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ON NEIGHBOURHOODS OF UNIVALENT CONVEX FUNCTIONS

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Introduction. Let A denote the class of analytic functions f in the unit disk $E = \{z \mid |z| < 1\}$ with $f(0) = f'(0) - 1 = 0$. For $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ in A and $\delta \geqq 0$ Ruscheweyh has defined the δ-neighbourhood $N_\delta(f)$ as follows:

$$N_\delta(f) = \{g \in A \mid g(z) = z + \sum_{k=2}^{\infty} b_k z^k \text{ and } \sum_{k=2}^{\infty} k|a_k - b_k| \leqq \delta\}.$$

He has shown in [3], among other results, that if $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \in C$, then

$$N_{d_n}(f) \subset S^* \text{ if } d_n = 2^{-2/n} \tag{1}$$

where $C(S^*)$ denotes the class of normalized convex (starlike) univalent functions in A. Ruscheweyh also asked in [3] if results analogous to (1) would hold if the class C were replaced by some of its subclasses.

Let $t > 1/2$. We consider the following subclasses of A:

$$(S^*)_t = \{f \in A \mid \left|\frac{zf'(z)}{f(z)} - t\right| < t, z \in E\}$$

and

$$(C)_t = \{f \in A \mid \left|\frac{zf''(z)}{f'(z)} + 1 - t\right| < t, z \in E\}.$$

It is clear that $(S^*)_t \subset S^*$ and $(C)_t \subset C$. The classes $(S^*)_t$ and $(C)_t$ have been studied by several authors (see for example [4], [5], [6]). We prove

THEOREM 1. *Let $t \geqq 1$ and $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \in (C)_t$. Then $N_{\delta_n}(f) \subset (S^*)_t$ if $\delta_n = (2 - 1/t)^{-(1/n)(2-1/t)/(1-1/t)}$. The value given to δ_n is the best possible.*

THEOREM 2. *Let $1/2 < t \leqq 2$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in (C)_t$. Then $N_\delta(f) \subset (S^*)_t$ if $\delta = \inf_{z \in E} |t(f(z)/z)| - |f'(z) - t(f(z)/z)|$.*

THEOREM 3. *Let $1/2 < t \leqq 1$ and $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \in (C)_t$. Then*

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$N_{\delta_n}(f) \subset (S^*)_1$ *if* $\delta_n = (2 - 1/t)^{-(1/n)(2-1/t)/(1-1/t)}$. *The value given to* δ_n *is the best possible.*

A special case of Theorem 1 has already been published in [2]. It is not clear that the value given to δ in Theorem 2 is best possible for each function $f \in (C)_t$ when $1/2 < t \leqq 2$. However, we are going to verify that

$$\inf_{z \in E,\, f \in (C)_t} \left| t\frac{f(z)}{z} \right| - \left| f'(z) - t\frac{f(z)}{z} \right| = \delta_1 \text{ when } 1 \leqq t.$$

It follows from (1) that $N_{1/4}(C) \subset S^*$, and it follows from Theorem 1 that $N_\delta((C)_t) \subset (S^*)_t$ if $\delta = (2 - 1/t)^{-(2-1/t)/(1-1/t)}$. Ruschewehy asked [3] for a geometric characterization of $N_{1/4}(C)$. We are unable to answer this question, but we can show

THEOREM 4. *Let* $t \geqq 1$, $w_t = (1/t) - 1$ *and* $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \in (C)_t$. *Let also* $\delta_n = (2 - 1/t)^{-(1/n)(2-1/t)(1-1/t)}$ *and* $g \in N_{\delta_n}(f)$. *Then* $(1/x)\, g(xz) \in (C)_t$ *where* x *is the unique root in the interval* $(0, 1)$ *of the equation*

$$(1 - x^n)(1 - w_t x^n)^{-1+\frac{1}{n}\frac{1-w_t}{w_t}} - \sup_{k \geqq 2} (kx^{k-1})\delta_n = 0. \tag{2}$$

