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THE INFLUENCE OF THE SPECTRUM OF THE SYMMETRIZED FOURTH ORDER OPERATOR ON THE GEOMETRY OF A RIEMANNIAN MANIFOLD

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1. Introduction. Let (M, g) be a compact Riemannian manifold of dimension n, $\dim M = n$. Let Δ be the Lapalace operator which acts on the algebra $C^\infty(M)$. We denote by $\mathrm{Sp}(M, g)$ the spectrum of Δ on $C^\infty(M)$. The influence of $\mathrm{Sp}(M, g)$ on the geometry on (M, g) is treated in [**6**] and [**16**].

There are other operators of higher order which act on the algebra $C^\infty(M)$. These operators have a spectrum which is related to the geometry on (M, g).

The aim of the present paper is to study the influence of the spectrum of a special fourth order operator, which is called the symmetrized fourth order Laplace operator on the geometry of a compact Riemannian manifold.

The whole paper contains seven paragraphs. The second paragraph gives the definition of the symmetrized fourth order Laplace operator and some properties. The coefficients of the asymptotic expansion of a function which is constructed by the spectrum of the operator are computed in the third paragraph. The fourth paragraph studies the influence of the spectrum of this symmetrized fourth order Laplace operator on the geometry of a compact Riemannian manifold. The fifth paragraph deals with special compact Riemannian manifolds whose geometry is determined by the spectrum of this Laplace operator.

The sixth paragraph deals with the influence of this symmetrized fourth order Laplace operator on the geometry of a compact Kahler manifold.

The last paragraph studies special compact Kahler manifolds whose geometry is determined by the spectrum of this symmetrized fourth order Laplace operator.

2. Let (M, g) be a compact Riemannian manifold whose dimension is n. Let P be a point of the manifold M. We consider a normal coordinate system $(x_1, \ldots, x_n)$ at the point P of M. This normal coordinate system covers an open neighborhood U of M with center the point P. Let $(y_1,$

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$y_2, \ldots, y_n)$ be any other coordinate system on U. The Laplace operator Δ on U is given by

$$(2.1) \qquad \Delta = \frac{1}{\sqrt{|g|}} \sum_{i,j=1}^{n} \frac{\partial}{\partial y_i}\left(g^{ij}\sqrt{g}\,\frac{\partial}{\partial y_j}\right),$$

where g_{ij} are the components of the metric tensor g relative to $(y_1, \ldots, y_n)$, (g^{ji}) is the inverse matrix of (g_{ij}), and $|g| = \det(g_{ij})$.

On U we consider another operator $\bar{\Delta}_P$ for the differential functions $D^\circ(U)$ on U, which, with respect to the normal coordinate system $(x_1, \ldots, x_n)$, can be written

$$(2.2) \qquad \bar{\Delta}_P = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_1^2}.$$

The operator $\bar{\Delta}_P$, defined by (2.2), is called the Euclidean Laplacian. It can be easily proved that

$$(2.3) \qquad (\Delta f)_P = (\bar{\Delta}_P f)_P,$$

but in general the relation

$$(\Delta^k f)_P = (\bar{\Delta}_P^k f)_P$$

is not valid.

From the above we obtain that the formula (2.1) is valid for any local coordinate system, in contrast to the formula (2.2) which is valid for a normal coordinate system with origin at the point P.

On the manifold M we consider the covariant derivative ∇ and denote by ∇^k the k^{th} power of ∇. Let f be a function on the manifold M. Now we can consider $\nabla^k f$ as a covariant tensor field of degree k. The value of this covariant tensor field on vector fields $X_1, \ldots, X_k$ is denoted by

$$(2.4) \qquad \nabla^k_{X_1, \ldots, X_k} f.$$

Let $(e_1, \ldots, e_n)$ be a local orthonormal frame. Below we use the expression

$$\nabla_{i_1 \cdots i_k}$$

instead of the expression

$$\nabla^k_{e_{i1}, \ldots, e_{ik}}, \quad 1 \leqq i_1, \ldots, i_k \leqq n.$$

The following formulas can be proved easily.

$$(2.5) \quad \nabla^k_{X_1 \cdots X_{k-1} X_k} f = \nabla^k_{X_1 \cdots X_{k-2} X_k X_{k-1}} f, \quad X_1, \ldots, X_{k-1}, X_k \in D^1(M), f \in D^\circ(M)$$

$$(2.6) \qquad \Delta^k f = \sum_{i_1, \ldots, i_k = 1}^{n} \nabla^{2k}_{i_1 i_1 \cdots i_k i_k} f.$$

It can be proved easily that the right hand side of (2.6) is independent of the choice of frame.

Therefore the operator Δ^k is globally defined on the whole Riemannian manifold (M, g).

Now we want to define the operator L of degree $2k$ on the algebra $D^\circ(M)$ which has the property [11]

(2.7) $$L_P f = (\bar{\Delta}_P f)_P.$$

The operator L can be defined as

(2.8) $$Lf = \frac{1}{1.3\ldots.(2k-1)} \sum_{i_1\cdots i_k=1}^{n} \{\nabla^{2k}_{i_1 i_1\cdots i_k i_k} f + \cdots + \nabla^{2k}_{i_1\cdots i_k i_k\cdots i_1} f\}.$$

The right hand side of (2.8) does not depend on the choice of orthonormal frame.

