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α-SEPARATION AXIOMS AND α-COMPACTNESS IN FUZZY TOPOLOGICAL SPACES

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ABSTRACT. In **[11]** Rodabaugh introduced the concept of α-Hausdorff fuzzy topological spaces which is compatible with α-compactness **[4]** and fuzzy continuity. It is the purpose of this paper to extend these concepts. We define and study $\alpha - T_i$ $(i = 0, 3, 4)$, $\alpha - T_i'$ $(i = 0, 1, 2, 3, 4)$, α-almost compact and α-nearly compact fuzzy topological spaces. Also, we define α-continuous mappings as a generalization of F-continuous mappings. Finally, we define αS-closed fuzzy spaces and study some of their properties.

1. Preliminaries. Let X be a set. If $A \subset X$, $\mu(A)$ will denote the characteristic function for A defined on X into the unit interval $I = [0, 1]$. A fuzzy topology τ on X is a family of fuzzy sets (functions from X into I) which is closed under arbitrary suprema and finite infima and which contains $0 = \mu(\phi)$ and $1 = \mu(X)$. A pair (X, τ), where τ is a fuzzy topology on X, is called a fuzzy topological space (abbreviated as fts). A fuzzy set u of an fts (X, τ) is regular open (resp. regular closed) if $u = \bar{u}^0$ (resp. $u = \bar{u}^0$), it is fuzzy semiopen if $u \leqq \bar{u}^0$. For notion and results used but not defined or shown we refer to **[3, 5, 13, 16, 18]**.

DEFINITION 1.1 **[11]**. Let (X, τ) be an fts and $A \subset X$. A point $x \in X$ is an α (resp. α^*)-cluster point of A if for each $u \in \tau$ with $u(x) > \alpha$ (resp. $u(x) \geqq \alpha$), $u \wedge \mu(X/A) \neq 0$, where $\alpha < 1$ (resp. $\alpha < 0$). The family of all α (resp. α^*)-cluster points of A will be denoted by A^α(resp. A^{α^*}). The α (resp. α^*) closure of A is the union of A and its α (resp. α^*) cluster points and will be denoted by $\mathrm{Cl}_\alpha(A)$ (resp. $\mathrm{CL}_{\alpha^*}(A)$). The subset A of X is α (resp. α^*)-closed if $\mathrm{Cl}_\alpha(A) \subset A$ (resp. $\mathrm{Cl}_{\alpha^*}(A) \subset A$).

PROPOSITION 1.2 [11]. *Let (X, τ) be an fts. Then*
(i) *a subset A of X is α (resp. α^*)-closed if and only if for each point $x \in X\backslash A$ there is $u \in T$ such that $u(x) > \alpha$ (resp. $u(x) \geqq \alpha$) and $u \wedge \mu(A) = 0$.*
(ii) *arbitrary intersection of α (resp. α^*)-closed sets is α (resp. α^*)-closed,*
(iii) *a finite union of α (resp. α^*)-closed sets is α (resp. α^*)-closed, and,*
(iv) *the inverse image of each α (resp. α^*)-closed set under an F-continuous mapping is α (resp. α^*)-closed.*

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DEFINITION 1.3. Let (X, τ) be an fts and let $A \subset X$. A is an α (resp. α^*)-open set if X/A is α (resp. α^*)-closed. Equivalently, A is α (resp. α^*)-open if, for each point $x \in A$ there is $u \in \tau$ with $u(x) > \alpha$ (resp. $u(x) \geqq \alpha$) and $u \wedge \mu(X/A) = 0$.

REMARK 1.4. The following notions are found in [7, **8**, **11**, **13**]. Let $\alpha \in I$ and (X, τ) be a given fts. Put $u_\alpha = \{x \in X: u(x) > \alpha\}$ and $\tau_\alpha = \{u_\alpha: u \in \tau\}$. Clearly, τ_α is a family of α-open sets in (X, τ). One may easily verify that τ_α is a topology on X.