THEOREM 5. *Let* $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \in C$. *also let* $d_n = 2^{-2/n}$ *and* $g \in N_{d_n}(f)$. *Then* $(1/x)\, g(xz) \in C$ *where* x *is the unique root in the interval* $(0, 1)$ *of the equation*

$$\frac{1 - x^n}{(1 + x^n)^{1+2/n}} - \sup_{k \geqq 2} (kx^{k-1}) d_n = 0. \tag{3}$$

It is not hard to see that the root in the interval $(0, 1)$ of the equation (2) in the case where $n = 1$ is, in fact, equal to the radius of convexity of the class $N_\delta((C)_t)$ when $\delta = (2 - 1/t)^{-(2-1/t)/(1-1/t)}$. It is also not hard to check that the equation (3) when $n = 1$ is equivalent to

$$\frac{1 - x}{(1 + x)^3} - \frac{x}{2} = 0.$$

This implies easily that the radius of convexity of the class $N_{1/4}(C)$ is equal to $\sqrt{2} - 1$. We would also like to indicate that the case $n = \infty$ of both Theorems 4 and 5 is just the following well-known result (see [1; p. 74, problem 24]). Let $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in A$ with $\sum_{k=2}^{\infty} k|b_k| \leqq 1$. Then $2g(2/2) \in C$.

Finally we point out that in establishing most of the above mentioned theorems our main tool is the Hadamard product (or convolution) of analytic functions. If the two functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ belong to A their Hadamard product is the function $f * g$ in A defined as

$$f*g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

It is not difficult to verify that many classes mentioned above can be defined in terms of convolution. For example

$$f \in S^* \Leftrightarrow \forall\, T \in \mathbf{R}\ \forall\, z \in E,\ \frac{f * h_T(z)}{z} \neq 0 \tag{4}$$

where

$$h_T(z) = \frac{z/(1-z)^2 + iT\, z/(1-z)}{1+iT},$$

and

$$f \in (S^*)_t \Leftrightarrow \forall\, \theta \in [0, 2\pi]\ \forall\, z \in E,\ \frac{f * h_\theta(z)}{z} \neq 0 \tag{5}$$

where

$$h_\theta(z) = \frac{z/(1-z)^2 - t(1+e^{i\theta})z/(1-z)}{1 - t(1+e^{i\theta})}.$$

Proof of Theorem 1. We first remark that in order to prove Theorem 1 it is enough to show that

$$\left|\frac{f * h_\theta(z)}{z}\right| > \delta_n,\ \theta \in [0, 2\pi],\ z \in E. \tag{6}$$

As a matter of fact if $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in N_{\delta_n}(f)$ we obtain

$$\left|\frac{g*h_\theta(z)}{z}\right| \geqq \left|\frac{f*h_\theta(z)}{z}\right| - \left|\frac{(g-f)*h_\theta(z)}{z}\right| > \delta_n - \left|\frac{(g-f)*h_\theta(z)}{z}\right| > 0 \tag{7}$$

because

$$\left|\frac{(g-f)*h_\theta(z)}{z}\right| = \left|\sum_{k=2}^{\infty} \frac{k - t(1+e^{i\theta})}{1 - t(1+e^{i\theta})}(b_k - a_k)z^{k-1}\right|$$

$$\leqq \sum_{k=2}^{\infty} \left|\frac{k - t(1+e^{i\theta})}{1 - t(1+e^{i\theta})}\right| |b_k - a_k| \tag{8}$$

$$\leqq \sum_{k=2}^{\infty} k|b_k - a_k| \tag{9}$$

$$\leqq \delta_n. \tag{10}$$

The passage from (8) to (9) is justified by the fact that

$$\forall\, \theta, \left|\frac{k - t(1+e^{i\theta})}{1 - t(1+e^{i\theta})}\right| \leqq k \text{ if } t \geqq 1.$$

The passage from (9) to (10) is justified by the fact that $g \in N_{\delta_n}(f)$. According to (5) the condition (7) means that $g \in N_{\delta_n}(f)$.