The relation

(2.9) $$(\bar{\Delta}_P f)_P = (\Delta f)_P = \left(\sum_{i=1}^{n} \nabla_{ii} f\right)_P$$

can be proved:

Let R_{ijkl} and p_{ij} be the components of the curvature and Ricci tensor fields with respect to an orthonormal basis of $T_P(M)$. We choose the signs so that

(2.10) $$R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$$

for all $X,\ Y \in D^1(M)$.

We denote by τ the scalar curvature on the Riemannian manifold.

It has been proved [11] that

(2.11) $$(\bar{\Delta}^2_P f)_P = \left(\Delta^2 f + \frac{1}{3}\langle \nabla f, \nabla\tau\rangle + \frac{2}{3}\langle \nabla^2 f, p\rangle\right)_P,$$

where

(2.12) $$\langle df, d\tau\rangle = \langle \nabla f, \nabla\tau\rangle = \sum_{i=1}^{n} (\nabla_i f)\,(\nabla_i \tau),$$

(2.13) $$\langle \nabla^2 f,\ p\rangle = \sum_{i,j=1}^{n} (\nabla^2_{ij} f) p_{ij},$$

and $(\nabla^2 f)_P$ is just the Hessian of f at m.

The given second powers of the Laplacians (other than Δ^2) are the following

(2.14) $$\sum_{i,j=1}^{n} \nabla^4_{ijij}, \qquad \sum_{i,j=1}^{n} \nabla^4_{ijji},$$

which are equal and can be expressed by

$$(2.15)\qquad \sum_{i,j=1}^{n} \nabla^4_{ijij} f = \sum_{i,j=1}^{n} \nabla^4_{ijji} f = \Delta^2 f + \frac{1}{2}\langle \nabla f, \nabla \tau\rangle + \langle \nabla^2 f, p\rangle.$$

From the above we conclude that we have the following natural fourth order operators with leading symbol given by the square of the matric tensor

$$(2.16)\qquad \Delta_{4,1}(f) = \Delta^2 f = \nabla^4_{iijj} f,$$

$$(2.17)\qquad \Delta_{4,2}(f) = \nabla^4_{ijij} f = \Delta_{4,3}(f) = \nabla^4_{ijji} f,$$

$$(2.18)\qquad \bar{\Delta}_2 f = \frac{1}{3}(\nabla^4_{iijj} f + \nabla^4_{ijij} f + \nabla^4_{ijji} f).$$

Therefore we have three distinct operators. The operator $\bar{\Delta}_2$ is called the symmetrized fourth order Laplacian.

3. From the above we obtain that

$$(3.1)\qquad \bar{\Delta}_2 \colon C^\infty(M) \longrightarrow C^\infty(M),$$

$$(3.2)\qquad \bar{\Delta}_2 \colon f \longrightarrow \bar{\Delta}_2(f), \quad f \in C^\infty(M).$$

If we have $\bar{\Delta}_2 f = \lambda f$, $\lambda \in \mathbf{R}$, then f is called an eigenfunction of $\bar{\Delta}_2$ and λ the eigenvalue of $\bar{\Delta}_2$ associated to f.

The set of all eigenvalues of $\bar{\Delta}_2$ is denoted by $\mathrm{Sp}(M, g, \bar{\Delta}_2)$ and therefore we have

$$(3.3)\qquad \mathrm{Sp}(M, g, \bar{\Delta}_2) = \{\lambda / \bar{\Delta}_2 f = \lambda f\}.$$

This has the form

$$(3.4)\qquad \mathrm{Sp}(M, g, \bar{\Delta}_2) = \{0 = \lambda_0 < \lambda_1 = \cdots = \lambda_1 < \lambda_2 = \cdots \lambda_2 < \cdots < \infty\}.$$

This means the spectrum is discrete and the multiplicity of each eigenvalue is finite, since $\bar{\Delta}_2$ is an elliptic operator.

Now we form the function

$$(3.5)\qquad f(t, \bar{\Delta}_2) = \sum_{i=0}^{\infty} k_i e^{-\lambda_i t},$$

where $k_0 = 1$ and k_i, $i \geqq 1$, is the multiplicity of the eigenvalue λ_i.

If we obtain the asymptotic expansion of the formula (3.5), then we have

$$(3.6)\qquad f(t, \bar{\Delta}_2) \approx_{t \to 0+} (4\pi t)^{-n/2}\{\alpha_0(\bar{\Delta}_2) + \alpha_1(\bar{\Delta}_2)t + \alpha_2(\bar{\Delta}_2)t^2 + \cdots\},$$

where $\alpha_0(\bar{\Delta}_2)$, $\alpha_1(\bar{\Delta}_2)$, $\alpha_2(\bar{\Delta}_2)$, ... are real numbers which can be computed by the formulas

$$(3.7)\qquad \alpha(\bar{\Delta}_2) = \int_M u_i(\bar{\Delta}_2) dM, \quad i = 0, 1, 2, \ldots,$$

where dM is the volume element of M and

$$u_i(\bar{\Delta}_2)\colon M \longrightarrow \mathbf{R}, \quad i = 0, 1, 2, \ldots, \tag{3.8}$$

are functions which are local Riemannian invariants.

The three first coefficients $\alpha_0(\bar{\Delta}_2)$, $\alpha_1(\bar{\Delta}_2)$ and $\alpha_2(\bar{\Delta}_2)$ are given by the following theorem **[10]**.