PROPOSITION 1.5. *Let (X, τ) be an fts. The family of all α-open sets in X is a topology on X coarser than τ_α.*

PROOF. $\{\alpha\text{-open sets}\} = W_\alpha$ [7].

EXAMPLE 1.6. Let $X = \{a, b, c\}$ and let u, v be fuzzy sets in X defined by

$$u(a) = 0.3, \qquad u(b) = 0.5, \qquad u(c) = 0.7$$
$$v(a) = 0.7, \qquad v(b) = 0.6, \qquad v(c) = 0.9.$$

Define the fuzzy topology $\tau = \{1, 0, u, v\}$ on X. For $\alpha = 0.4$, $\{b, c\}$ is a τ_α-open set which is not α-open.

DEFINITION 1.7. Let (X, τ) be an fts and let $A \subset X$. A point $x \in X$ is an α (resp. α^*)-weak cluster point (αw (resp. $\alpha^* w$)-cluster point, for short) of A if for every $u \in \tau$ with $u(x) > \alpha$ (resp. $u(x) \geqq \alpha$), $\bar{u} \wedge \mu(A \backslash \{x\}) \neq 0$. The set of all αw(resp. $\alpha^* w$)-cluster points of A is denoted by $A^{\alpha w}$(resp. $A^{\alpha^* w}$). The αw(resp. $\alpha^* w$)-closure of A is the union of A and its αw(resp. $\alpha^* W$)-cluster points. A is $\alpha w(\alpha^* w)$-closed if $A^{\alpha w} \subset A$ (resp. $A^{\alpha^* w} \subset A$).

REMARK 1.8. If a point $x \in X$ is an α (resp. α^*)-cluster point of a subset A of an fts X, then it is an αw(resp. $\alpha^* w$)-cluster point of A. Consequently, if A is αw(resp. $\alpha^* w$)-closed, then A is α (resp. α^*)-closed. The following example indicates that the converse is not true.

EXAMPLE 1.9. Let $X = \{a, b, c\}$ and let u, v be fuzzy sets in X defined by

$$u(a) = 0.5, \qquad u(b) = 0.6, \qquad u(c) = 0$$
$$v(a) = 0.4, \qquad v(b) = 0, \qquad v(c) = 0.5.$$

Define the fuzzy topology $\tau = \{1, 0, u, v, u \vee v, u \wedge v\}$ on X. For $\alpha < 0.5$, the point a is an αw-cluster point of the set $\{a, c\}$, but it is not an α-cluster point. The set $\{c\}$ is an α-closed set which is not αw-closed.

DEFINITION 1.10. Let (X, τ) be an fts and let $A \subset X$. A is an α (resp. α^*)-strongly open (αs (resp. $\alpha^* s$)-open, for short) if $X \backslash A$ is αw(resp. $\alpha^* w$)-closed. Equivalently, A is αs (resp. $\alpha^* s$)-open if, for every point $x \in A$, there is $u \in \tau$ such that $u(x) > \alpha$ (resp. $u(x) \geqq \alpha$) and $\bar{u} \wedge \mu(X \backslash A) = 0$.

REMARK 1.11. Let (X, τ) be an fts and let $A \subset X$. If A is an αs (resp. $\alpha^* s$)-open set, then it is α (resp. α^*)-open. In Example 1.10, $\{a, b\}$ is an α-open set which is not αs-open.

2. α-Separation axioms. We use the concepts of α-closed sets αw-closed sets to define the following separation axioms.

DEFINITION 2.1. An fts (X, τ) is called:

1. An $\alpha - T_0$ (resp. $\alpha - T_0'$) space if for each two distinct points $x, y \in X$ there exists $u \in \tau$ such that $u(x) > \alpha$ and $u(y) = 0$ (resp. $\bar{u}(y) = 0$), or $u(y) > \alpha$ and $u(x) = 0$ (resp. $\bar{u}(x) = 0$).
2. An $\alpha - T_1$(resp. $\alpha - T_1'$) space if for each two distinct points $x, y \in X$ there exist $u, v \in \tau$ such that $u(x) > \alpha$, $v(y) > \alpha$ and $u(y) = v(x) = 0$ (resp. $\bar{u}(y) = \bar{v}(x) = 0$).
3. An $\alpha - T_2$(resp. $\alpha - T_2'$) space if for each two distinct point $x, y \in X$ there exist $u, v \in \tau$ such that $u(x) > \alpha$, $v(y) > \alpha$ and $u \wedge v = 0$ (resp. $\bar{u} \wedge \bar{v} = 0$).
4. An α-regular (resp. α-weakly regular) space if for each α-closed (resp. αw-closed) subset A of X and each point $x \in X \backslash A$ there exist $u, v \in \tau$ such that $u(x) > \alpha$, $v(y) > \alpha$ on A and $u \wedge v = 0$.

An α-regular (resp. α-weakly regular) space which is also $\alpha - T_1$ (resp. $\alpha - T_1'$) is called an $\alpha - T_3$ (resp. $\alpha - T_3'$) space.

5. An α-normal (resp. α-weakly normal) space if for each two disjoint α-closed (resp. αw-closed) subsets A, B of X there exist $u, v \in \tau$ such that $u(x) > \alpha$ on A, $v(y) > \alpha$ on B and $u \wedge v = 0$.

An α-normal (resp. α-weakly normal) space which is also $\alpha - T_1$ (resp. $\alpha - T_1'$) is called an $\alpha - T_4$ (resp. $\alpha - T_4'$) space.

The separation axioms $\alpha - T_1$ and $\alpha - T_2$ have been defined in **[11, 12]** and greatly generalized in **[14]**. The proof of the following theorem is routine and it is omitted.

THEOREM 2.2. *An fts (X, T) is $\alpha - T_1$ (resp. $\alpha - T_1'$) if and only if every one point subset $\{x\}$ of X is α-closed (resp. αw-closed).*

Definition 2.1 and Theorem 2.2 yield the following diagram

$$\begin{array}{ccccccccc}
\alpha - T_4 & \Longrightarrow & \alpha - T_3 & \Longrightarrow & \alpha - T_2 & \Longrightarrow & \alpha - T_1 & \Longrightarrow & \alpha - T_0 \\
\Uparrow & & \Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\
\alpha - T_4' & \Longrightarrow & \alpha - T_3' & \Longrightarrow & \alpha - T_2' & \Longrightarrow & \alpha - T_1' & \Longrightarrow & \alpha - T_0'
\end{array}$$

EXAMPLE 2.3. Let $X = \{a, b, c\}$ and let u, v be fuzzy sets in X defined by

$$u(a) = 0.5, \qquad u(b) = 0, \qquad u(c) = 0.6$$
$$v(a) = 0, \qquad v(b) = 0.4, \qquad v(c) = 0.5.$$

Define the fuzzy topology $\tau = \{1, 0, u, v, u \vee v, u \wedge v\}$ on X. The fts (X, τ) is $\alpha - T_0$ but it is not $\alpha - T_1$. Also (X, τ) is α-normal but it is not α-regular.

REMARK 2.4. The above diagram and Corollary 7.2 of **[11]** imply that the I-fuzzy unit interval I(I) [5] and the I-fuzzy real line **R**(I) **[4]** are not $\alpha - T_i$ nor $\alpha - T_i'$ for i = 1, 2, 3, 4.

3. α -Compactness.

DEFINITION 3.1 **[4]**. Let X be a nonempty set and $\alpha \in I$. A family $\mathcal{U} \subset I^X$ is an α-shading of X if, for each point $x \in X$, there is $u \in \mathcal{U}$ such that $u(x) > \alpha$. A subfamily $\mathcal{V}$ of an α-shading $\mathcal{U}$ of X, that is also an α-shading of X, is called an α-subshading of $\mathcal{U}$. An α-shading $\mathcal{U}$ of an fts X is an open (resp. closed, . . .) α-shading if each member of $\mathcal{U}$ is a fuzzy open (resp. closed, . . .) set. An fts X is said to be α-compact if every open α-shading of X has a finite α-subshading.

