ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 16, Number 4, Fall 1986

A COMPLETE CHARACTERIZATION OF THE LEVEL SPACES OF R (I) AND I (I)

R. LOWEN

ABSTRACT. By means of T_0-modifications we are able to precisely describe all level spaces of $\mathbf{R}$ (I) and I (I) and to show that there are only 3 nonhomeomorphic level spaces of $\mathbf{R}$ (I) and only 4 nonhomeomorphic level spaces of I (I). A large list of α-properties of both $\mathbf{R}(I)$ and $I(I)$ is deduced, and an open problem with regard to α-compactness is solved.

1. Preliminaries. I denotes the unit interval and $I_1 = [0, 1[$. If X is a topological space and $x \in X$ we denote its neighborhoodfilter $\mathcal{N}(x)$.

We recall that a topological space is called hyperconnected (resp. ultraconnected) if no disjoint open (resp. closed) sets exist [**12**].

If $\mathscr{B}$ is a filterbase then the generated filter is denoted $[\mathscr{B}]$. If (X, Δ) is a fuzzy topological space then for any $\alpha \in I_1$ the α-level space, denoted $\iota_\alpha(X)$, is the topological space $(X, \iota_\alpha(\Delta))$ where $\iota_\alpha(\Delta) = \{\mu^{-1}]\,\alpha, 1] \mid \mu \in \Delta\}$ (see [**5**], [**6**]). We use the simplified version of $\mathbf{R}$ (I) and I (I) introduced in [7]. That means that throughout this paper $\mathbf{R}$ (I) is the set of all non-increasing left continuous maps from $\mathbf{R}$ to I with supremum equal 1 and infimum equal 0.

The fuzzy topology considered on this set is determined by the subbasis $\{L_x, R_y \mid x, y \in \mathbf{R}\}$ where L_x and R_y are defined as

$$L_x(\lambda) = 1 - \lambda(x)$$
$$R_y(\lambda) = \lambda(y+)$$

for any $\lambda \in \mathbf{R}$ (L).

We also recall that I (I) is the subspace of $\mathbf{R}$ (I) defined by $\mu \in I$ (I) if and only if $\mu(0) = 1$ and $\mu(t) = 0$ for all $t > 1$. For more information on these spaces see [**1**], [**2**], [**3**], [7], [**9**] and [**10**]. In [**10**] the numbers $a(\mu, \alpha)$, $b(\mu, \alpha)$, $a^*(\mu, \alpha)$, $b^*(\mu, \alpha)$ for any $\mu \in \mathbf{R}(I)$ and $\alpha \in I$ were introduced. In [7] we showed the following proposition which we require in the sequel.

PROPOSITION 1.1. *For any* $\mu \in \mathbf{R}$ (I) *and* $\alpha \in I$

(i) $a(\mu, \alpha) = \inf \mu^{-1}[0, 1 - \alpha[$,

Received by the editors on January 14, 1982, and in revised form on January 31, 1985
Copyright © 1986 Rocky Mountain Mathematics Consortium

(ii) $b(\mu, \alpha) = \sup \mu^{-1}]\alpha, 1]$,
(iii) $a^*(\mu, \alpha) = \inf \mu^{-1}[0, 1 - \alpha]$,
(iv) $b^*(\mu, \alpha) = \sup \mu^{-1}[\alpha, 1]$,
(v) $a^*(\mu, \alpha) = b(\mu, 1 - \alpha)$,
(vi) $b^*(\mu, \alpha) = a(\mu, 1 - \alpha)$.

We also recall that $\mathbf{R}\,(I)$ contains a subspace homeomorphic to $\mathbf{R}$, namely $\{\lambda_x \mid x \in \mathbf{R}\}$ where $\lambda_x = 1_{]-\infty, x]}$, and that we shall denote this subspace also by $\mathbf{R}$.

Finally, the reader who knows our previous papers, will know that in our opinion the fuzzy topology of $\mathbf{R}\,(I)$ must be saturated for the constant maps, i.e., to the subbasis given higher up the constants must be added (see also **[11]**). However since for the consideration of this paper the choice one makes with regard to the constants is totally irrelevant we leave it up to reader to interpret the results in whichever framework he chooses.

2. $\mathbf{T_0}$-modifications. As far back as 1936 M.H. Stone **[13]** showed that each topological space can be modified into a T_0-space by a suitable identification of points. This simple technique turns out to be especially useful in describing the level topologies of $\mathbf{R}\,(I)$ and $I\,(I)$. For any topological space X let R denote the equivalence relation defined by, xRy if and only if $\overline{\{x\}} = \overline{\{y\}}$. Put $X^0 = X|R$ the quotient topological space and $\phi: X \to X^0$ the quotient map. X^0 is called the *T_0-modification of X*.

PROPOSITION 2.1. *For any topological space X the following properties hold*

(1) *xRy if and only if $\mathcal{N}(x) = \mathcal{N}(y)$,*
(2) *X^0 is a T_0-space,*
(3) *ϕ induces a lattice isomorphism between the open sets in X and the open sets in X^0,*
(4) *Each open (resp. closed) subset of X is saturated,*
(5) *X has the weak topology for the pair (X^0, ϕ),*
(6) *If $K \subset X$ (resp. $H \subset X^0$) then K (resp. H) is compact if and only if $\phi(K)$ (resp. $\phi^{-1}(H)$) is compact,*
(7) *ϕ is an open proper map,*
(8) *X^0 is densely embedded in X,*
(9) *X^0 is C-embedded in X,*
(10) *If Y is a T_0-space and $f: X \to Y$ is a continuous map then f is constant on equivalence classes.*

PROOF. (1). This is clear.

(2) and (3). These were shown in **[13]**.

(4). For open sets this follows from (1) and for closed sets from the fact that the complement of a saturated set is saturated.

(5) and (6). These follow by easy verification using (4).

(7). That ϕ is continuous follows by definition, that it is open and closed was shown in **[14]** and that it has compact fibers is but a special case of (6).

(8), (9) and (10). These were shown in **[15]**.