THEOREM 3.1. *Let (M, g) be an orientable compact Riemannian manifold. The three first coefficients $\alpha_0(\bar{\Delta}_2)$, $\alpha_1(\bar{\Delta}_2)$ and $\alpha_2(\bar{\Delta}_2)$ of the symmetrized fourth order Laplacian are given by*

$$\alpha_0(\bar{\Delta}_2) = \frac{1}{2}(4\pi)^{-n/2}\,\Gamma^{-1}\Big(\frac{n}{2}\Big)\,\Gamma\Big(\frac{n}{4}\Big)\int_M dM, \tag{3.9}$$

$$\alpha_1(\bar{\Delta}_2) = -\frac{1}{2}(4\pi)^{-n/2}\,\Gamma\Big(\frac{n-2}{2}\Big)^{-1}\Gamma\Big(\frac{n-2}{4}\Big)\frac{n+2}{6n}\int \tau\, dM, \tag{3.10}$$

$$\begin{aligned}\alpha_2(\bar{\Delta}_2) = {} & \frac{1}{2}(4\pi)^{-n/2}\,\Gamma\Big(\frac{n-4}{2}\Big)^{-1}\Gamma\Big(\frac{n-4}{4}\Big)\frac{1}{360(n-1)(n+2)} \\ & \cdot \int ((5n^2 + 20n + 40)\tau^2 \\ & - (2n^2 + 40n + 32)\,|p|^2 + 2(n^2 - 4)\,|R|^2)\,dM,\end{aligned} \tag{3.11}$$

where $\Gamma(k)$ is the gamma function defined by

$$\Gamma(k) = \int_0^\infty t^{k-1}e^{-t}dt \tag{3.12}$$

which satisfies the functional relationship

$$k\Gamma(k) = \Gamma(k + 1) \tag{3.13}$$

and defines a meromorphic function with simple poles $k = 0, -1, \ldots$ and $|R|$, $|p|$ are the norms of R and p respectively with respect to g.

REMARK 3.2. From the above we conclude that the coefficients in the formulas (3.10) and (3.11) depend in a non-trivial fashion on the dimension n of the manifold.

The coefficients

$$\Gamma\Big(\frac{n-2}{2}\Big)^{-1}\Gamma\Big(\frac{n-2}{4}\Big) = \sigma_1(n), \tag{3.14}$$

$$\Gamma\Big(\frac{n-4}{2}\Big)^{-1}\Gamma\Big(\frac{n-4}{4}\Big)\frac{1}{n-1} = \sigma_2(n) \tag{3.15}$$

are regular holomorphic functions if n since the poles cancel the zero at the exceptional value of $n = 2$. In this case we use the limiting values.

The formulas (3.9), (3.10) and (3.11), by means of (3.14) and (3.15), can be written

$$\alpha_0(\bar{\Delta}_2) = q_0(n) \int_M dM, \tag{3.16}$$

$$\alpha_1(\bar{\Delta}_2) = - q_1(n) \int_M \tau dM, \tag{3.17}$$

$$\alpha_2(\bar{\Delta}_2) = q_2(n) \int_M (A_1(n)\, \tau^2 - A_2(n)\, |p|^2 + A_3(n)\, |R|^2) dM, \tag{3.18}$$

where

$$q_0(n) = \frac{1}{2} (4\pi)^{-n/2}\, \Gamma\left(\frac{n}{2}\right)^{-1} \Gamma\left(\frac{n}{2}\right), \tag{3.19}$$

$$q_1(n) = \frac{1}{2} (4\pi)^{-n/2}\, 6_1(n) \frac{n+2}{6n}, \tag{3.20}$$

$$q_2(n) = \frac{1}{2} (4\pi)^{-n/2}\, 6_2(n) \frac{1}{360(n+2)}, \tag{3.21}$$

$$A_1(n) = 5n^2 + 20n + 40, \quad A_2(n) = 2n^2 + 40n + 32, \quad A_3(n) = 2(n^2 - 4) \tag{3.22}$$

4. Let (M, g) be an orientable, compact Riemannian manifold. For this manifold we have

PROPOSITION 4.1. *We consider an orientable, compact Riemannian manifold of dimension* 2. *Then the spectrum* $\mathrm{Sp}(M, g, \bar{\Delta}_2)$ *determines the Euler–Poincare characteristic* $x(M)$ *of* M.

PROOF. The form (3.17), for $n = 2$, implies

$$\alpha_1 = - q_1(2) \int \tau dM. \tag{4.1}$$

Let $x(M)$ be the Euler–Poincare characteristic of the two-dimensional orientable compact Riemannian manifold (M, g). This is given by the Gauss-Bonnet formula

$$x(M) = \frac{1}{2\pi} \int \tau dM. \tag{4.2}$$

From (4.1) and (4.2) we conclude that

$$x(M) = - \frac{1}{2\pi q_1(2)} \alpha_1. \tag{4.3}$$

Therefore $x(M)$ is isospectral invariant.

Let (M, g), (M', g') be two orientable compact Riemannian manifolds. If we have the condition

$$\text{(4.4)} \qquad \mathrm{Sp}(M, g, \bar{\Delta}_2) = \mathrm{Sp}(M', g', \bar{\Delta}_2'),$$

then the two manifolds are called isospectral with respect to the symmetrized fourth order Laplace operator.