DEFINITION 3.2 **[1]**. An fts (X, τ) is α-almost (resp. α-nearly) compact if for each open α-shading $\mathcal{U}$ of X there is a finite subfamily $\mathcal{V}$ of $\mathcal{U}$, the fuzzy closures (resp. fuzzy interiors of the fuzzy closures) of whose members are α-shading of X.

One may notice that α-compactness $\Rightarrow$ α-nearly compactness $\Rightarrow$ α-almost compactness. These implications do not reverse **[1]**.

THEOREM 3.3. *An fts (X, τ) is α-nearly compact if and only if each regular open α-shading of X has a finite α-subshading.*

PROOF. A simple combination of Definition 3.2 and the definition of a regular open α-shading yields the result.

THEOREM 3.4. *An αw-closed subset of an α-almost (resp. α-nearly) compact fts is α-almost (resp. α-nearly) compact.*

PROOF. Let A be an αw-closed subset of an τ-almost compact fts (X, τ). Let $\mathcal{U}$ be an open α-shading of A. Since A is αw-closed, then for each point $x \in X \backslash A$ there exists $v_x \in \tau$ such that $v_x(x) > \alpha$ and $\bar{v}_x \wedge \mu(A) = 0$, i.e., $\bar{v}_x(y) = 0$ on A. Then $\mathcal{V} = \{v_x \colon x \in X \backslash A\} \cup \mathcal{U}$ is an open α-shading of X. Consequently there exists a finite subfamily $\{V_{x_1}, \ldots, V_{x_n}\} \cup \{u_1, \ldots, u_m\}$ of $\mathcal{V}$ such that $\{V_{x_1}, \ldots, V_{x_n}, \bar{u}_1, \ldots, \bar{u}_m\}$ is an α-shading of X. Consequently $\{\bar{u}_1, \ldots, \bar{u}_m\}$ is an α-shading of A and A is α-almost compact. The α-nearly case has a similar proof.

A partial converse of Theorem 3.4 is the following theorem which has a routine proof.

THEOREM 3.5. *Any α-almost compact (crisp) subset of an $\alpha - T_2'$ space is αw-closed.*

COROLLARY 3.6.

(i) *Any α-nearly compact (crisp) subset of an $\alpha - T_2'$ space is αw-closed.*

(ii) *A (crisp) subset of an α-almost (resp. α-nearly) compact $\alpha - T_2'$ space is α-almost (resp. α-nearly) compact if and only if it is αw-closed.*

(iii) *The intersection of any family of α-almost (resp. α-nearly) compact crisp subsets of an α-almost (resp. α-nearly) compact $\alpha - T_2'$ space is α-almost (resp. α-nearly) compact.*

THEOREM 3.7. *Let (X, τ) be an $\alpha - T_2'$ space and let A be an α-almost (resp. α-nearly) compact crisp subset of X. For each point $x \in X\backslash A$ there exist $u, v \in \tau$ such that $u(x) > \alpha$, $\bar{v}(y) > \alpha$ (resp. $v(y) > \alpha$) on A and $\bar{u} \wedge \bar{v} = 0$.*

PROOF. We give a proof considering A as an α-almost compact crisp subset of X; the other part has a similar proof. For each $y \in A$ there exist $u_y, v_y \in \tau$ such that $u_y(x) > \alpha$, $v_y(y) > \alpha$ and $\bar{u}_y \wedge \bar{v}_y = 0$. Then $\mathscr{U} = \{v_y: y \in A\}$ is an open α-shading of A. Consequently, there is a finite subfamily $\{v_{y_1}, \ldots, v_{y_n}\}$ of $\mathscr{U}$ such that $\{\bar{v}_{y_1}, \ldots, \bar{v}_{y_n}\}$ is an α-shading of A. Set $u = \wedge_{i=1}^{n} u_{y_i}$ and $v = \vee_{i=1}^{n} v_{y_i}$. Then $u, v \in \tau$, $u(x) > \alpha$, $\bar{v}(y) > \alpha$ on A and $\bar{u} \wedge \bar{v} = 0$.