In the sequel we shall often use the T_0-modification of specific topological spaces. Since we are interested in the properties of the original space and not of its T_0-modification we want to know which propreties are *T_0-preserved* (i.e., if X has the property then so does X^0) and-or *inversely T_0-preserved* (i.e., if X^0 has the property then so does X). We shall call a topological property which is both T_0-preserved and inversely T_0-preserved for any space X a *T_0-property.*

The following proposition makes no attempt at exhaustiveness but is restricted to those properties, the knowledge of which as to whether they are T_0-properties or not we shall use later on.

PROPOSITION 2.2. *In the following enumeration only T_0 fails to be a T_0-property.* $\{$*T_0, regular, normal, paracompact, compact, pseudocompact, σ-compact, locally compact, 2^{nd} countable, 2^{nd} category, Baire, path connected, locally path connected, hyperconnected, ultraconnected*$\}$.

PROOF. $[T_0]$ Let X be a completely regular space which is not T_0 then X^0 is completely regular and T_2.

[Regular, 2^{nd} countable] Both are already preserved by proper maps **[8]**. That they are inversely T_0-preserved follows by use of Proposition 2.1 (4) and (7).

[Normal, paracompact] From Proposition 2.1 (4) and (7).

[Compact] This is a special case of Proposition 2.1 (6).

[Pseudocompact] To show it is inversely T_0-preserved remark that if f: $X \to \mathbf{R}$ is continuous then from Proposition 2.1 (10) there exists a continuous factorization g: $X^0 \to \mathbf{R}$ which is thus bounded. The boundedness of f follows.

[σ-compact, locally compact] From Proposition 2.1 (4), (6) and (7).

[2^{nd} category, Baire] From Proposition 2.1 (4) and (7) and the fact that for ϕ the image and preimage of dense sets is dense.

[Pathwise connected] To show the property is inversely T_0-preserved let $x, y \in X$. If $\phi(x) = \phi(y)$ then f: $I \to X$, $f|_{[0,1/2[} = x$, $f|_{[1/2,1]} = y$ is a path connecting x and y. If $\phi(x) \neq \phi(y)$ let φ: $I \to X^0$ be a path connecting $\phi(x)$ and $\phi(y)$. From the surjectivity of ϕ there exists a factorization θ: $I \to X$ which moreover can be chosen such that $\theta(0) = x$ and $\theta(1) = y$. Continuity of θ follows from the fact that X carries the weak topology.

[Locally pathwise connected] Analogous and by use of Proposition 2.1 (7).

[Hyperconnected, ultraconnected] Immediate from Proposition 2.1 (4) and (7).

3. Characterization of the level spaces of R (I) and I (I). Actually this entire chapter is devoted to the proof of Theorem 3.1. In the process of doing so however we shall discover what the level spaces of $\mathbf{R}$ (I) and I (I), or rather their T_0-modifications, precisely look like.

THEOREM 3.1. *There are only* 3 *non-homeomorphic level spaces of* $\mathbf{R}$ (I) *and only* 4 *non-homeomorphic level spaces of* $I(I)$. *More precisely*

(1) $\iota_\alpha(\mathbf{R}(I))$ *is homeomorphic to* $\iota_\beta(\mathbf{R}(I))$ *if and only if* $(\alpha, \beta) \in \{0\}^2 \cup]0, 1/2[^2 \cup [1/2, 1[^2$.

(2) $\iota_\alpha(I(I))$ *is homeomorphic to* $\iota_\beta(I(I))$ *if and only if* $(\alpha, \beta) \in \{0\}^2 \cup]0, 1/2[^2 \cup \{1/2\}^2 \cup]1/2, 1[^2$

The proof of Theorem 3.1 shall be divided in three parts. In the first part we prove the if-statements of both (1) and (2) for pairs $(\alpha, \beta) \in]0, 1/2[^2 \cup]1/2, 1[^2$.

Actually this will be shown in Theorem 3.2 and Corollary 3.3 where it is proved that in those cases very natural homeomorphisms can be constructed.

(The if statements in (1) for $(\alpha, \beta) \in \{0\}^2$ and in (2) for $(\alpha, \beta) \in \{0\}^2 \cup \{1/2\}^2$ obviously require no proof).

In the second part we deduce the only if part in (1) for pairs $(\alpha, \beta) \in \{0\}^2 \cup]0, 1/2[^2 \cup]1/2, 1[^2$ and in (2) for pairs $(\alpha, \beta) \in]0, 1/2[^2 \cup]1/2, 1[^2$. This will be a consequence of Theorem 3.5, a T_0-modified version of Theorem 3.1 itself.

In the third and final part of the proof we settle the status of the three remaining level spaces $\iota_{1/2}(\mathbf{R}(I))$, $\iota_0(I(I))$ and $\iota_{1/2}(I(I))$. This part of the proof will be based on Theorems 3.5 and 3.6.

Proof of Theorem 3.1. (*first part*)

THEOREM 3.2. *For any pair* $(\alpha, \beta) \in]0, 1/2[^2 \cup]1/2, 1[^2$ *let* $\theta_{\beta\alpha}: I \to I$ *be an order isomorphism fulfilling* $\theta_{\beta\alpha}(\alpha) = \beta$ *and* $\theta_{\beta\alpha}(1 - \alpha) = 1 - \beta$. *Then the map*

$$f_{\beta\alpha}: \iota_\alpha(\mathbf{R}(I)) \to \iota_\beta(\mathbf{R}(I)): \mu \to \theta_{\beta\alpha} \circ \mu$$

is a homeomorphism and order isomorphism.

REMARK. As an example of such maps $\theta_{\beta\alpha}$ for $(\alpha, \beta) \in]0, 1/2[^2$ we have for instance

$$\theta_{\beta\alpha}(t) = \begin{cases} \dfrac{\beta}{\alpha} t & 0 \leqq t \leqq \alpha \\ \dfrac{1 - 2\beta}{1 - 2\alpha}(t - \alpha) + \beta & \alpha \leqq t \leqq 1 - \alpha \\ \dfrac{\beta}{\alpha} t + 1 - \beta & 1 - \alpha \leqq t \leqq 1 \end{cases}$$

and for $(\alpha, \beta) \in]1/2, 1[^2$ by letting $\theta_{\beta\alpha} = \theta_{(1-\beta)(1-\alpha)}$.