From the definition of the isospectral orientable compact Riemannian manifolds (M, g) and (M', g') with respect to $\bar{\Delta}_2$ we obtain

$$\text{(4.5)} \qquad \alpha_i(\bar{\Delta}_2) = \alpha_i'(\bar{\Delta}_2'), \quad i = 0, 1, \ldots,$$

where these coefficients are defined by (3.7).

Now, from the above, we have the following theorem.

THEOREM 4.2. *Let (M, g), (M', g') be two orientable compact Riemannian manifolds with the property*

$$\text{(4.6)} \qquad \mathrm{Sp}(M, g, \bar{\Delta}_2) = \mathrm{Sp}(M', g', \bar{\Delta}_2').$$

Then we have

(i) $\mathrm{Vol}(M) = \mathrm{Vol}(M')$,
(ii) $\dim M = \dim M'$.

PROOF. From (4.6) we conclude that

$$\text{(4.7)} \quad (4\pi t)^{-n/2}\{\alpha_0+\alpha_1 t+\alpha_2 t^2+\cdots\}=(4\pi t)^{-n'/2}\{\alpha_0'+\alpha_1' t+\alpha_2' t^2+\cdots\},$$

where $n = \dim M$ and $n' = \dim M'$.

The relation (4.7) by means of (4.5) implies

$$\text{(4.8)} \qquad n = \dim M = n' = \dim M'.$$

From (4.6), by virtue of (4.5), we obtain

$$\text{(4.9)} \qquad \alpha_0(\bar{\Delta}_2) = \alpha_0'(\bar{\Delta}_2'),$$

which, by means of (3.16) and (4.8), gives

$$\text{(4.10)} \qquad q_0(n)\int_M dM = q_0(n)\int_{M'} dM'.$$

Equation (4.10) finally gives

$$\text{(4.11)} \qquad \int_M dM = \mathrm{Vol}\, M = \int_{M'} dM' = \mathrm{Vol}\, M'.$$

REMARK 4.2. The formula (3.18), for $n = 4$, becomes

$$\text{(4.12)} \qquad \alpha_2(\bar{\Delta}_2) = q_2(4)\int_M (200\tau^2 - 224\,|p|^2 + 24|R|^2)dM.$$

The Euler-Poincare characteristic $x(M)$ of (M, g), for dimension 4, is given by

$$x(M) = \frac{1}{32\pi^2} \int (|R|^2 - |4|p^2 + \tau^2) dM. \tag{4.13}$$

From (4.12) and (4.13) we conclude that

$$\frac{1}{q_2(4)} \alpha_2(\bar{\Delta}_2) - 768x(M) = \int_M (176\tau^2 - 128|p|^2) dM. \tag{4.14}$$

The second member of (4.14) depends on the Riemannian metric on the manifold. Therefore it is not topologically invariant.

From this we conclude that $\alpha_2(\bar{\Delta}_2)$ is not topologically invariant in contrast to the fact that $\alpha_1(\bar{\Delta}_2)$ is topologically invariant.

5. Let (M, g), (M', g') be two orientable compact Riemannian manifolds with the property

$$\mathrm{Sp}(M, g, \bar{\Delta}_2) = \mathrm{Sp}(M', g', \bar{\Delta}_2'). \tag{5.1}$$

We study the condition (5.1) with some other relations under which the Riemannian manifolds (M, g) and (M', g') are isometric.

THEOREM 5.1. *Let (M, g), (M', g') be two orientable compact Riemannian manifolds of dimension 2 with the property (5.1). (M, g) has constant Gauss curvature if and only if (M', g') has constant Gauss curvature.*

PROOF. From the condition (5.1) we obtain

$$\alpha_0(\bar{\Delta}_2) = \alpha_0'(\bar{\Delta}_2'), \qquad \alpha_1(\bar{\Delta}_2) = \alpha_1'(\bar{\Delta}_2'), \tag{5.2}$$

$$\alpha_2(\bar{\Delta}_2) = \alpha_2'(\bar{\Delta}_2'). \tag{5.3}$$

From (5.2) and (5.3), by means of (3.16), (3.17), (3.18) and (3.22) and taking under consideration that $n = 2$, we obtain

$$\int_M dM = \mathrm{Vol}\, M' = \int_{M'} dM' = \mathrm{Vol}\, M', \tag{5.4}$$

$$\int_M \tau dM = \int_{M'} \tau' dM', \tag{5.5}$$

$$\int_M (100\tau^2 - 120|p|^2) dM = \int_{M'} (100\tau'^2 - 120|p'|^2) dM'. \tag{5.6}$$

Since the Riemannian manifolds (M, g) and (M', g') are of dimension two we have

$$|p|^2 = \frac{\tau^2}{2}, \quad |R|^2 = 2|p|^2 = \tau^2, \tag{5.7}$$

$$|p'|^2 = \frac{\tau'^2}{2}, \quad |R'|^2 = 2|p'|^2 = \tau'^2. \tag{5.8}$$

By use of (5.7) and (5.8), formula (5.6) takes the form

$$\int_M 40\tau^2 dM = \int_{M'} 40\tau'^2\, dM'. \tag{5.9}$$

We assume that the Riemannian manifold (M', g') has constant Gauss curvature, and therefore

$$\tau' = k' = \text{constant}. \tag{5.10}$$

From the relations (5.5) and (5.10), we conclude the inequality

$$\int_M \tau^2 dM \geqq \int_{M'} \tau'^2\, dM'. \tag{5.11}$$

The equality holds if $\tau = \text{constant}$.