Using the same arguments one may prove the following result.

THEOREM 3.8. *An α-nearly compact $\alpha - T_2'$ space is $\alpha - T_4'$.*

4. α-Continuous mappings.

DEFINITION 4.1. The following is found in [7, **8**, **11**, **13**] etc. Let $\alpha \in I$ and (X, τ) be a given fts. Put $u_\alpha = \{x: u(x) > \alpha\}$, $\tau_\alpha = \{u_\alpha: u \in \tau\}$, $J_\alpha(X, \tau) = (X, \tau_\alpha)$ and $J_\alpha(f) = f$, where $f: X \to Y$.

PROPOSITION 4.2 [7, **8**, **11**]. *J_α is a functor from the category of I-fts to TOP; hence f is* F*-continuous implies $J_\alpha(f)$ is continuous.*

REMARK 4.3. J_α is the α-level functor.

REMARK 4.4. The implication of Proposition 4.2 does not reverse (see Theorem 4.8 (2, 3) below). Hence Proposition 4.2 yields a generalization of Proposition 1.2 (iv) as follows: if $J_\alpha(f)$ is continuous, then f^{-1}(α-open) is α-open and so f^{-1}(α-closed) is α-closed.

THEOREM 4.5. *Let (X, τ), (Y, σ) be fts and $f: X \to Y$. If $J_\alpha(f): (X, \tau_\alpha) \to (Y, \sigma_\alpha)$ is continuous, then X is α-compact implies $f(X)$ is α-compact.*

PROOF. This is a corollary of two results of [**11**], namely Theorem 3.1 (2) and Proposition 3.1 (1).

REMARK 4.6. Theorem 4.5 generalizes Theorem 2.9 of [4] because of Proposition 4.2 and Theorem 4.8 (2, 3) below.

DEFINITION 4.7. A mapping f: $(X, \tau) \to (Y, \sigma)$ is α-continuous if, for every point $x \in X$ and for every $v \in \sigma$ with $f(x) \in v_\alpha$, there exists $u \in \tau$ such that $x \in u_\alpha$ and $f(u) \leqq v$.

THEOREM 4.8. *Let (X, τ), (Y, σ) be fts and f: $X \to Y$. The following hold:*
(i) *f is F-continuous $\Rightarrow$ f is α-continuous.*
(ii) *The implication of* (1) *does not reverse.*
(iii) *f is α-continuous $\Rightarrow$ $J_\alpha(f)$ is continuous.*
(iv) *The implication of* (3) *does not reverse.*

PROOF.
(i). Straight-forward.
(ii). Let $X = \{a, b\}$ and let u, v be fuzzy sets in X defined by

$$u(a) = 0.5, \qquad u(b) = 0.6,$$
$$v(a) = 0.7, \qquad v(b) = 0.8.$$

Define the fuzzy topologies $\tau = \{1, 0, u\}$ and $\sigma = \{1, 0, v\}$ on X. The identity map $i_X: (X, \tau) \to (Y, \sigma)$ is not F-continuous, yet is α-continuous for $\alpha < 0.5$.
(iii). Let $v \in \sigma$ and $f(x) \in V_\alpha$. From α-continuity there is $u \in \tau$ such that $x \in u_\alpha$ and $f(u) \leqq v$. We show that $f(u_\alpha) \subset v_\alpha$. Let $z \in u_\alpha$ and observe

$$v(f(z)) \geqq f(u)\ (f(z)) = U\{u(w):\ f(w) = f(z)\} \geqq u(z) > \alpha.$$

So $f(z) \in v_\alpha$.
(iv). Let $X = \{a, b\}$ and let u, v be fuzzy sets in X defined by

$$u(a) = 0.3, \qquad u(b) = 0.5$$
$$v(a) = 0.6 \qquad v(b) = 0.7$$

Define the fuzzy topologies $\tau = \{1, 0, u\}$ and $\sigma = \{1, 0, v\}$ on X. The identity map $i_X: (X, \tau) \to (X, \sigma)$ is not α-continuous but $J(f): (X, \tau_\alpha) \to (Y, \sigma_\alpha)$ is continuous for $0.5 \leqq \alpha < 0.6$.