PROOF OF THEOREM 3.2. Since $\theta_{\beta\alpha}$ is an order isomorphism it is clear that also $f_{\beta\alpha}$ is an order isomorphism. To show $f_{\beta\alpha}$ is continuous one verifies that for any $x \in \mathbf{R}$

$$f_{\beta\alpha}^{-1}(L_x^{-1}]\,\beta, 1]) = L_x^{-1}]\,\alpha, 1]$$

and analogously

$$f_{\beta\alpha}^{-1}(R_x^{-1}]\,\beta, 1]) = R_x^{-1}]\,\alpha, 1].$$

That $f_{\beta\alpha}$ is open follows from the bijectiveness of $f_{\beta\alpha}$ and from the two previous equalities.

COROLLARY 3.3. *For any pair* $(\alpha, \beta) \in\,]0, 1/2[^2 \cup\,]1/2, 1[^2$ *the map*

$$f'_{\beta\alpha}: \iota_\alpha(I(I)) \to \iota_\beta(I(I)): \mu \to \theta_{\beta\alpha} \circ \mu$$

is a homeomorphism and order isomorphism.

PROOF. Since $I(I)$ is a subspace of $\mathbf{R}\,(I)$ it is clear that all that needs to be shown is that $f'_{\beta\alpha}(I(I)) = I(I)$. This however is trivial since the only condition on the elements μ of $I(I)$ is that $\mu(0) = 1$ and $\mu(t) = 0$ if $t > 1$ and since for any pair (α, β) we have $\theta_{\beta\alpha}(0) = 0$ and $\theta_{\beta\alpha}(1) = 1$.

It is clear that the cases $\alpha = 0$, $\beta \neq 0$; $\alpha = 1/2$, $\beta \neq 1/2$, $\alpha < 1/2 < \beta$ or vice versa can not be treated in this way. There does not exist an order isomorphism θ on I fulfilling $\theta(\alpha) = \beta$ and $\theta(1 - \alpha) = 1 - \beta$ for any of those pairs (α, β).

To overcome this difficulty we first look at the T_0-modification of all the level spaces in question.

It will turn out that for these T_0-modifications we can more easily prove a modified version of Theorem 3.1.

The reason for this being that they can be characterised as subsets of an extended plane with the product topology of two well known topologies.

Consider on $\mathbf{R} \cup \{-\infty\} \times \mathbf{R} \cup \{\infty\}$ the product topology of respectively the left order topology

$$\{[-\infty, x[\ \mid x \in \overline{\mathbf{R}}\}$$

on the first factor and the right order topology

$$\{]x, +\infty]\mid x \in \overline{\mathbf{R}}\}$$

on the second factor.

Next define the following 5 subspaces, each equipped with the subspace topology

$$P_u^* := \{(a, b) \in \overline{\mathbf{R}} \times \overline{\mathbf{R}} \mid a \leqq b\} \setminus \{(-\infty, -\infty), (\infty, \infty)\},$$
$$P_u := \{(a, b) \in \mathbf{R} \times \mathbf{R} \mid a \leqq b\},$$
$$P_1 := \{(a, b) \in \mathbf{R} \times \mathbf{R} \mid b \leqq a\},$$

$$D_u := P_u \cap (I \times I)$$
$$D_1 := P_1 \cap (I \times I).$$

Then let P_α denote P_u^* if $\alpha = 0$, P_u if $\alpha \in]0, 1/2[$, P_1 if $\alpha \in [1/2, 1[$ and let D_α denote D_u if $\alpha \in [0, 1/2[$ and D_1 if $\alpha \in [1/2, 1[$.

It is our intention first of all now to prove the following theorem.

THEOREM 3.4. *For any* $\alpha \in I$
(1) $\iota_\alpha(\mathbf{R}(I))^0$ *is homeomorphic to* P_α
(2) $\iota_\alpha(I(I))^0$ *is homeomorphic to* D_α.

The proof of this theorem is based on a number of lemmas.

LEMMA 3.4.1. *For any* $\alpha \in I_1$, $\mu \in \mathbf{R}(I)$ *and* $x \in \mathbf{R}$
(1) $\alpha < L_x(\mu)$ *if and only if* $a(\mu, \alpha) < x$,
(2) $\alpha < R_x(\mu)$ *if and only if* $x < b(\mu, \alpha)$.

PROOF. (1) If $\alpha < L_x(\mu)$ then by left continuity of μ and using Proposition 1.1 it follows there exists $\delta > 0$ such that $a(\mu, \alpha) \leqq x - \delta < x$. Conversely if $a(\mu, \alpha) = \inf \mu^{-1}[0, 1 - \alpha[< x$ then clearly $\mu(x) < 1 - \alpha$.

(2) $\alpha < R_x(\mu) \Leftrightarrow \exists\, z$, $x < z$ such that $\alpha < \mu(z) \Leftrightarrow \mu^{-1}]\alpha, 1] \cap]x, +\infty[\neq \emptyset$. Since $]-\infty, b(\mu, \alpha)[\subset \mu^{-1}]\alpha, 1] \subset]-\infty, b(\mu, \alpha)]$ equivalence with $x < b(\mu, \alpha)$ follows.

LEMMA 3.4.2. *For any* $\alpha \in I_1$ *let* R_α *(resp.* R'_α*) denote the qeuivalence relation associated with* T_0*-modification in* $\iota_\alpha(\mathbf{R}(I))$ *(resp.* $\iota_\alpha(I(I))$*). Then for any* $\mu, \lambda \in \mathbf{R}(I)$ *(resp.* $\mu, \lambda \in I(I)$*) we have* $\mu R_\alpha \lambda$ *(resp.* $\mu R'_\alpha \lambda$*) if and only if both* $a(\mu, \alpha) = a(\lambda, \alpha)$ *and* $b(\mu, \alpha) = b(\lambda, \alpha)$.

PROOF. We only comment upon the case of $\mathbf{R}(I)$, the one of $I(I)$ being perfectly analogous.

Since a basis for the open sets in $\iota_\alpha(\mathbf{R}(I))$ is given by $\{(L_x \wedge R_y)^{-1}]\alpha, 1] \mid x, y \in \mathbf{R}\}$ it follows from Lemma 3.4.1 that for any $\mu \in \mathbf{R}(I)$ its neighborhoodfilter in $\iota_\alpha(\mathbf{R}(I))$ is given by

$$\mathcal{N}_\alpha(\mu) = [\{(L_x \wedge R_y)^{-1}]\alpha, 1] \mid a(\mu, \alpha) < x, y < b(\mu, \alpha)\}]$$

The lemma now is an immediate consequence of Proposition 2.1 (1).