From (5.9) and (5.11) we conclude that

$$\int_M \tau^2\, M = \int_{M'} \tau'^2\, dM', \tag{5.12}$$

which implies $\tau = \text{const} = k$.

The equality (5.5) implies

$$\tau' = k' = \tau = k, \tag{5.13}$$

From Theorem 5.1., we obtain the following propositions.

PROPOSITION 5.2. *Let (M, g) be an orientable compact simply connected Riemannian manifold of dimension 2. Let (S^2, g_0) be the standard Euclidean sphere of dimension 2 with the standard metric g_0 of Gauss curvature 1. We assume*

$$\mathrm{Sp}(M, g, \bar{\Delta}_2) = \mathrm{Sp}(S^2, g_0, \bar{\Delta}_2). \tag{5.14}$$

Then (M, g) is isometric to the sphere (S^2, g_0).

PROOF. By the assumption (5.14) and by Theorem 5.1, we conclude that the Gauss curvature of (M, g) is constant and equal to 1. Therefore (M, g) is locally isometric to (S^2, g_0). From the fact that (M, g) is simply connected, we conclude that (M, g) is isometric to (S^2, g_0).

PROPOSITION 5.3. *Let (M, g) be an orientable compact Riemannian manifold of dimension 2 whose first homotopy group is $\mathbf{Z}_2$. If we have the condition*

$$\mathrm{Sp}(M, g, \bar{\Delta}_2) = \mathrm{Sp}(\mathbf{P}^2(\mathbf{R}), g_0', \bar{\Delta}_2'), \tag{5.15}$$

then (M, g) is isometric to $(\mathbf{P}^2(\mathbf{R}), g_0')$, which is the real projective space of dimension 2 with the standard metric g_0'.

PROOF. From (5.15), we conclude that (M, g) has constant sectional curvature $k = k' = 1$, where k' is the sectional curvature of $(\mathbf{P}^2(\mathbf{R}), g_0')$. Since by assumption, we have

$$\pi_1(M) = \pi_1(\mathbf{P}^2(\mathbf{R})) = \mathbf{Z}_2, \tag{5.16}$$

we conclude that (M, g) is isometric to $(P^2(\mathbf{R}), g_0')$.

From the above propositions we conclude the following corollaries.

COROLLARY 5.4. $\mathrm{Sp}(S^2, g_0, \Delta_2)$ *determines completely the geometry on* (S^2, g_0).

COROLLARY 5.5. $\mathrm{Sp}(\mathbf{P}^2(\mathbf{R}), g_0', \Delta_2)$ *determines completely the geometry on* $(\mathbf{P}^2(\mathbf{R}), g_0')$.

Let (M, g) be an orientable compact Riemannian manifold of dimension n. Let (U, φ) be a chart on the manifold with local coordinate system $(x^1, \ldots, x^n)$. Let C, G be the Weyl conformal curvature tensor field and the Einstein tensor field on (M, g). The components (C_{ijkl}) and (G_{ij}) of C and G respectively with respect to $(x^1, \ldots, x^n)$, are given by

$$\begin{aligned} C_{ijkl} = R_{ijkl} &- \alpha(p_{jk}g_{il} - p_{jl}g_{ik} - p_{il}g_{jk} - p_{ik}g_{jl}) \\ &+ \beta(g_{jk}g_{il} - g_{jl}g_{ik})\,\tau, \end{aligned} \tag{5.17}$$

$$G_{ij} = p_{ij} - \gamma\, g_{ij}\, \tau, \tag{5.18}$$

where

$$\alpha = \frac{1}{n-1}, \qquad \beta = \frac{1}{(n-1)(n-2)}, \qquad \gamma = \frac{1}{n}. \tag{5.19}$$

From (5.17) and (5.18) we obtain

$$|C|^2 = |R|^2 - \frac{4|p|^2}{n-2} + \frac{2\tau^2}{(n-1)(n-2)}, \tag{5.20}$$

$$|G|^2 = |p|^2 - \frac{\tau^2}{n}, \tag{5.21}$$

Using (5.20) and (5.21), (3.18) takes the from

$$\alpha_2(\Delta_2) = q_2(n) \int_M (B_1\tau^2 + B_2|G|^2 + B_3\,|C|^2)\,dM, \tag{5.22}$$

where

$$B_1 = A_1 - \frac{A_2}{n} + \frac{2A_3}{n(n-1)}, \tag{5.23}$$

$$B_2 = -A_2 + \frac{4A_3}{n-2}, \tag{5.24}$$

$$B_3 = A_3, \tag{5.25}$$

which by virtue of (3.22) take the form

$$B_1 = \frac{1}{n(n-1)}((n-1)(5n^3 + 18n^2 - 32) + 8(n^2 - 4)), \tag{5.26}$$

$$B_2 = -2(n^2 + 18n + 14), \tag{5.27}$$

$$B_3 = 2(n^2 - 4) \tag{5.28}$$

From (5.26), (5.27) and (5.28) we can conclude that

$$B_1 > 0, \text{ if } n \geqq 2, \tag{5.29}$$

$$B_2 < 0, \text{ if } n \geqq 2, \tag{5.30}$$

$$B_3 > 0, \text{ if } n > 2. \tag{5.31}$$

From (5.29), (5.30) and (5.31) we find that, in order to study the influence of Sp $(M, g, \bar{\Delta}_2)$ on the geometry of (M, g), we must assume that the Riemannian structure on (M, g) is an Einstein structure. Einstein structures are characterized by the fact that

$$|G|^2 = 0. \tag{5.32}$$

By (5.32), (5.22) becomes

$$\alpha_2(\bar{\Delta}_2) = q_2(n) \int_M (B_1\tau^2 + B_3 |C|^2)\, dM. \tag{5.33}$$

Now we can state the following theorem.

