in $H^{4n}(BU_{2n})$.

Finally we can combine (2.10) and (2.13) to deduce

$$\frac{1}{2} c_{2n}(\zeta_{2n}) \equiv (-1)^n \alpha.1 \text{ (mod decomposables)} \tag{2.14}$$

in $H^{4n}(BU_{2n})$.

This allows us to define $y_{2n} \in H^{4n}(BU_{2n+1})$ as

$$\begin{aligned} y_{2n} &= \frac{1}{2} c_{2n}(\zeta_{2n+1}) \\ &\equiv \alpha \text{ (mod decomposables)}. \end{aligned}$$

To complete the proof requires a straightforward verification that the elements of $c_1, c_2, \ldots, c_n, y_{n+1}, \ldots, y_{2n-1}, y_{2n}, y_{2n+1}, y_{2n+3}, \ldots$ are indeed a set of polynomial generators.

A number of immediate corollaries follow from Theorem (2.2). In particular results of **[13]** for mod p cohomology and the following form of "splitting principle" for complex bundles with lift to BU_N.

Consider the bundle $\gamma_n \times \chi^* \gamma \to BU(n) \times SO/U$; since $\chi^*\gamma$ is trivial as a real bundle (it is the pullback of the composition $\chi \cdot r{:}SO/U \to BU \to BSO$ which is trivial) there is a lift g of $\gamma_n \times \chi^*\gamma$ to BU_{2n}; set $g' = g_{2n} \cdot g$: $BU(n) \times SO/U \to BU_{2n+1}$. Notice that

$$g^*w_n = \frac{1}{2} g^*[c_n(\zeta_{2n}) - e(\xi_{2n})]$$

$$= c_{n-1}(\gamma_n) \otimes y_1' + \cdots + 1 \otimes \frac{1}{2} c_n(\chi^*\gamma).$$

THEOREM 2.15. *$g^*{:}H^*(BU_{2n}) \to H^*(BU(n) \times SO/U)$ is a monomorphism onto a direct summand; similarly for g'^*.*

The proof is direct from (2.2). Note that this result allows, for example, calculation of the action of the Steenrod algebra on $H^*(BU_N)$ since this is known on $H^*(BU(n) \times SO/U) = H^*(BU(n) \otimes H^*(SO/U)$ with (mod p) coefficients; cf, **[13]**. We can also calculate the Pontrjagin classes of ξ_N in terms of our generators.

3. In this section we will investigate the construction of algebra generators for $E^*(BU_N)$ for a "complex oriented ring spectrum" (E, x^E) with $x^E \in E^2(\mathbf{C}P^\infty)$. The reader is referred to **[1]** for a detailed exposition of the relevant notions. In particular, we have that $E_*(\mathbf{C}P^\infty)$ is the free E_* module on a basis $\{\beta_n^E: n \geqq 0\}$, and that $E_*(\mathbf{C}P^\infty)$ is the power series ring on x^E, $E^*[[x^E]]$ over $E^* = E_{-*}$; moreover, with respect to the E-theory Kronecker product, we have

$$\langle (x^E)^i, \beta_j^E \rangle = \delta_{ij}$$

hence, $\{(x^E)^i\}$ and $\{\beta_j^E\}$ are dual bases over E_*. We also have a canonical formal group law $F^E(X, Y) \in E_*[[X, Y]]$ associated to the pair (E, x^E) and hence a unique "formal inverse" series $[-1]_E(X) \in E_*[[X]]$ with

$$F^E(X, [-1]_E X) = 0.$$

The canonical map $i\colon \mathbf{C}P^\infty = BU(1) \to BU$ induces a monomorphism

$$i_*\colon E_*(\mathbf{C}P^\infty) \to E_*(BU)$$

and this allows us to identify $i_* \beta_n^E$ with β_n^E and obtain

$$E_*(BU) = E_*[\beta_n^E: n \geqq 1].$$

Dually we have

$$E^*(BU) = E^*[[c_n^E : n \geqq 1]]$$

where c_n^E is the n-th universal E-theory Chern-Conner-Floyd class, for which

$$\begin{aligned} i^* c_n^E &= 0, \quad n > 1 \\ &= x^E, \quad n = 1. \end{aligned}$$

These satisfy the Cartan formula, as do the β_n^E, with respect to the canonical diagonals in E-(co)homology. For a complex bundle $\xi \to X$, with classifying map $f: X \to BU$ it is usual to write

$$c_n^E(\xi) = f^* c_n^E.$$

If $\xi \to X$ is a k-dimensional real bundle which is MU-orientable (for example, stably complex) then there is a canonical E-Euler class $e^E(\xi) \in E^k(X)$; if ξ is in fact an n-dimensional complex bundle, then $e^E(\xi) = c_n^E(\xi)$.