LEMMA 3.4.3. *For any* $\alpha \in I_1$ ψ_α (resp. ψ'_α) *defined as follows*

$$\psi_\alpha\colon \mathbf{R}(I) \to P_\alpha\colon \mu \to (a(\mu, \alpha), b(\mu, \alpha))$$
$$(\textit{resp. } \psi'_\alpha\colon I(I) \to D_\alpha\colon \mu \to (a(\mu, \alpha), b(\mu, \alpha)))$$

is a well defined surjective map.

PROOF. Again we give the proof only for the case $\mathbf{R}(I)$. That for any $\mu \in \mathbf{R}(I)$ we have $a(\mu, \alpha) \leqq b(\mu, \alpha)$ if $\alpha \in [0, 1/2[$ and $b(\mu, \alpha) \leqq a(\mu, \alpha)$ if $\alpha \in [1/2, 1[$ the reader can verify himself using Proposition 1.1.

Furthermore if $\alpha = 0$ and $a, b \in \mathbf{R}$ then for instance $\psi_0(\mu) = (-\infty,$

b) for $\mu(x) = (1 - e^{x-b}) \vee 0$, $\psi_0(\mu) = (a, +\infty)$ for $\mu(x) = (1 - e^{a-x}) \vee 0$ and $\psi_0(\mu) = (-\infty, +\infty)$ for $\mu(x) = (1/\pi)(\pi/2 - \mathit{arc\ tg}\ x)$.

Next for any $\alpha \in [0, 1/2[$ and $a, b \in \mathbf{R}$ such that $a \leqq b$ $\psi_0(\mu) = (a, b)$ for $\mu = 1_{]-\infty, a]} \vee (1/2)\, 1_{]a,b]}$ and analogously for $\alpha \in [1/2, 1[$ and $b \leqq a$. That for $\alpha = 0$ neither $(-\infty, -\infty)$ nor (∞, ∞) can occur is evident since for all $\mu \in \mathbf{R}(I)$, $\sup \mu = 1$ and $\inf \mu = 0$. And finally for the same reason for any $\alpha \in]0, 1/2[$ no points with either coordinates $\pm\infty$ can occur.

PROOF OF THEOREM 3.4. We again comment only upon the case $\mathbf{R}(I)$, leaving $I(I)$ to the reader. Consider the following diagram

$$\begin{tikzcd}
\iota_\alpha(\mathbf{R}(I)) \arrow[r, "\varphi_\alpha"] \arrow[dr, "\psi_\alpha"'] & \iota_\alpha(\mathbf{R}(I))^0 \arrow[d, "j_\alpha"] \\
 & P_\alpha
\end{tikzcd}$$

where φ_α is the quotientmap and j_α is the unique factorization. That j_α is a bijection follows from the fact that ψ_α is onto and from Lemma 3.4.2. To show j_α is continuous it suffices to show that ψ_α is continuous which follows from Lemma 3.4.1. And analogously to show j_α is open it suffices to show ψ_α is open (recall that open sets in $\iota_\alpha(\mathbf{R}(I))$ are saturated!) which also follows from Lemma 3.4.1.

REMARK. In the sequel, whenever convenient, we shall not differentiate between $\iota_\alpha(\mathbf{R}(I))^\circ$ (resp. $\iota_\alpha(I(I))^\circ$) and P_α(resp. D_α) for any $\alpha \in I_1$.

THEOREM 3.5. (T_0-MODIFICATION OF THEOREM 3.1.) *There are only* 3 *nonhomeomorphic T_0-modifications of level spaces of* $\mathbf{R}(I)$ *and only* 2 *nonhomeomorphic T_0-modifications of level spaces of $I(I)$. More precisely*

(1) $\iota_\alpha(\mathbf{R}(I))^0$ *is homeomorphic to* $\iota_\beta(\mathbf{R}(I))^0$ *if and only if* $(\alpha, \beta) \in \{0\}^2 \cup]0, 1/2[^2 \cup [1/2, 1[^2$

(2) $\iota_\alpha(I(I))^0$ *is homeomorphic to* $\iota_\beta(I(I))^0$ *if and only if* $(\alpha, \beta) \in [0, 1/2[^2 \cup [1/2, 1[^2$.

PROOF. The if part of both (1) and (2) is precisely the result of Theorem 3.4.

To show the only if part it is obviously sufficient to prove that none of the spaces P_u^*, P_u and P_1 are homeomorphic to each other and that D_u and D_1 are not homeomorphic.

Although there are simple reasons proving these assertments directly we prefer to refer the reader to the results of chapter 4 from which they follow in an evident way.

Proof of Theorem 3.1. (*second part*)

Consider the following classes

$$\mathscr{C}^1_{\mathbf{R}} := \{\iota_0(\mathbf{R}\,(I))\},$$

$$\mathscr{C}^2_{\mathbf{R}} := \{\iota_\alpha(\mathbf{R}\,(I)) \mid \alpha \in]0, \tfrac{1}{2}[\ \},$$

$$\mathscr{C}^3_{\mathbf{R}} := \{\iota_\alpha(\mathbf{R}\,(I)) \mid \alpha \in]\tfrac{1}{2}, 1[\ \},$$

$$\mathscr{C}^1_I := \{\iota_0(I(I))\},$$

$$\mathscr{C}^2_I := \{\iota_\alpha(I(I)) \mid \alpha \in]0, \tfrac{1}{2}[\ \},$$

$$\mathscr{C}^3_I := \{\iota_{1/2}(I(I))\},$$

$$\mathscr{C}^4_I := \{\iota_\alpha(I(I)) \mid \alpha \in]\tfrac{1}{2}, 1[\ \}.$$

Then for all non-singleton classes we already know, from Theorem 3.2 and Corollary 3.3, that all spaces within one class are homeomorphic to one another while for the singleton classes this is evident.