THEOREM 5.6. *Let (M, g), (M', g') be two orientable compact Einstein manifolds of dimension $n \geqq 3$. We assume that*

$$\mathrm{Sp}(M, g, \bar{\Delta}_2) = \mathrm{Sp}(M', g', \bar{\Delta}_2'). \tag{5.34}$$

The Einstein manifold M has constant sectional curvature k if and only if (M', g') has constant sectional curvature $k' = k$.

PROOF. We assume that (M', g') has constant sectional curvature k'. This implies

$$C' = 0 \Leftrightarrow |C'|^2 = 0. \tag{5.35}$$

From (5.34) we obtain

$$\alpha_0(\bar{\Delta}_2) = \alpha_0(\bar{\Delta}_2'), \qquad \alpha_1(\bar{\Delta}_2) = \alpha_1'(\bar{\Delta}_2'), \tag{5.36}$$

$$\alpha_2(\bar{\Delta}_2) = \alpha_2(\bar{\Delta}_2'). \tag{5.37}$$

By (3.17) the second equation of (5.36) gives

$$\int_M \tau dM = \int_{M'} \tau' dM'. \tag{5.38}$$

By (5.35), (5.33) implies

(5.39) $$\alpha_2'(\bar{\Delta}_2') = q_2(2) \int_{M'} B_1 \tau'^2 \, dM'.$$

By (5.33) and (5.39), (5.37) implies

(5.40) $$\int_M (B_3 |C|^2 + B_1 \tau^2) \, dM = \int_{M'} B_1 \tau'^2 \, dM'.$$

From (5.38), since $\tau' =$ const. we conclude that

(5.41) $$\int_M \tau^2 \, dM \geqq \int_{M'} \tau'^2 \, dM.$$

Equations (5.29), (5.31) when $n \geqq 3$, (5.40) and (5.41) imply

(5.42) $$|C|^2 = 0, \qquad C = 0.$$

Therefore the Einstein manifold (M, g) has constant sectional curvature k. Equation (5.38) implies $k = k'$.

THEOREM 5.7. *Let (M, g) be an orientable compact simply connected Einstein manifold of dimension $n \geqq 3$. If*

(5.43) $$\mathrm{Sp}(M, g, \bar{\Delta}_2) = \mathrm{Sp}(S^n, g_0, \bar{\Delta}_2)$$

where (S^n, g_0) is the standard Euclidean sphere with the Riemannian metric of constant sectional curvature 1, *then (M, g) is isometric to (S^n, g_0).*

PROOF. From Theorem 5.6. and assumption (5.43) we conclude that (M, g) has constant sectional curvature 1. Since (M, g) is orientable compact and simply connected we conclude that (M, g) is isometric to (S^n, g_0).

6. Let (M, J, g) be a compact Kahler manifold. Let (U, φ) be a chart on M on which we obtain a local coordinate system $(x^1, \ldots, x^n)$. Let B be the Bochner tensor field on (M, J, g). We denote by (B_{ijkl}) the components of B with respect to $(x^1, \ldots, x^n)$. These are given by

(6.1) $$\begin{aligned} B_{ijkl} = R_{ijkl} - \alpha^*(&p_{jk}g_{il} - p_{jl}g_{ik} + g_{jk}p_{il} - g_{jl}p_{ik} + p_{jr}J_k^r J_{il} \\ &- p_{jr}J_l^r J_{ik} + J_{jk}p_{ir}J_l^r - J_{jl}p_{ir}J_k^r - 2p_{kr}J_l^r J_{ij} - 2p_{ir}J_j^r J_{kl}) \\ &+ \beta^*(g_{jk}g_{il} - g_{jl}g_{ik} + J_{jk}J_{il} - J_{jl}J_{ik} - 2J_{kl}J_{ij})\tau, \end{aligned}$$

where

(6.2) $$\alpha^* = \frac{1}{n+4}, \qquad \beta^* = \frac{1}{(n+2)(n+4)}.$$

It is known that J is parallel and satisfies the relations

(6.3) $$J_j^i J_k^j = -\delta_k^i, \qquad g_{ij} J_r^i J_s^j = g_{rs}.$$

From (6.1), by (6.2) and (6.3), we obtan

$$|B|^2 = |R|^2 - \frac{16}{n+4}|p|^2 + \frac{8}{(n+2)(n+4)}\tau^2. \tag{6.4}$$

By (3.18), (5.21) and (6.4), we obtain

$$\alpha_2(\bar{\Delta}_2) = q_2(n)\int_M (L_1\tau^2 + L_2|G|^2 + L_3|B|^2)dM, \tag{6.5}$$