Our main result is the following, where we leave the precise definitions of various classes to the body of this section.

THEOREM 3.1. *For a complex oriented ring spectrum* (E, x^E) *we have classes*

$$c_k^E = c_k^E(\zeta_N^E) \in E^{2k}(BU_N).$$

$$w_n^E \in E^{2n}(BU_{2n}),$$

and $y_k^E \in E^{2k}(BU_N)$ *for certain values of* k *depending on* N, *such, that.*

$$\begin{aligned} E^*(BU_{2n}) = E^*[[&c_1^E, \ldots, c_n^E, w_n^E, y_{n+1}^E, y_{n+2}^E, \ldots, \\ &y_{2n-1}^E, y_{2n+1}^E, y_{2n+3}^E, \ldots, y_{2r+1}^E, \ldots]], \end{aligned}$$

$$\begin{aligned} E^*(BU_{2n+1}) = E^*[[&c_1^E, \ldots, c_n^E, y_{n+1}^E, y_{n+2}^E, \ldots, \\ &y_{2n-1}^E, y_{2n}^E, y_{2n+1}^E, y_{2n+3}^E, \ldots, y_{2r+1}^E, \ldots]]. \end{aligned}$$

Moreover, we have

$$2w_n^E \equiv c_n^E(\zeta_{2n}) - e^E(\xi_{2n}) \qquad (\text{mod } \textit{filtration})$$

$$c_k^E(\zeta_N) \equiv 2y_k^E \qquad (\text{mod } \textit{filtration, decomposables})$$

and $h_N^* y_k^E = y_k^E \in E^{2k}(SO/U)$ *whenever this class is defined.*

In this statement we make use of the composites

$$h_N: SO/U \xrightarrow{h} BU_1 \longrightarrow BU_N$$

(see (1.10)). The statements "mod filtration" refers to the skeletal filtration on $E^*(BU_N)$ which agrees with that of the associated Atiyah-Hirzebruch Spectral Sequence (AHSS); similarly, "mod decomposables" refers to the decomposables in the E^*-algebra structure of $E^*(BU_N)$.

We begin by observing that the overall form of (3.1) is suggested by the triviality

PROPOSITION 3.2. *We have*

$$E^*(BU_N) \cong E^* \otimes H^{**}(BU_N),$$

hence $E^*(BU_N)$ is the power series ring on a set of generators in one-to-one correspondence with those of $H^*(BU_N)$.

This is a consequence of a standard AHSS argument. The subtlety is of course to provide good explicit generators.

From now on we will assume given a pair (E, x^E) and often delete E from the notation if no confusion is likely to result. We will also tacitly assume that E_* is torsion free, since by universality of the pair (MU, x^{MU}) this will make the statements of some results rather simpler.

We begin by investigating $E^*(SO/U)$; recall that $BU_1 \cong SO/U$. Our technique will involve a calculation of $E_*(SO/U)$ suggested to the author by Francis Clarke, and then the construction of elements dual to certain basis elements for this algebra.

Consider the bundle $(\gamma - \gamma^*) \to BU$; since the realification $r(\gamma - \gamma^*)$ is trivial there is a lift to the fibre of Bj: $BU \to BSO \to$ but this fibre is SO/U included by χ; $SO/U \to BU$. Indeed, such a lift ϕ: $BU \to SO/U$ is unique up to homotopy. We can take both χ and ϕ to be infinite loop maps, which induce E_* algebra maps on E-homology.