From Theorem 3.5 we now further deduce that no space of $\mathscr{C}^i_{\mathbf{R}}$ is homeomorphic to a space of $\mathscr{C}^j_{\mathbf{R}}$ for $i \neq j$ and that no space of $\mathscr{C}^2_I$ is homeomorphic to a space of $\mathscr{C}^4_I$.

The third and final part of the proof of Theorem 3.1 will consist in showing that $\iota_{1/2}(\mathbf{R}(I))$ can be added to the class $\mathscr{C}^3_{\mathbf{R}}$, i.e., is homeomorphic to any space in $\mathscr{C}^3_{\mathbf{R}}$, and that $\iota_0(I(I))$ and $\iota_{1/2}(I(I))$ both are in a class of their own, i.e., are homeomorphic to no other level space of $I(I)$. To be able to show this we need the following result.

THEOREM 3.6. *Let X and Y be arbitrary topological spaces and let φ_X: $X \to X^0$ and φ_Y: $Y \to Y^0$ be the quotient maps onto their respective T_0-modifications. Then X and Y are homeomorphic if and only if there exists a homeomorphism f: $X^0 \to Y^0$ such that for all $x^0 \in X^0$.*

$$\text{(H)} \qquad \operatorname{Card} \varphi_X^{-1}(x^0) = \operatorname{Card} \varphi_Y^{-1}(f(x^0)).$$

PROOF. To show the if part, for all $x^0 \in X$ let

$$F_{x^0}\colon \varphi_X^{-1}(x^0) \to \varphi_Y^{-1}(f(x^0))$$

be a bijection and define

$$F\colon X \to Y\colon x \to F_{\varphi_X(x)}(x).$$

Clearly F is a bijection. To show it is continuous it suffices to verify that for any $G \subset Y$ open $\varphi_X^{-1}(f^{-1}(\varphi_Y(G))) = F^{-1}(G)$ and to show it is open it suffices to verify that for any $G \subset X$ open $\varphi_Y^{-1}(f(\varphi_X(G))) = F(G)$. This we leave to the reader. To show the only if part consider the diagram where F is a homeomorphism.

$$\begin{tikzcd} X \arrow[rr, "F"] \arrow[d, "\varphi_X"'] \arrow[drr, "\varphi_Y \circ F"] & & Y \arrow[d, "\varphi_Y"] \\ X^0 \arrow[rr, dashed, "f"'] & & Y^0 \end{tikzcd}$$

From Proposition 2.1 (10) it follows that $\varphi_Y \circ F$ is constant on classes which combined with the surjectivity of φ_X proves there exists a unique factorization f: $X \to Y$.

To show f is surjective let $y^0 \in Y^0$, choose $y \in \varphi_Y^{-1}(y^0)$ and $x \in X$ such that $F(x) = y$. Then $f(\varphi_X(x)) = y^0$.

To show f is injective let u^0, $v^0 \in X^0$ such that $f(u^0) = f(v^0)$ and choose $u \in \varphi_X^{-1}(u^0)$, $v \in \varphi_X^{-1}(v^0)$ then $\varphi_Y F(u) = \varphi_Y F(v)$. Now since, again from Proposition 2.1 (10), also $\varphi_X \circ F^{-1}$ is constant on classes we have $u^0 = \varphi_X \circ F^{-1}(F(u)) = \varphi_X \circ F^{-1}(F(v)) = v^0$.

The continuity of f now follows from the continuity of $\varphi_Y \circ F$ and the continuity of f^{-1} from that of $\varphi_X \circ F^{-1}$.

To show that f fulfills condition (H) it suffices to remark that for any $x^0 \in X^0$ we have

$$F^{-1}(\varphi_Y^{-1}(f(x^0))) = \varphi_X^{-1} \circ f^{-1}(f(x^0)) = \varphi_X^{-1}(x^0).$$

Proof of Theorem 3.1. (*final part*)

(A) $\iota_{1/2}(\mathbf{R}\,(I))$ *is homeomorphic to* $\iota_\alpha(\mathbf{R}(I))$ *for any* $\alpha \in [1/2, 1[$

This will follow from Theorem 3.5 and 3.6 if we can prove that there exists a homeomorphism f: $P_{1/2} \to P_\alpha$ which fulfills condition (H). Actually we can prove a lot more. Namely we can show that any such homeomorphism fulfills condition (H). This indeed follows from the following.

ASSERTION. $\forall\, \alpha \in [1/2, 1[$, $\forall\, (a, b) \in P_\alpha$. Card $\psi_\alpha^{-1}(a, b) =$ Card $\mathbf{R}(\mathrm{I})$

Indeed $\leqq$ is clear. To show the other inequality it suffices to verify that for all $(a, b) \in P_\alpha$

$$h_{(a,b)}: \mathbf{R}(I) \to \psi_\alpha^{-1}(a, b)$$

defined by

$$h_{(a,b)}(\mu)(x) = \begin{cases} 1 & x \leqq b \\ \dfrac{1}{2} & b < x \leqq a \\ \left(\dfrac{1-\alpha}{2}\right)\mu(ln(x-a)) & a < x \end{cases}$$

for any $\mu \in \mathbf{R}(I)$, is welldefined and injective. This we leave to the reader. Consequently for any homeomorphism f: $P_{1/2} \to P_\alpha$ and any $(a, b) \in P_{1/2}$ we have

$$\text{card } \psi_{1/2}^{-1}(a, b) = \text{card } \mathbf{R}(I) = \text{Card } \psi_\alpha^{-1}(f(a, b)),$$

thus (H) is fulfilled, and (A) is proved.

(B) *$\iota_0(I(I))$ is not homeomorphic to any $\iota_\alpha(I(I))$ $\alpha \in]0, 1[$.*

From Theorem 3.5 it follows that it suffices to show this for $\alpha \in]0, 1/2[$. Again by use of Theorem 3.6, this follows from the

ASSERTION.
(1) $\forall\, \alpha \in]0, 1/2[$, $\forall (a, b) \in D_\alpha$: Card $\psi_\alpha^{-1}(a, b) =$ Card $I(I)$,
(2) $\forall\, a \in I$: Card $\psi_0^{-1}(a, a) = 1$.