In order to prove (6) we need two lemmas (stated here without proof) about bounded analytic functions in the disk.

LEMMA 1.1. *Let the function $w(z)$ be analytic in the unit disk E and let $|w(z)| < 1$ if $z \in E$. Then if $w(z) = w(0) + \sum_{k=1}^{\infty} c_k z^k$,*

$$\forall\, z \in E\ \mathrm{Re}(w(z) - w(0)) \geqq -(1-|w(0)|^2)\,|z|^n \frac{1+|z|^n \mathrm{Re}(w(0))}{1-|z|^{2n}|w(0)|^2}.$$

LEMMA 1.2. *Under the hypothesis of Lemma 1.1 we have*

$$\forall\, z \in E\ \forall\, \theta \in [0, 2\pi], \left|\frac{w(z) - e^{i\theta}}{w(0) - e^{i\theta}}\right| \geqq \frac{1-|z|^n}{1+|w(0)|\,|z|^n}.$$

Lemma 1.1 will be used to obtain a sharp lower bound on $|f'(z)|$. According to the definition of $(C)_t$ we have

$$(11)\qquad ln(f'(z)) = \frac{1}{1+w_t}\int_0^z \frac{w(\xi) - w_t}{\xi}\,d\xi = \frac{1}{1+w_t}\int_0^1 \frac{w(\rho z) - w_t}{\rho}\,d\rho$$

where $w(z) = (1/t)\,(zf''(z)/f'(z)) + (1/t) - 1$ is a function of the type described in Lemma 1.1 with $w(0) = w_t = (1/t) - 1 \leqq 0$. By comparing the real parts in (11) it will follow from lemma 1.1 that

$$|f'(z)| \geqq (1 - w_t\,|z|^n)^{\frac{1-w_t}{nw_t}}, \quad z \in E.$$

We are now in position to prove (6). Put

$$F(z) = f * h_\theta(z) = \frac{zf'(z) - t(1+e^{i\theta})f(z)}{1 - t(1+e^{i\theta})}.$$

A simple calculation will show that

$$(1 - w_t e^{-i\theta})\frac{F'(z)}{f'(z)} = 1 - e^{-i\theta}\left(\frac{1}{t}\frac{zf''(z)}{f'(z)} + w_t\right)$$

and this last statement, together with the definition of the class $(C)_t$, means that the function $F(z)$ is a univalent close-to-convex function. Moreover

$$F'(z) = f'(z)\frac{w(z) - e^{i\theta}}{w(0) - e^{i\theta}}$$

where $w(z)$ is a function satisfying the hypothesis of Lemma 1.2 with $w(0) = w_t$. We therefore obtain, using (12) and Lemma 1.2,

$$(13)\qquad |F'(z)| \geqq (1 - w_t\,|z|^n)^{\frac{1-w_t}{nw_t}} \frac{1-|z|^n}{(1-w_t\,|z|^n)}, \quad z \in E.$$

Since the function F is univalent we can integrate this last inequality to obtain

$$|F(z)| \geqq \int_0^{|z|} \frac{1-\rho^n}{(1-w_t\rho^n)^{1-\frac{1-w_t}{nw_t}}} d\rho = |z|(1-w_t|z|^n)^{\frac{1-w_t}{nw_t}}$$

and an application of the maximum principle to the non-vanishing function $F(z)/z$ will then give

$$\left|\frac{f * h_\theta(z)}{z}\right| = \left|\frac{F(z)}{z}\right| > (1-w_t)^{\frac{1-w_t}{nw_t}} = \delta_n, \quad z \in E.$$

This completes the proof of Theorem 1. The value given to δ_n is the best possible, as is seen from the functions

$$f(z) = \int_0^z (1+w_t e^{i\alpha}\xi^n)^{\frac{1-w_t}{nw_t}} d\xi, \ w_t = \frac{1}{t} - 1, \quad \alpha \in \mathbf{R}.$$

Some simple calculations will show that $f \in (C)_t$, $f^{(k)}(0) = 0$ if $1 < k \leqq n$ and

$$f'(z) + \delta_n z^{n-1} = \frac{(f(z) + \frac{\delta_n}{n} z^n) * h_\pi(z)}{z} = 0$$

for a good choice of z with $|z| = 1$ and $\alpha \in \mathbf{R}$. It means therefore that $N_\delta(f) \not\subset (S^*)_t$ if $\delta > \delta_n$.