COROLLARY 4.9. *Each α-continuous map preserves α-compactness.*

PROOF. Theorems 4.5 and 4.8 (iii).

5. αs-closed spaces. In 1976, T. Thompson [16] induced the concept of S-closed spaces. The literature includes [2, 9, 10]. We now use the concept of an α-shading to define αS-closed spaces in fuzzy topology.

DEFINITION 5.1. An fts (X, τ) is called αS-closed if for each semiopen α-shading $\mathcal{U}$ of X there is a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ such that the fuzzy closures of its members are an α-shading of X.

REMARK 5.2. It is clear that if (X, τ) is an αS-closed fts it is also α-almost compact.

THEOREM 5.3. *An fts X is αS-closed if and only if every regular closed α-shading has a finite α-subshading.*

PROOF. Necessity follows from the fact that a fuzzy regular closed set is semiopen and closed.

Sufficiency follows from the fact that if u is a fuzzy semiopen set, then $\bar{u}$ is regular closed.

DEFINITION 5.4. A fuzzy set u of an fts (X, τ) is called a fuzzy regular semiopen set if there exists a fuzzy regular open set v of x such that $v \leqq u \leqq \bar{v}$.

REMARK 5.5. If u is a fuzzy regular semiopen set, it is also a fuzzy semiopen set but the converse is not true in general, as we can show from Example 5.6, below. This example indicates also that a fuzzy open set need not be a fuzzy regular semiopen set.

EXAMPLE 5.6. Let $X = \{a, b\}$ and let u_1, u_2, u_3 be fuzzy sets in X defined by

$$u_1(a) = 0.4, \qquad u_1(b) = 0.5, \qquad u_2(a) = 0.5$$

$$u_2(b) = 0.6, \qquad u_3(a) = 0.5, \qquad u_3(b) = 0.5$$

Define a fuzzy topology $\tau = \{1, 0, u_1, u_2\}$ on X. The fuzzy set u_3 is a fuzzy regular semiopen but it is not fuzzy open. The fuzzy set u_2 is a fuzzy open set but it is not fuzzy regular semiopen.

THEOREM 5.7. *An fts X is αS-closed if and only if for every regular semiopen α-shading $\mathcal{U}$ of X, there is a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ such that the fuzzy closures of its members are an α-shading of X.*

PROOF. If X is αS-closed, then the result follows directly from the above definition. If X is not an αS-closed fts, then there exists a fuzzy semiopen α-shading $\{u_j\colon j \in J\}$ which has no finite subfamily such that the fuzzy closures of its members are an α-shading. Then the family $\{\bar{u}_j^0 \vee u_j\colon j \in J\}$ is a fuzzy regular semiopen α-shading of X which has no finite subfamily such that the fuzzy closures of its members are an α-shading, since $u_j \leqq \bar{u}_j^0 \vee u_j \leqq \bar{u}_j$. This completes the proof.

Now we extend the concept of an extremely disconnected space **[10]** to fuzzy topology.

DEFINITION 5.8. An fts (X, τ) is called a fuzzy extremely disconnected space (abbreviated as FED-space) if the fuzzy closure of every fuzzy open set in X is fuzzy open.

THEOREM 5.9. *An FED-space X is αS-closed if and only if it is α-almost compact.*

PROOF. It follows from Theorem 5.7 and the fact that, in FED-space, a fuzzy regular open set is fuzzy open as well as fuzzy closed.

DEFINITION 5.10 [6]. An fts (X, τ) is called a regular fuzzy space if every fuzzy open set u of X can be written as the supremum of fuzzy open sets u_j's of X such that $\bar{u}_j \leqq u$ for each j.