To show (1) it is obviously again sufficient to prove $\geqq$. For any $(a, b) \in D_\alpha$ define $h_{(a,b)}: I(I) \to \psi_\alpha^{-1}(a, b)$ by

(i) in case $a < b$

$$h_{(a,b)}(\mu)(x) = \begin{cases} 1 & x \leqq a \\ \frac{1}{4}(2\alpha + 1) + \frac{1}{2}(1 - 2\alpha)\,\mu\left(\frac{x-a}{b-a}\right) & a < x \leqq b \\ 0 & b < x, \end{cases}$$

(ii) in case $a = b < 1$

$$h_{(a,b)}(\mu)(x) = \begin{cases} 1 & x \leqq a \\ \alpha\,\mu\left(\frac{x-a}{1-a}\right) & a < x \leqq 1 \\ 0 & 1 < x, \end{cases}$$

(iii) in case $a = b = 1$

$$h_{(a,b)}(\mu)(x) = \begin{cases} 1 & x \leqq 0 \\ (1 - \alpha) + \alpha\mu(x) & 0 < x \leqq 1 \\ 0 & 1 < x \end{cases}$$

We again leave it to the reader to verify that, for any choice $(a, b) \in D_\alpha$, $h_{(a,b)}$ is well defined and injective which then proves the other inequality.

To show (2) the reader can also very easily convince himself of the fact that for any $a \in I$ $\psi_0^{-1}(a, a) = \{\lambda_a\}$. Thus it is clear that no map for no $\alpha \in]0, 1/2[$, let alone a homeomorphism, can exist between D_0 and D_α which fulfills condition (H). It follows that (B) is proved.

(C) *$\iota_{1/2}(I(I))$ is not homeomorphic to any $\iota_\alpha(I(I))$ $\alpha \in [0, 1[\backslash\{1/2\}$.*

From Theorem 3.5 it again follows that it is sufficient to show this for $\alpha \in]1/2, 1[$. As in (B) this follows from Theorem 3.6 and the

ASSERTION.
(1) $\forall\, \alpha \in]1/2, 1[$, $\forall (a, b) \in D_\alpha$: Card $\psi_\alpha^{-1}(a, b) =$ Card $I(I)$,
(2) Card $\psi_{1/2}^{-1}(1, 0) = 1$.

To show (1) it again sufficies to prove $\geqq$ which as in (B) follows from the fact that for any $(a, b) \in D_\alpha$ $h_{(a,b)} : I(I) \to \psi_\alpha^{-1}(a, b)$, as defined further on, is welldefined and injective. This verification too is left to the reader.

(i) in case $b < a$

$$h_{(a,b)}(\mu)(x) = \begin{cases} 1 & x \leqq b \\ (1 - \alpha) + (2\alpha - 1)\ \mu\left(\dfrac{x - b}{a - b}\right), & b < x \leqq a \\ 0 & a < x, \end{cases}$$

(ii) in case $0 < a = b$

$$h_{(a,b)}(\mu)(x) = \begin{cases} 1 & x \leqq 0 \\ \dfrac{1}{2}(1 + \alpha) + \dfrac{1}{2}(1 - \alpha)\ \mu\left(\dfrac{x}{a}\right) & 0 < x \leqq a \\ 0 & a < x, \end{cases}$$

(iii) in case $0 = a = b$

$$h_{(a,b)}(\mu)(x) = \begin{cases} 1 & x \leqq 0 \\ \left(\dfrac{1 - \alpha}{2}\right)\mu(x) & 0 < x \leqq 1 \\ 0 & 1 < x. \end{cases}$$

To show (2) the reader is asked, finally, to verify that $\psi_{1/2}^{-1}(1, 0) = \{\mu_0\}$ where μ_0 is defined by

$$\mu_0(x) = \begin{cases} 1 & x \leqq 0 \\ \dfrac{1}{2} & 0 < x \leqq 1 \\ 0 & 1 < x \end{cases}$$

As in (B) this shows that no map fulfilling (H) can exist between $D_{1/2}$ and D_α for any $\alpha \in]1/2, 1[$, and (C) is proved. This ends the proof of Theorem 3.1.

4. α-Properties of R (I) and I (I)

NOTE ON TERMINOLOGY. The first property for fuzzy topological spaces which was called an α-property was that of α-compactness introduced by T.E. Gantner, R.C. Steinlage and R.H. Warren in [1]. Later it was shown by the author in [5] that this concept was equivalent to the compactness (in the ordinary sense) of the α-level space of the fuzzy topological space in question. Still later in [10] S.E. Rodabaugh introduced a notion, called α-Hausdorffness, as follows: (X, Δ) is α-Hausdorff if for all $x \neq y \in X$, there exist $\mu, \nu \in \Delta$ such that $\mu(x) \wedge \nu(y) > \alpha$ and $\mu \wedge \nu = 0$. This notion thus actually involves two levels, containment of x and y in μ and ν respectively at α-level and empty intersection of μ and ν at 0-level. We would like to suggest that such a notion might better be called $(\alpha, 0)$-Hausdorffness and to reserve the α-notation for properties of the α-level space, in consistency with the first α-property as introduced in [1].

DEFINITION 4.1. For any topological property P we shall say that a given fuzzy topological space X has $\alpha - P$ if and only if $\iota_\alpha(X)$ has P.

In the following theorem $+$ (resp. $-$) means that the corresponding property is (resp. is not) fulfilled by the corresponding space.

THEOREM 4.1. *The following table gives a list of α-properties which are either fulfilled or not fulfilled by* $\mathbf{R}(I)$ *and* $I(I)$.

	$\mathbf{R}(I)$			$I(I)$	
	$\alpha=0$	$0<\alpha<(1/2)$	$(1/2)\leqq\alpha<1$	$0\leqq\alpha<(1/2)$	$(1/2)\leqq\alpha<1$
T_0	$-$	$-$	$-$	$-$	$-$
Regular	$-$	$-$	$-$	$-$	$-$
Normal	$-$	$-$	$+$	$-$	$+$
Paracompact	$-$	$-$	$-$	$+$	$+$
Compact	$-$	$-$	$-$	$+$	$+$
Pseudocompact	$+$	$+$	$+$	$+$	$+$
σ-compact	$+$	$+$	$+$	$+$	$+$
Locally compact	$+$	$+$	$+$	$+$	$+$
2^{nd} countable	$+$	$+$	$+$	$+$	$+$
2^{nd} category	$+$	$-$	$+$	$+$	$+$
Baire	$+$	$-$	$+$	$+$	$+$
Path connected	$+$	$+$	$+$	$+$	$+$
Locally path connected	$+$	$+$	$+$	$+$	$+$
Hyperconnected	$+$	$+$	$-$	$+$	$-$
Ultraconnected	$-$	$-$	$+$	$-$	$+$

PROOF. That neither $\mathbf{R}(I)$ nor $I(I)$ is α-T_0 for any $\alpha \in I_1$ follows from the fact that no T_0-modification of an α-level space reduces to that α-level space, which in turn follows at once by easy verification.