It is also interesting to note that the result given by (1) is in fact a simple consequence of Theorem 1. Let $f(z) \in C$ with $f^{(k)}(0) = 0$ if $1 < k \leqq n$ and let $0 < r < 1$; there must exist a real number $t_0(r) > 1$ such that

$$t \geqq t_0(r) \Rightarrow \frac{1}{r} f(rz) \in (C)_t,$$

and, according to Theorem 1,

$$t \geqq t_0(r) \Rightarrow N_{\delta_n}\left(\frac{1}{r} f(rz)\right) \subset (S^*)_t \subset S^* \text{ if } \delta_n = (2 - 1/t)^{-\frac{1}{n}\frac{2-1/t}{1-1/t}}.$$

Therefore, if we let $t \to \infty$ for fixed r, we have

$$N_{d_n}\left(\frac{1}{r} f(rz)\right) \subset S^* \text{ if } d_n = 2^{-1/n},$$

and now letting $r \to 1$, we obtain Ruscheweyh's result.

Proof of Theorem 2. Let $f(z) = z + \sum_{k=2}^\infty a_k z^k \in (C)_t$ where $1/2 < t \leqq 2$ and $g(z) = z + \sum_{k=2}^\infty b_k z^k \in N_\delta(f)$ with $\delta = \inf_{z \in E} |t(f(z)/z)| - |f'(z) - t(f(z)/z)|$. In order to show that $g \in (S^*)_t$ it is enough to verify that

$$\left|t\frac{g(z)}{z}\right| - \left|g'(z) - t\frac{g(z)}{z}\right| > 0, \quad z \in E.$$

But we have

$$\begin{aligned}
&\left|t\frac{g(z)}{z}\right| - \left|g'(z) - t\frac{g(z)}{z}\right| \\
&\geqq \left|t\frac{f(z)}{z}\right| - \left|f'(z) - t\frac{f(z)}{z}\right| \\
&\quad - \left(t\left|\frac{g(z)}{z} - \frac{f(z)}{z}\right| + \left|(g'(z) - f'(z)) - t\left(\frac{g(z)}{z} - \frac{f(z)}{z}\right)\right|\right) \\
&\geqq \delta - \left(t\left|\frac{g(z)}{z} - \frac{f(z)}{z}\right| + \left|(g'(z) - f'(z)) - t\left(\frac{g(z)}{z} - \frac{f(z)}{z}\right)\right|\right) > 0
\end{aligned}$$

because for $z \in E$

$$\begin{aligned}
&t\left|\frac{g(z)}{z} - \frac{f(z)}{z}\right| + \left|(g'(z) - f'(z)) - t\left(\frac{g(z)}{z} - \frac{f(z)}{z}\right)\right| \\
&\quad = t\left|\sum_{k=2}^{\infty}(b_k - a_k)z^{k-1}\right| + \left|\sum_{k=2}^{\infty}(k - t)(b_k - a_k)z^{k-1}\right| \\
&\quad \leqq \sum_{k=2}^{\infty}(t + |k - t|)\,|b_k - a_k|\,|z|^{k-1} \\
&\quad < \sum_{k=2}^{\infty} k|b_k - a_k| \leqq \delta, \text{ if } f \not\equiv g.
\end{aligned}$$

This complete the proof of Theorem 2.

We are unable to decide in general if the value given to δ is the best possible. However we are going to show that in the case where $1 \leqq t$ we have

$$\inf_{\substack{z \in E \\ f \in (C)_t}} \left|t\frac{f(z)}{z}\right| - \left|f'(z) - t\frac{f(z)}{z}\right| = (1 - w_t)^{\frac{1-w_t}{w_t}}, \; w_t = \frac{1}{t} - 1. \tag{14}$$

The statement (14) together with the fact that the value given to δ_1 in Theorem 1 is best possible will show, at least, that Theorem 2 is sharp with respect to the complete class $(C)_t$ if $1 \leqq t \leqq 2$.