PROPOSITION 3.3. *We have* $E_*(SO/U) = E_*[\theta_n^E | n \geqq 1]/\langle\theta^E(T)\theta^E([-1]_E(T)) = 1\rangle$ *where* $\theta_n^E = \phi_*(\beta_n^E)$ *and*

$$\theta^E(T) = \sum_{0\leq n} \theta_n^E T^n \in E^*(SO/U)[[T]].$$

The notation

$$\langle\theta^E(T)\theta^E([-1]_E(T)) = 1\rangle$$

signifies the ideal in $E_*(SO/U)$ generated by the coefficients of the series

$$\theta^E(T)\theta^E([-1]_E(T)) - 1.$$

PROOF OF (3.3). We have $E_*(BU) = E_*\,[\beta_n | 1 \leqq n]$. Also we have that χ_*: $E_*(SO/U) \to E_*(BU)$ is a monomorphism (by the corresponding result in the case $E = H$ and the collapsing of the relevant AHSS — see **[11]**, **[12]**. Similarly ϕ_*: $E_*(BU) \to E_*(SO/U)$ is onto. Note that $\chi \cdot \phi$ classifies $\gamma - \gamma^*$ and hence

$$\chi_*\phi_*(\beta(T)) = \beta(T)\beta([-1](T))^{-1}.$$

This follows from basic properties of the series $\beta(T) = \sum_{0\leq n} \beta_n T^n$ related to the actions of the diagonal, Whitney sum and bundle conjugation (the last relies on the fact that $\beta(T)$ arises in $E^*(CP^\infty)[[T]]$ together with a formula to be found in [10]). Hence

$$\chi_*(\theta(T)\theta([-1](T)) = 1.$$

Since χ_* is a monomorphism we get the relation

$$\theta(T)\theta([-1](T)) = 1.$$

The remaining details make use of the collapsing of the relevant AHSS.

For a space x with $E_*(X \times X) \cong E_*(X) \otimes E_*E_*(X)$ let $\Delta: X \to X \times X$ denote the diagonal and let

$$PE_*(X) = \{x \in \tilde{E}_*(X) | \Delta_*(x) = x \otimes 1 + 1 \otimes x\}$$

be the E_*-module of primitives.

COROLLARY 3.4. *$PE_*(SO/U)$ is a free E_* summand of $E_*(SO/U)$ with basis $\{\pi^E_{2k+1} | k \geqq 0\}$ where*

$$\pi^E_k = \phi_*(\sigma^E_k)$$

and $\sigma^E_k = \beta^E_1\sigma^E_{k-1} - \beta^E_2\sigma^E_{k-2} + \cdots + (-1)^{k-1}k\beta^E_k$ is the Newton polynomial in the β^E_j's.

Of course, $PE_*(BU)$ is the free E_*-module on the σ^E_k's. The proof of this Corollary involves a careful bookeeping exercise in the AHSS for $E_*(SO/U)$ preceded by a separate verification for the case $E = H$.

It is worthwhile observing that we can identify $E_*(BU)$ with the E_*-algebra of symmetric functions on indeterminates $u_1, u_2, \ldots$; then $\beta_k \equiv \sum u_1u_2 \ldots u_k$ and $\sigma_k \equiv \sum u_1^k$. Hence we can interpret $E_*(SO/U)$ as a suitable quotient of $E_*(BU) = E_*[\beta_1, \beta_2, \ldots]$. Upon setting $\theta_k \equiv \sum u_1u_2 \ldots u_k$ and $\pi_k \equiv \sum u_1^k$ we have the relations amongst the generators θ_k of $E_*(SO/U)$

$$\prod_i (1 + u_iT)(1 + u_i[-1](T)) = 1. \tag{3.5}$$

On applying the natural logarithm function ln to the series of relations (3.5) we obtain

$$\sum_{1\leq n} \frac{(-1)^{n-1}}{n} \pi_n T^n = \ln \theta(T) \tag{3.6}$$

and hence

$$\sum_{1\leq n} \frac{(-1)^{n-1}}{n} [\pi_n T^n + \pi_n([-1](T))^n] = 0. \tag{3.7}$$

This gives for example the recursive formulae

$$\pi_{2k} = -2k[\sum_{1\leq j\leq 2k-1} \frac{(-1)^{j-1}}{j} \pi_j([-1](T))^j]_{2k} \tag{3.8}$$

where the notation $[f(T)]_r$ signifies the coefficient of T^r in the power series $f(T)$. More generally, any symmetric function with occurrences of the u_i's to even degree can be expressed in terms of those with only odd degree occurrences using the relations of (3.5) repeatedly; the main formula required for this is $[-1](T) = -T + \ldots$ and so a basis for $E_*(SO/U)$ consists of the functions

$$\pi_{(2s_1+1, 2s_2+1, \ldots, 2s_m+1)} = \sum u_1^{2s_1+1} \ldots u_m^{2s_m+1} \tag{3.9}$$

which can be expressed as E_*-polynomials in the θ_j's.