THEOREM 5.11. *Let (X, τ) be a regular fuzzy space. If X is αS-closed, then it is α-compact.*

PROOF. Let $\mathscr{U}$ be an open α-shading of X. For each point $x \in X$ there is a fuzzy open set $u_x \in \mathscr{U}$ with $u_x(x) > \alpha$. Thus the family $\{u_x: x \in X\}$ is an open α-shading of X. Since X is a regular fuzzy space, $u_x = \vee_{j_x} u_{j_x}$ with $u_{j_x} \in \tau$ and $\bar{u}_{j_x} \leqq u_x$ for each j. Since $u_x(x) > \alpha$, there is u_{j_x} such that $u_{j_x}(x) > \alpha$. Thus $\{u_{j_x}: x \in X\}$ is an open α-shading of X. Hence there exists a finite subfamily $\{u_{j_{x1}}, \ldots, u_{j_{xn}}\}$ of $\{u_{j_x}: x \in X\}$ such that $\{\bar{u}_{j_{x1}}, \ldots, \bar{u}_{j_{xn}}\}$ is an α-shading of X. Therefore $\{u_{x_1}, \ldots, u_{x_n}\}$ is a finite α-subshading of $\mathscr{U}$ and X is α-compact.

THEOREM 5.12. *In a regular FED-space, the following are equivalent.*
(i) *X is α-compact.*
(ii) *X is α-nearly compact.*
(iii) *X is α-almost compact.*
(iv) *X is αS-closed.*

PROOF. It is a direct consequence of Theorems 5.9 and 5.11.

THEOREM 5.13. *Let an fts X have the property that for each α-shading $\mathscr{U}$ of X, $\mathscr{U}^0 = \{u^0: u \in \mathscr{U}\}$ is an α-shading of X. Then X is α-almost compact if and only if X is αS-closed.*

PROOF. For sufficiency, see Remark 5.2. For necessity, let $\mathscr{U}$ be a semi-open α-shading of X. Then $\mathscr{U}^0 = \{u^0: u \in \mathscr{U}\}$ is an open α-shading of X and there exists a finite subfamily $\{u_1^0, \ldots, u_n^0\}$ of $\mathscr{U}^0$ such that $\{u_1^{0-}, \ldots, u_n^{0-}\}$ is an α-shading of X. This means that $\{u_1, \ldots, u_n\}$ is a finite subfamily of $\mathscr{U}$ such that $\{\bar{u}_1, \ldots, \bar{u}_n\}$ is an α-shading of X. Consequently X is αS-closed.

THEOREM 5.14. *An αw-closed subset of an αS-closed fts is αS-closed.*

PROOF. It is obvious.

DEFINITION 5.15. A function $f: X \to Y$ is said to be F-almost open if $f^{-1}(\bar{u}) \leqq \overline{f^{-1}(u)}$ for every fuzzy open set u of Y.

REMARK 5.16. If f: $X \to Y$ is an F-open mapping, in the sense of Wong [**18**], then it is also F-almost open. But the converse is not true as one may notice from the following example.

EXAMPLE 5.17. Let $X = \{a, b\}$ and $Y = \{c, d\}$. Define $u \in I^x$ and $v \in I^y$ as follows

$$u(a) = 0.5, \qquad u(b) = 0.3, \qquad v(c) = 0.5, \qquad v(d) = 0.4.$$

Let $\tau = \{1, 0, u\}$, $\sigma = \{1, 0, v\}$ and f: $(X, \tau) \to (Y, \sigma)$, where $f(a) = c$, $f(b) = d$. Then f is F-almost open but it is not F-open.

LEMMA 5.18. *If f: $X \to Y$ is F-continuous and F-almost open, then f^{-1} (semiopen) is semiopen.*

PROOF. If u is a fuzzy semiopen set of Y, then there is v fuzzy open such that $v \leqq u \leqq \bar{v}$. Hence $f^{-1}(v) \leqq f^{-1}(u) \leqq \overline{f^{-1}(v)}$, i.e., $f^{-1}(u)$ is semiopen.

THEOREM 5.19. *Let f: $X \to Y$ be F-continuous and F-almost open. If X is S-closed, so is $f(X)$.*

PROOF. A combination of Definition 5.1 and Lemma 5.18 yields the proof.

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