Let $t \geqq 1$, $f \in (C)_t$ and $w_t = 1/t - 1 \leqq 0$. Define

$$F(z) = zf'(z) - t(1 + e^{i\theta})f(z), \; \theta \in [0, 2\pi].$$

The identity

$$\frac{F'(z)}{tf'(z)} = \frac{1}{t}\frac{zf''(z)}{f'(z)} + \frac{1}{t} - 1 - e^{i\theta}$$

shows clearly that F is a univalent (non-normalized) close-to-convex function; it shows also that

$$\left|\frac{F'(z)}{f'(z)}\right| \geqq t\left(1 - \frac{|z| - w_t}{1 - w_t|z|}\right) = \frac{1 - |z|}{1 - w_t|z|}, \; z \in E.$$

Using the estimate (12), we obtain

$$|F'(z)| \geqq (1 - w_t|z|)^{\frac{1-w_t}{nw_t}} \frac{1 - |z|}{1 - w_t|z|}, \; z \in E$$

and just as in The proof of Theorem 1

$$\left|\frac{F(z)}{z}\right| = \left|f'(z) - t(1 + e^{i\theta}) \frac{f(z)}{z}\right| > (1 - w_t)^{\frac{1-w_t}{w_t}}, \; z \in E, \; \theta \in [0, 2\pi].$$

Now since the value of θ in the last inequality is arbitrary we obtain

$$\left|t\frac{f(z)}{z}\right| - \left|f'(z) - t\frac{f(z)}{z}\right| > (1 - w_t)^{\frac{1-w_t}{w_t}}, \; z \in E. \tag{15}$$

Simple calculations would show that the above inequality becomes an equality if we choose $f(z) = (1 + w_t z)^{1/w_t} - 1 \in (C)_t$ and $z = -1$. This, together with (15), mean that the statement (14) is valid.

Finally we would like to insist on the fact that, contrary to what might be suggested by (14), Theorem 1 is not valid when $1/2 < t < 1$; otherwise we would obtain, letting $n \to \infty$,

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in (C)_t \text{ if} \sum_{k=2}^{\infty} |kb_k| \leqq 1 \text{ and } \frac{1}{2} < t < 1. \tag{16}$$

But this last statement is seen to be false by a careful study of the polynomials $g(z) = z + (e^{i\theta}/n)\, z^n$ with n large enough. The correct "version" of (16) was first established in **[6]** where it is shown that the condition $\sum_{k=2}^{\infty} k|b_k| \leqq 1$ should be replaced by the more restrictive condition $\sum_{k=2}^{\infty} k|b_k| \leqq 2t - 1$. An extension of Theorem 1 to the case where $1/2 < t < 1$ is given in Theorem **3**.

Proof of Theorem 3. As in the case of Theorem 1 it is sufficient to show

$$\left|\frac{f * h_\theta(z)}{z}\right| > \delta_n, \; z \in E, \; \theta \in [0, 2\pi],$$

where

$$h_\theta(z) = \frac{z/(1 - z)^2 - (1 + e^{i\theta})z/(1 - z)}{-e^{i\theta}}.$$

We define $F(z) = f * h_\theta(z)$ and obtain the identity

$$\frac{F'(z)}{f'(z)} = 1 - e^{-i\theta}\frac{zf''(z)}{f'(z)}. \tag{17}$$

By the definition of the class $(C)_t$ we have that the function $zf''(z)/f'(z)$ is subordinate to the function $(1 - w_t)z^n/(1 + w_t z^n)$ and it follows from (17) that

$$\operatorname{Re}\left(\frac{F'(z)}{f'(z)}\right) \geqq 1 - \frac{(1 - w_t)|z|^n}{1 - w_t|z|^n} = \frac{1 - |z|^n}{1 - w_t|z|^n} > 0, z \in E, \tag{18}$$

and by (12)

$$|F'(z)| \geqq (1 - w_t|z|^n)^{\frac{1-w_t}{nw_t}} \frac{1 - |z|^n}{1 - w_t|z|^n}, \; z \in E. \tag{19}$$