For all the remaining properties it follows from Proposition 2.2 that it is sufficient to verify them on the T_0-modifications of the α-level spaces which in turn is equivalent, by Theorem 3.4 to verifying them on the spaces P_u^*, P_u, P_1, D_u and D_1. We shall leave this to the reader, and rather comment upon some consequences of Theorem 4.1.

COMMENTS. (1) In [5] we introduced the notion of strong compactness meaning α-compact for all $\alpha \in I_1$. If, as in Definition 4.1, we adopt this terminology for all topological properties then by simply looking at the table in Theorem 4.1 the reader can for himself formulate which strong properties are or are not fulfilled by $\mathbf{R}(I)$ and $I(I)$.

(2) It is known that both $\mathbf{R}(I)$ and $I(I)$ are completely regular in the sense of [4]. Thus the entries at regular in the table of Theorem 4.1 gives us

Completely regular [4] $\not\Rightarrow$ α-completely regular for some $\alpha \in I$,

or even stronger

Completely regular [4] $\not\Rightarrow$ α-regdlar for some $\alpha \in I_1$.

(3) Analogously $\mathbf{R}(I)$ and $I(I)$ are normal [9] in the sense of [4] and from Theorem 4.1, $\iota_1(\mathbf{R}(I))$ and $\iota_\alpha(I(I.)$ are normal only if $\alpha \in [1/2, 1[$, thus we can formulate

Normal [4] $\not\Rightarrow$ Strongly normal.

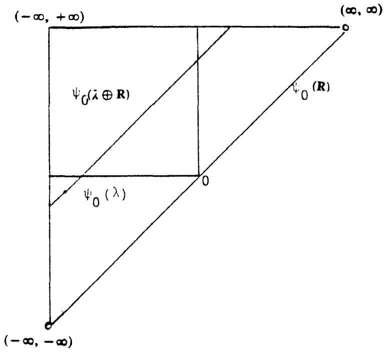

FIG. 1: P_u^*

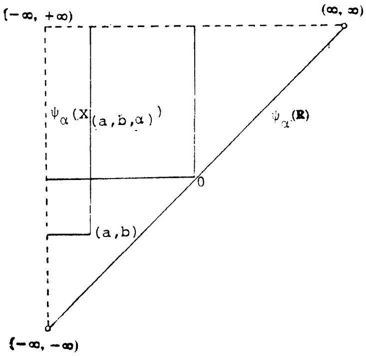

FIG. 2: P_u

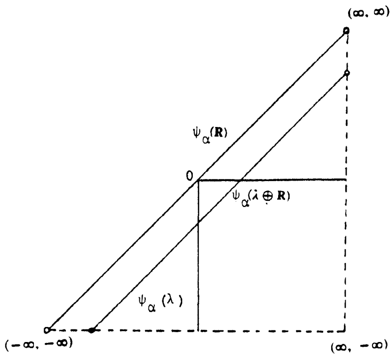

FIG. 3: P_1

Recall that
$P_u^* \cong \iota_0(\mathbf{R}(I))^0$
$P_u \cong \iota_0(\mathbf{R}(I))^0$ if $0 < \alpha < 1/2$
$P_1 \cong \iota_\alpha(\mathbf{R}(I))^0$ if $1/2 < \alpha < 1$

Remark moreover that $\iota_\alpha(\mathbf{R}\,(I))$ and $\iota_\alpha(I(I))$ for $\alpha \in [1/2, 1[$ are normal only in the rather vacuous sense that no disjoint closed subsets exist.

5. Final comments. In this section we point out how the previous results answer some open questions, we give some more consequences and we discuss the generalization of our technique to arbitrary lattices and to α^*-level spaces.

I. In [9] S. Rodabaugh asked a question as to the possibility of characterizing compact (in any sense) subspaces of $\mathbf{R}(I)$, using the $a(\mu, \alpha)$'s and $b(\mu, \alpha)$'s (i.e., the function ψ_α!).

For α-compactness and strong compactness our results provide a positive answer. Indeed we have (from Proposition 2.2 and Theorem 3.4)

THEOREM 5.1. *$X \subset \mathbf{R}\,(I)$ is strong compact (resp. α-compact for some $\alpha \in I_1$) if and only if $\psi_\alpha(X)$ is compact for each $\alpha \in I_1$ (resp. $\psi_\alpha(X)$ is compact).*

This produces some remarkable α-compact subspaces. To illustrate this without too much formalism we draw some pictures representing P_u^*, P_u and P_1.

A FEW EXAMPLES. (1) For all $(a, b, \alpha) \in P_u^* \times \{0\} \cup P_u \times]0, 1/2[\cup P_1 \times [1/2, 1[$ any subspace

$$Y \subset X_{(a,b,\alpha)} := \{\mu \in \mathbf{R}\,(I) \mid a(\mu, \alpha) \leqq a,\ b \leqq b(\mu, \alpha)\}$$

such that $\psi_\alpha(Y) = \psi_\alpha(X_{(a,b,\alpha)})$ is α-compact (see Fig. 2).

(2) Although $\mathbf{R} \subset \mathbf{R}\,(I)$ is not 0-compact, remark that any translation of $\mathbf{R}$ over any element $\lambda \in \mathbf{R}(I)\backslash\mathbf{R}$ is 0-compact! Indeed by the fundamental identities [9]

$$a(\mu \oplus \lambda, \alpha) = a(\mu, \alpha) + a(\lambda, \alpha) \text{ and } b(\mu \oplus \lambda, \alpha) = b(\mu, \alpha) + b(\lambda, \alpha)$$

we have for any $\alpha \in I_1$ $\psi_\alpha(\lambda \oplus \mathbf{R}) = \psi_\alpha(\lambda) + \psi_\alpha(\mathbf{R})$. For $\alpha = 0$ (see Fig. 1) we thus see that $\psi_0(\lambda \oplus \mathbf{R}) \cong \mathbf{R}$ and thus that $\lambda \oplus \mathbf{R} \cong \bar{\mathbf{R}}$ as subspace of $\iota_0(\mathbf{R}(I))$.