where

$$L_1 = A_1 - \frac{A_2}{n} + \frac{8A_3}{n(n+2)}, \tag{6.6}$$

$$L_1 = -A_2 + \frac{16A_3}{n+4}, \tag{6.7}$$

$$L_3 = A_3. \tag{6.8}$$

By (3.22), the relations (6.6), (6.7) and (6.8) become

$$L_1 = \frac{5n^3 + 18n^2 - 16n - 64}{n}, \tag{6.9}$$

$$L_2 = \frac{2(-20(n+4)(n+1)-(n^2-4)(n-2))}{n+4}, \tag{6.10}$$

$$L_3 = 2(n^2-4). \tag{6.11}$$

From the formulas (6.9). (6.10) and (6.11), we obtain

$$L_1 > 0, \text{ if } n \geqq 2, \tag{6.12}$$

$$L_2 < 0, \text{ if } n \geqq 2, \tag{6.13}$$

$$L_3 > 0, \text{ if } n \geqq 3, \tag{6.14}$$

The relations (6.12), (6.13) and (6.14) imply that, in order to study the influence of Sp $(M, g, \bar{\Delta}_2)$ on the geometry of the compact Kahler manifold, we must assume that (M, g) is an Einstein manifold which yields

$$|G|^2 = 0, \quad G = 0. \tag{6.15}$$

Theorefore, (6.5), by means of (6.15), takes the form

$$\alpha_2(\bar{\Delta}_2) = q_2(n)\int_M (L_1\tau^2 + L_3|B|^2)\, dM. \tag{6.16}$$

7. We consider two compact Kahler manifolds (M, g, J) and (M', g', J') which satisfy the relation

$$\mathrm{Sp}(M, J, g, \bar{\Delta}_2) = \mathrm{Sp}(M, J, g, \bar{\Delta}_2'). \tag{7.1}$$

The relation (7.1) implies

$$(7.2) \qquad \alpha_1(\bar{\Delta}_2) = \alpha_1(\bar{\Delta}'_2),$$

$$(7.3) \qquad \alpha_2(\bar{\Delta}_2) = \alpha_2(\bar{\Delta}'_2).$$

Equations (7.2) and (3.17) give

$$(7.4) \qquad \int_M \tau dM = \int_{M'} \tau' \, dM'.$$

Equations (7.3) and (3.18) imply

$$(7.5) \qquad \int_M (L_1\tau^2 + L_3 \, |B|^2) \, dM = \int_{M'} (L_1\tau'^2 + L_3 \, |B'|^2) \, dM'.$$

Now we can prove the following theorem.

THEOREM 7.1. *Let (M, J, g), (M', J', g') be two compact Kahler and Einstein manifolds of real dimension $n = 2m$. We assume*

$$(7.6) \qquad \mathrm{Sp}(M, J, g, \bar{\Delta}_2) = \mathrm{Sp}(M', J', g', \bar{\Delta}'_2).$$

The Kahler manifold (M, J, g) has constant holomorphic sectional curvature h if and only if (M', g', h') has constant holomorphic sectional curvature h' and $h = h'$.

PROOF. We assume that the Kahler Einstein manifold (M', J', g') has constant holomorphic sectional curvature $|h'|$. Therefore, we obtain

$$(7.7) \qquad |B'|^2 = 0,$$

hence $B' = 0$.

The relation (7.5), by means of (7.7), implies

$$(7.8) \qquad \int_M (L_1\tau^2 + L_3 \, |B|^2) \, dM = \int_{M'} L_1 \, \tau'^2 \, dM.$$

Since the Kahler manifold (M', J', g') is an Einstein manifold we have

$$(7.9) \qquad \tau' = \text{const.}$$

The equality (7.4), by virtue of (7.9), implies the inequality

$$(7.10) \qquad \int_M \tau^2 \, dM \geqq \int_{M'} \tau'^2 \, dM'.$$

From (7.8) and (7.10), by means of (6.12) and (6.14), we conclude that

$$(7.11) \qquad |B|^2 = 0,$$

so $B = 0$, which implies that the Kahler Einstein manifold (M, J, g) has constant holomorphic sectional curvature h.

The relation (7.4) implies

$$(7.12) \qquad h = h'.$$

Now we can prove

THEOREM 7.2. *Let (M, J, g) be a compact simply connected Kahler Einstein manifold of real dimension $n = 2m$. We assume*

$$\mathrm{Sp}(M, J, g, \bar{\Delta}_2) = \mathrm{Sp}(\mathbf{P}^m(\mathbf{C}), J_0, g_0, \bar{\Delta}_2) \tag{7.13}$$

where $(\mathbf{P}^m(\mathbf{C}), J_0, g_0)$ is the complex projective space with the Fubini-Study metric. Then (M, J, g) is holomorphically isometric to $(\mathbf{P}^m(\mathbf{C}), J_0, g_0)$.

PROOF. It is known that $(\mathbf{P}^m(\mathbf{C}), J_0, g_0)$ has constant holomorphic sectional curvature h. From this, (7.13), and Theorem 7.1., we conclude that (M, J, g) has the same constant holomorphic sectional curvature. Since (M, J, g) is simply connected we conclude that this is holomorphically isometric to $(\mathbf{P}^m(\mathbf{C}), J_0, g_0)$.

We consider the complex projective space $\mathbf{P}^m(\mathbf{C})$ with the almost complex structure J_0 which comes from the complex structure on $\mathbf{P}^n(\mathbf{C})$. On the complex manifold $\mathbf{P}^m(\mathbf{C})$ we consider two Kahler metrics g and g_0, where g_0 is the Fubini-Study metric whose holomorphic sectional curvature h_0 is constant.