The inequality (18) means that F is a univalent close-to-convex function and just as in the proof of Theorem 1,

$$\left|\frac{f * h_\theta(z)}{z}\right| = \left|\frac{F(z)}{z}\right| > (1 - w_t)^{\frac{1-w_t}{nw_t}} = \delta_n, \; z \in E.$$

This completes the proof of Theorem 3. For the same reasons as in Theorem 1, the value given to δ_n is the best possible.

Proof of Theorem 4. Let $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \in (C)_t$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in N_{\delta_n}(f)$. We have to show that $(1/x)\, g(xz) \in (C)_t$ where x is the only root in the interval $(0, 1)$ of the equation (2). It is easily checked that $(1/x)\, g(xz) \in (C)_t \Leftrightarrow zg'(xz) \in (S^*)_t$, and in order to prove Theorem 4 it will be sufficient, according to (5), to show that

$$\frac{zg'(xz) * h_\theta(z)}{z} \neq 0, \quad \theta \in [0, 2\pi], \; z \in E.$$

Since

$$\left|\frac{zg'(xz) * h_\theta(z)}{2}\right| \geqq \left|\frac{zf'(xz) * h_\theta(z)}{z}\right| - \left|\frac{(zg'(xz) - zf'(xz)) * h_\theta(z)}{z}\right|$$

it will be enough to verify, in view of equation (2), that

$$\left|\frac{(zg'(xz) - zf'(xz)) * h_\theta(z)}{z}\right| \leqq \sup_{k \geqq 2} (kx^{k-1})\delta_n, \quad \theta \in [0, 2\pi], \; z \in E, \tag{20}$$

and

$$\left|\frac{zf'(xz) * h_\theta(z)}{z}\right| > (1 - x^n)(1 - w_t x^n)^{-1+\frac{1-w_t}{nw_t}}, \quad \theta \in [0, 2\pi], \; z \in E. \tag{21}$$

The truth of (20) follows from

$$\left|\frac{(zg'(xz) - zf'(xz))*h_\theta(z)}{z}\right| = \left|\sum_{k=2}^{\infty} \frac{k - t(1 + e^{i\theta})}{1 - t(1 + e^{i\theta})} kx^{k-1}(b_k - a_k)z^{k-1}\right|$$

$$\leqq \sum_{k=2}^{\infty} k^2 x^{k-1} |b_k - a_k|$$

$$\leqq \sup_{k\geqq 2} (kx^{k-1})\delta_n,$$

since $g \in N_{\delta_n}(f)$.

To establish (21) we remark that $zf'(z) * h_\theta(z)/z = (f * h_\theta(z))'$ and according to (13),

$$\left|\frac{zf'(xz) * h_\theta(z)}{z}\right| \geqq (1 - x^n|z|^n)(1 - w_t x^n |z|^n)^{-1+\frac{1-w_t}{nw_t}}$$

$$> (1 - x^n)(1 - w_t x^n)^{-1+\frac{1-w_t}{nw_t}}.$$

This completes the proof of Theorem 4. The value given to x is the best possible, as can be seen from the function $f(z) = \int_0^z (1 + w_t \xi^n)^{(1-w_t)/nw_t} d\xi$. In fact, if $\sup_{k\geqq 2}(kx^{k-1}) = mx^{m-1}$ where m is an integer $\geqq 2$ and if $g(z) = f(z) + e^{i\alpha}(\delta_n/m)z^m \in N_{\delta_n}(f)$, simple calculations show that

$$\frac{zg'(xz)*h_\pi(z)}{z} = (1 + w_t x^n z^n)^{\frac{1-w_t}{nw_t}}$$

$$+ (1 - w_t)(1 + w_t x^n z^n)^{\frac{1-w_t}{nw_t} - 1} x^n z^n + (mx^{m-1})\delta_n e^{i\alpha} z^{m-1}$$