II. On the other hand for $\alpha \in]0, 1[$ we see that for any $\lambda \in \mathbf{R}\,(I)$, $\psi_\alpha(\lambda + \mathbf{R}) \simeq \mathbf{R}$ (Fig. 3). This leads to the following remark, Card $\mathbf{R}(I) = c$. Indeed from $\mathbf{R} \subset \mathbf{R}\,(I)$ we have $\geqq$. Conversely since any $\mu \in \mathbf{R}\,(I)$ is completely determined by its values on Q

$$\mathbf{R}\,(I) \longrightarrow I^{\mathbf{Q}}$$
$$\mu \longrightarrow (\mu\,(q))_{q \in \mathbf{Q}}$$

is an injection and $\leqq$ follows from Card $I^{\mathbf{Q}} = c^{\aleph_0} = c$.

Now it can be shown (partly done in Proof of Theorem 3.1 (final part)

(A)) that for all $(a, b, \alpha) \in P_u \times]0, 1/2[\cup P_1 \times [1/2, 1[$: Card $\psi_\alpha^{-1}(a, b) =$ Card $\mathbf{R}$ (I). Thus if we take $a \in \mathbf{R}^+$ and consider $(-a, a) + \psi_\alpha(\mathbf{R})$, then by choosing each time one representative in each equivalence class we can write $\psi_\alpha^{-1}((-a, a) + \psi_\alpha(\mathbf{R}))$ as a disjoint union of c copies of $\mathbf{R}$.

Since there are also c choices for $a \in \mathbf{R}^+$ and all the spaces $(-a, a) + \psi_\alpha(\mathbf{R})$ are mutually disjoint it follows that we can write $\iota_\alpha(\mathbf{R}(I))$ as the disjoint union of $c^2 = c$ subspaces each of them homeomorphic to $\mathbf{R}$, and this in $c.\ c^c = 2^c$ different ways.

III. Another remarkable feature is that for all $\alpha \in [0, 1/2[$, $\psi_\alpha(\mathbf{R})$ is closed in P_α while for an $\alpha \in [1/2, 1[$ it is dense. Thus analogously as in II there are c closed copies of $\mathbf{R}$ in $\iota_\alpha(\mathbf{R}(I))$ for $\alpha \in [0, 1/2[$ and c dense copies of $\mathbf{R}$ in $\iota_\alpha(\mathbf{R}(I))$ if $\alpha \in [1/2, 1[$.

Obviously by simply checking the spaces P_α and D_α the reader can draw many more interesting conclusions regarding α-properties of subspaces of $\mathbf{R}$ (I).

IV. We have restricted our attention to $L = I$ for at least two reasons.

(1) The unit interval is clearly the most important lattice in fuzzy set theory as a survey of the literature easily shows.

(2) It is our experience that the unit interval as the underlying lattice usually permits a coherent development in which lattice-theoretic considerations do not obscure the topological questions.

V. It is obvious that a^*-analogue of our paper can be written without too much hardship.

α^*-level spaces were defined in [**5**] as follows.

If (X, Δ) is a fuzzy topological space then $\iota_\alpha^*(X)$ is the topological space $(X, \iota_\alpha^*(\Delta))$ where $\iota_\alpha^*(\Delta)$ is generated by the basis $\{\mu^{-1}[\alpha, 1] \mid \mu \in \Delta\}$.

Instead of using the $a(\mu, \alpha)$'s and $b(\mu, \alpha)$'s simply use the $a^*(\mu, \alpha)$'s and $b^*(\mu, \alpha)$'s and proceed.

References

1. T.E. Gantner, R. C. Steinlage and R.H. Warren, *Compactness in fuzzy topological spaces*, J. Math. Anal. Appl. **63** (1978), 347–362.

2. B. Hutton, *Normality in fuzzy topological spaces*, J. Math. Anal. Appl. **50** (1975), 74–79.

3. B. Hutton, *Uniformities on fuzzy topological spaces*, J. Math. Anal. Appl. **58** (1977), 559–571.

4. B. Hutton and I. Reilly Separation axioms in fuzzy topological spaces, Fuzzy Sets and Systems **3** (1980), 93–104.

5. R. Lowen, *A comparison of different compactness notions in fuzzy topological spaces*, J. Math. Anal. Appl. **64** (1978), 446–454.

6. R. Lowen, *Compact Hausdorff fuzzy topological spaces are topological*, Topology and its Applications **12** (1981), 65–74.

7. R. Lowen, *On* (R (L), $\oplus$), Fuzzy Sets and Systems **10** (1983), 203–209.

8. J.I. Nagata, *Modern general topology*, North Holland Publishing Company (1968).

9. S.E. Rodabaugh, *The L-fuzzy real line and its subspaces*, in Fuzzy Set and Possibility theory, Pergamon Press, (1982), 402–418.

10. S.E. Rodabaugh, *The Hausdorff separation axiom for fuzzy topological spaces*, Topology and its Applications **11** (1980), 319–334.

11. S.E. Rodabaugh, *A categorical accomodation of various notions of fuzzy topology*, Fuzzy Sets and Systems **9** (1983), 241–265.

12. L.A. Steen and J.A. Seebach, Jr., *Counterexamples in topology*, Holt, Rinehart and Winston, Inc. (1970).

13. M.H. Stone, *Application of Boolean algebras to topology*, Mat. Sb. **1** (1936), 765–771.

14. W.J. Thron, *Topological structures*, Holt, Rinehart and Winston (1966).

15. R.H. Warren, *Identification spaces and unique uniformity*, Pacific Journal of Mathematics, **95** (1981), 483–492.

Wiskundige Analyse, University of Antwerp, R.U.C.A., Groenenborgerlaan 171, 2020 Antwerpen, Belgium