Now we return to $E^*(SO/U)$. By the definition of the indecomposable quotient, $QE^*(SO/U) = \tilde{E}^*(SO/U)^2$. Then

$$QE^*(SO/U) \cong HomE_*(PE_*(SO/U), E_*). \tag{3.10}$$

To obtain a basis for $QE^*(SO/U)$ we can dualise the basis $\{\pi_{2k+1} | k \geqq 0\}$ of $PE_*(SO/U)$; more precisely, we define $y^E_{2k+1} \in E^{4k+2}(SO/U)$ by

$$\begin{aligned} \langle y^E_{2k+1}, \pi_{(2r_1+1, \ldots, 2r_m+1)} \rangle &= 0, \text{ unless } m = 1 \text{ and } r_1 = k; \\ \langle y^E_{2k+1}, \pi_{2k+1} \rangle &= 1. \end{aligned} \tag{3.11}$$

PROPOSITION 3.12 $E^*(SO/U) = E^*[[y^E_{2k+1} | k \leqq 0]]$.

The proof again involves first the verification of the case $E = H$ and then the use of the collapsing AHSS for $E^*(SO/U)$. Beware — the generators y^H_{2k+1} only agree with the y'_{2k+1} mod decomposables!

Now we can attempt to compare $\chi^* c^E_n$ with our given generators of $E^*(SO/U)$.

PROPOSITION 3.13. *In* $E^*(SO/U)$,

$$\chi^* c^E_n = y^E_n - \sum_{1\leqq k\leqq n} [([-1](T))^k]_n y^E_k \text{ (mod } \textit{decomposables})$$

where y^E_k *denotes the dual of* π^E_k *with respect to the basis of* $E_*(SO/U)$ *described earlier in the section.*

PROOF. First set $c(T) = \sum_{0\leqq j} c_j T^j \in E^*(BU)\ [[T]]$. Then

$$\begin{aligned} \phi^*\chi^* c(T) &= c(T)c([-1](T))^{-1} \\ &\equiv \sum_{0\leqq n} c_n(T^n + ([-1](T))^n) \text{ (mod } \textit{decomposables}). \end{aligned}$$

Now $\langle z\pi_k \rangle = \langle \phi^* z, \sigma_k \rangle$ for $z \in E^*(SO/U)$ and so since

$$\langle c_j, \sigma_k \rangle = \delta_{j,k} \text{ (Kronecker delta)}$$
$$\langle z, \sigma_k \rangle = 0 \textit{ if } z \text{ is decomposable,}$$

we have the statex result.

Notice that this yields in particular

(3.14) $\chi^* c^E_{2k+1} = 2y^E_{2k+1}$ + (higher filtration and decomposable terms).

N.B. This result involves formidable recursive formulae for the elements y^E_n even mod decomposables. It is not clear that even for a theory such as $KU^*(\ )$ these generators are easy to work with. It may be that there are more convenient systems of generators with simpler formulae taking the place of those in (3.13) (cf. **[6]**).

Notice that

(3.15) $$e^E(\xi_{2n+1}) = 0 \text{ in } E^*(BU_{2n+1}),$$

since when $E = MU$, $MU^{2n+1}(BU_{2n+1}) = 0$ and from this the general case follows by universality of MU.

We can now prove Theorem (3.1) by mimicking the proof of (2.2) using as starting point the description of $E^*(SO/U)$ given in (3.12).

4. In this section we will rederive some results of Crabb and Steer **[5]**, and Astey and Gitler **[2]** which have been used to obtain several non-sectioning conditions for bundles. Their approach is to work "intrinsically" with respect to a given stably complex bundle, whilst we will use our "universal" results on $KU^*(BU_n)$. Our approach also reveals an interesting connection between their modified total Chern classes and a certain "universal unit" of **[12]**.

Recall that for a complex bundle $\zeta \to X$ (of dimension $m \leqq \infty$ say) there are characteristic classes $\gamma^n(\zeta) \in KU^0(X)$ such that

(4.1) $$c_n^{KU}(\zeta) = t^n \gamma^n(\zeta)$$

where $KU_* = KU^{-*} = \mathbf{Z}[t, t^{-1}]$ with $t \in KU_2$ the Bott generator and $c_n^{KU}(\ )$ the KU-theory Chern class as in §3. We also have the total γ class $\bar\gamma(\zeta) = 1 + \gamma^1(\zeta) + \gamma^2(\zeta) + \dots \in KU^0(X)$. Note that each $\gamma^n(\zeta)$ is a reduced class whilst $\bar\gamma(\zeta)$ is a one dimensional virtual bundle. We will denote the universal γ's by $\gamma^n = \gamma^n(\gamma)$ and $\bar\gamma = \bar\gamma(\gamma) \in KU^0(BU)$. We will need the following identification:

(4.2) $$\bar\gamma(\zeta) = \det \zeta$$

where $\det \zeta$ is the line bundle obtained from the principal $U(m)$-bundle of ζ with the aid of the determinant representation det: $U(m) \to U(1)$.