$$= (1 - x^n)(1 - w_t x^n)^{-1+\frac{1-w_t}{nw_t}} - \sup_{k\geqq 2} (kx^{k-1})\delta_n = 0.$$

if $z^n = -1$ and α is a real number correctly chosen. This means that $(1/y)g(yz) \notin (C)_t$ if $y > x$. We also remark that since

$$\frac{zg'(xz) * h_\pi(z)}{z} = g'(xz)\left(1 + \frac{xzg''(xz)}{g'(xz)}\right),$$

the value given for x is, in fact, the radius of convexity of the class

$$\bigcup_{\substack{f \in (C)_t \\ f^{(k)}(0)=0,\, 1<k\leqq n}} N_{\delta_n}(f) \subset (S^*)_t, \text{ for fixed } t \leqq 1.$$

Proof of Theorem 5. The proof of Theorem 5 is very similar to the proof of Theorem 4 and for that reason only the main steps will be supplied. We need the following lemma due to Ruscheweyh [3]. Here

$$h_T(z) = \frac{z/(1 - z)^2 + iT\, z/(1 - z)}{1 + iT} = \sum_{n=1}^{\infty} \frac{n + iT}{1 + iT} z^n$$

where T is a real number.

Lemma 5.1. *Let* $F(z) = z + \sum_{k=n+1}^{\infty} c_k z^k \in S^*$. *Then*

$$\left|\frac{F * h_T(z)}{z}\right| \geqq \frac{1 - |z|^n}{(1 + |z|^n)^{1+2/n}}, \quad z \in E,\ T \in \mathbf{R}.$$

To prove Theorem 5 it will be enough to verify that, for $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \in C$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in N_{d_n}(f)$, we have

$$\left|\frac{(zg'(xz) - zf'(xz)) * h_T(z)}{z}\right| \leqq \sup_{k \geqq 2} (kx^{k-1}) d_n, \quad z \in E,\ T \in \mathbf{R}, \tag{22}$$

and

$$\left|\frac{zf'(xz) * h_T(z)}{z}\right| > \frac{1 - x^n}{(1 + x^n)^{1+2/n}}, \quad z \in E,\ T \in \mathbf{R}. \tag{23}$$

Here x is the unique root in (0, 1) of the equation (4).

The truth of (22) follows mainly from the fact that $\max_{T \in \mathbf{R}} |(k + iT)/(1 + iT)| = k$. The truth of (23) follows from an application of Lemma 5.1 to the starlike function $zf'(xz)$. This completes the proof of Theorem 5. The value given to x is best possible as seen from the functions $f(z) = \int_0^z (1 - \xi^n)^{-2/n} d\xi \in C$ and $g(z) = f(z) + d_n e^{i\alpha}/m\, z^m \in N_{d_n}(f)$ where $\sup_{k \geqq 2} (kx^{k-1}) = mx^{m-1}$, m is an integer $\geqq 2$ and α is an appropriately chosen real number.

Conclusion. As a conclusion we would like to mention that some of the main results of this paper can be extended to some classes of non-convex univalent functions. For example if

$$H = \{f \in A \mid \operatorname{Re}(f'(z)) > 0,\ z \in E\},$$
$$\tilde{H} = \{f \in A \mid \operatorname{Re}(f'(z) + zf''(z)) > 0,\ z \in E\}$$

we can prove that

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \in \tilde{H} \Rightarrow N_{\delta_n}(f) \subset H \text{ if } \delta_n = \int_0^1 \frac{1 - \rho^n}{1 + \rho^n} d\rho$$

and

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \in \tilde{H} \text{ and } g \in N_{\delta_n}(f) \Rightarrow \frac{1}{x} g(xz) \in \tilde{H}$$

where x is the unique root in (0, 1) of the equation

$$\frac{1 - x^n}{1 + x^n} - \sup_{k \geqq 2} (kx^{k-1}) \delta_n = 0.$$

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