Now we shall prove the following theorem.

THEOREM 7.3. *We consider the two complex Kahler manifolds* $(\mathbf{P}^m(\mathbf{C}), J_0, g_0)$ *and* $(\mathbf{P}^m(\mathbf{C}), J_0, g)$ *which satisfy the relation*

$$\mathrm{Sp}(\mathbf{P}^m(\mathbf{C}), J_0, g_0, \bar{\Delta}_2) = \mathrm{Sp}(\mathbf{P}^m(\mathbf{C}), J_0, g, \bar{\Delta}_2'). \tag{7.14}$$

If $m \geqq 4$, then $(\mathbf{P}^m(\mathbf{C}), J_0, g)$ has constant holomorphic sectional curvature h equal to the constant holomorphic sectional curvature h_0 of the complex projective space $(\mathbf{P}^n(\mathbf{C}), J_0, g_0)$ with the Fubini-Study metric g_0. Hence $(\mathbf{P}^m(\mathbf{C}), J_0, g) = (\mathbf{P}^m(\mathbf{C}), J_0, g_0)$.

PROOF. From the relation (7.14) we obtain

$$\alpha_1(\bar{\Delta}_2) = \alpha_1(\bar{\Delta}_2'), \tag{7.15}$$

$$\alpha_2(\bar{\Delta}_2) = \alpha_2(\bar{\Delta}_2'). \tag{7.14}$$

The relations (7.15), (7.16), (3.17), and (3.18) imply

$$\int_{\mathbf{P}^m(\mathbf{C})} \tau dM = \int_{\mathbf{P}^m(\mathbf{C})} \tau' dM', \tag{7.17}$$

$$\begin{aligned}&\int_{\mathbf{P}^m(\mathbf{C})} (A_1\tau^2 - A_2|p|^2 + A_3|R|^2)dM \\ &= \int_{\mathbf{P}^m(\mathbf{C})} (A_1\tau'^2 - A_2|p'|^2 + A_3|R'|^2)\, dM',\end{aligned} \tag{7.18}$$

where dM and dM' are the volume elements on $\mathbf{P}^m$ $(\mathbf{C})$ with respect to the metrics g_0 and g, respectively. In the formulas (3.22), $n = 2m$.

It is known that, for these Kahler manifolds, we have the condition [**1**, p. 149]

$$(7.19)\qquad \int_{\mathbf{P}^m(\mathbf{C})} (\tau^2 - 4\,|p^2| + |R|^2)\, dM = \int_{\mathbf{P}^m(\mathbf{C})} (\tau'^2 - 4|p'|^2 + |R'|^2) dM'.$$

From (7.18) and (7.19) we conclude

$$(7.20)\qquad \int_{\mathbf{P}^m(\mathbf{C})} (\Gamma_1\tau^2 + \Gamma_2|p|^2) dM = \int_{\mathbf{P}^m(\mathbf{C})} (\Gamma_1\tau'^2 + \Gamma_2|p'|^2)\, dM'$$

where

$$(7.21)\qquad \Gamma_1 = A_1 - A_3,$$

$$(7.22)\qquad \Gamma_2 = 4A_3 - A_2.$$

Using (3.22), (7.21) and (7.22) take the form

$$(7.23)\qquad \Gamma_1 = 3n^2 + 20n + 48,$$

$$(7.24)\qquad \Gamma_2 = 6n^2 - 40n - 64,$$

where $n = 2m$.

By (7.20) and (5.21),

$$(7.25)\qquad \int_{\mathbf{P}^m(\mathbf{C})} (\Lambda_1\, \tau^2 + \Lambda_2\, |G|^2)\, dM = \int_{\mathbf{P}^m(\mathbf{C})} (\Lambda_1\, \tau'^2 + \Lambda_2\, |G'|^2)\, dM',$$

where

$$(7.26)\qquad \Lambda_1 = \Gamma_1 + \frac{\Gamma_2}{n},$$

$$(7.27)\qquad \Gamma_2 = \Lambda_2,$$

which, by virtue of (7.23) and (7.24), become

$$(7.28)\qquad \Lambda_1 = \frac{3n^3 + 26n^2 + 8n - 64}{n},$$

$$(7.29)\qquad \Lambda_2 = 60n^2 - 40n - 64.$$

It can be proved earily that if

$$(7.30)\qquad n = 2m \geqq 8,$$

then

$$(7.31)\qquad \Lambda_1 > 0, \qquad \Lambda_2 > 0.$$

Since the Kahler manifold ($\mathbf{P}^m(\mathbf{C})$, J_0, g_0) has constant holomorphic sectional curvature h_0,

$$(7.32)\qquad |G|^2 = 0 \text{ and } G = 0.$$

Therefore, (7.25) and (7.32) give

$$\int_{P^m(C)} \Lambda_1 \tau^2 \, dM = \int_{P^m(C)} [\Lambda_1 \tau'^2 + \Lambda_2 |G'|^2] \, dM'. \tag{7.33}$$

From (7.15) and the fact that τ = constant, we obtain

$$\int_{P^m(C)} \tau^2 \, dM \leqq \int_{P^m(C)} \tau'^2 \, dM'. \tag{7.34}$$

The relations (7.31), (7.33), and (7.34) imply

$$|G'|^2 = 0 \text{ and } G' = 0.$$

This completes the proof of the theorem.

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