DEFINITION 4.3. $\Gamma(\zeta) = \sum_{0 \leq j}((1/2)^j)\gamma^j(\zeta) \in KU^0(X) \otimes \mathbf{Z}[1/2]$. (This is denoted $\tilde c^{KU}(\zeta)$ in **[2]**.)

Then Γ has the properties

$$\Gamma(\zeta_1 + \zeta_2) = \Gamma(\zeta_1)\Gamma(\zeta_2),$$
$$\Gamma(\lambda) = \frac{1}{2}(1 + \lambda) \text{ if } \lambda \to x \text{ is a line bundle.}$$

THEOREM 4.4. ([2], [5]) *Let $\zeta \to X$ be a stably compler bundle such that there is a real bundle $\theta \to X$ of dimension $2s + 1$ with $r\zeta \cong \theta$; then*

$$2^s\Gamma(\zeta) \in KU^0(X).$$

PROOF. From (2.15) together with the familiar AHSS argument we have

$$g'^*: KU^0(BU_{2s+1}) \to KU^0(BU(s) \times SO/U) \cong KU^0(BU(s)) \otimes KU^0(SO/U)$$

is monomorphic onto a direct summand. Observe that

$$g'^*\Gamma(\zeta_{2s+1}) = \Gamma(\gamma_s) \otimes \Gamma(\chi^*\gamma) \tag{4.4}$$

and since $2^s\Gamma(\gamma_s)$ is manifestly in $KU^0(BU(s))$ we need only show that $\Gamma(\chi^*\gamma)$ is in $KU^0(SO/U)$. To do this we will use the splitting principle in the form of a map

$$\psi: (\prod CP^\infty) \longrightarrow SO/U \text{ (with a countably infinite product)}$$

inducing a monomorphism onto a summand in E-cohomology where E is any complex oriented theory; hence $\chi \cdot \psi$ also has this property. But on each factor of the product of projective spaces this composite classifies the bundle $\eta - \eta^*$ and so we have

$$\begin{aligned}\psi^*\text{ch } \Gamma(\chi^*\gamma) &= \prod \text{ch } \Gamma(\eta - \eta^*)\\ &= \prod \text{ch } (1 + \eta)(1 + \eta^*)^{-1}\\ &= \prod \text{ch } \eta, \text{ since } \eta^* = \eta^{-1}.\end{aligned}$$

Hence we have ψ^* ch $\Gamma(\chi^*\gamma)$ = ch (det γ) and so since ch is a monomorphism we have

$$\psi^*\chi^*\gamma = \det \gamma. \tag{4.5}$$

A calculation with first Chern classes shows that we even have

$$(\Gamma(\chi^*\gamma))^2 = \det \chi^*\gamma. \tag{4.6}$$

The connection with the universal unit of [12] arises as follows. There is an orientation A^{KU} for $\chi^*\gamma$ in $KU^0(SO/U)$ determined by the choice of x^{KU} as $t^{-1}(\eta - 1) \in KU^2(CP^\infty)$. Then calculating ch A^{KU} we see that

$$A^{KU} = \Gamma(\chi^*\gamma). \tag{4.7}$$

There is a (universal) universal unit $A^{MU} \in MU^0(SO/U)$ which has the interesting property that if

$$A^{MU} = \sum_{\omega} \alpha_{\omega} (y^{MU})^{\omega}$$

then the ideal in MU_* generated by the coefficients α_ω for sequences ω with $(y^{MU})^{\omega} \neq 1$ is equal to the ideal of all elements which can be given new U-structures to bound; this in turn in equal to the kernel of the forgetful homomorphism $F_*\colon MU_* \to MSO_*$ (see [11], [12]). It would be interesting to know if there is a characteristic class in $MU^*(BU_n)$ analogous to Γ with a reasonably simple description in terms of the Conner-Floyd classes c_j^{MU}.

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