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AFFINE GEOMETRY AND THE FORM OF THE EQUATION OF A HYPERSURFACE

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Introduction. In classical geometry, a subset M of $\mathbf{R}^{n+1}$ is said to be a hypersurface if it is the zero-set of some (appropriately restricted) function. This function is not uniquely determined by M; for example, if M is the zero-set of F: $\mathbf{R}^{n+1} \to \mathbf{R}$, then it is also the zero-set of $G = h \cdot F$ for any nowhere-vanishing function h: $\mathbf{R}^{n+1} \to \mathbf{R}$. In other words, the "equation of M" can assume many different forms.

QUESTION. Is there a "canonical form" for the equation of M?

We do not answer this question here, but we do single out a class of "preferred forms" for the equation of any nondegenerate hypersurface $\mathbf{R}^{n+1}$. (See §1.)

The preferred forms for the equation of M give rise to certain geometric objects. Prescribing a volume element on $\mathbf{R}^{n+1}$ normalizes these objects, which then turn out to be well-known quantities from affine geometry: the Berwald-Blaschke (affine) metric, the Fubini-Pick form, and the affine normal. Our approach to these quantities is coordinate-free and seems simpler than the standard treatments which focus on the special linear group (see, for example, Blaschke [**2**], Guggenheimer [**5**], or Spivak [**6**]). (In particular, our approach does not require verifying the invariance of these quantities under change of parameters, since no parametric representation is used in the definitions.) Indeed, this paper could be used as a quick introduction to the basic notions of affine hypersurface-geometry.

Much of our formalism makes sense in spaces of infinite dimension, and our main result (Theorem C, in §2) characterizes the nondegenerate quadratic hypersurfaces in a Banach space. (The finite-dimensional version is Berwald's theorem [**1**] in affine geometry.)

Only minor changes are needed to make our discussion applicable to arbitrary level sets (and not just to zero-sets). In §3 we briefly consider those functions F such that on each level set of F, the equation $F = c$ describing that set is one of the "preferred forms" for the equation of that

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hypersurface. These functions must satisfy an affine version of the eikonal equation in Euclidean geometry.

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0. Conventions and notation. Throughout this paper, V denotes a Banach space over $\mathbf{R}$ and M is a connected hypersurface embedded in V, with inclusion map $i\colon M \to V$. (The "Banach" hypothesis allows us to use the infinite-dimensional differential calculus as described in, say, Dieudonné [3]. We never use the norm directly.)

We often identify V with its tangent spaces, and we identify "algebraic" tensors on the vector space V with "constant" tensor fields on the manifold V. The symbol $\lrcorner$ denotes the contraction operator; that is, if θ is a covariant tensor of rank r and if v is a vector, then $v \lrcorner \theta$ is the tensor of rank $r-1$ defined by $(v \lrcorner \theta)(v_1, \ldots, v_{r-1}) = \theta(v, v_1, \ldots, v_{r-1})$. The directional derivatives of a function F on V are written DF, D^2F, etc., so D^kF_p is a symmetric k-linear map $(A_1, \ldots A_k) \mapsto D^kF_p(A_1, \ldots, A_k)$ on $T_pV \cong V$. Sometimes we write D_AF_p instead of $DF_p(A)$. The operator D acts as the standard connection in V when applied to V-valued maps.

All manifolds, maps, etc., are assumed to be smooth (i.e., of class C^∞).

1. Basic constructions.

DEFINITION. An M-function is a smooth map $F\colon U \to R$, on some neighborhood U of M in V, such that $F(p) = 0$ and $DF_p \neq 0$ at each point p of M. The restrictions of D^2F and D^3F to vectors tangent along M are the fundamental forms, II^F and III^F, associated with F. More formally, if F is an M-function, then

$$\mathrm{II}^F = i^*D^2F \text{ and } \mathrm{III}^F = i^*D^3F.$$

REMARK. If F and G are both M-functions then there is a nowhere-vanishing function h such that $G = h\cdot F$ on a neighborhood $U' \subset U$. (See Lemma 3 in the Appendix.)

LEMMA 1. *Suppose F, G, and h are functions with $G = h\cdot F$ as above. Then for p in U' and A, B, C in T_pV, we have*

$$DG_p(A) = h(p)DF_p(A) + Dh_p(A)F(p), \tag{1}$$

$$\begin{aligned} D^2G_p(A, B) &= h(p)D^2F_p(A, B) \\ &\quad + Dh_p(A)DF_p(B) + Dh_p(B)DF_p(A) + D^2h_p(A, B)F(p), \end{aligned} \tag{2}$$

$$D^3G_p(A, B, C) = h(p)D^3F_p(A, B, C) + Dh_p(A)D^2F_p(B, C) \tag{3}$$
$$+ Dh_p(B)D^2F_p(A, C) + Dh_p(C)D^2F_p(A, B) + D^2h_p(B, C)DF_p(A)$$
$$+ D^2h_p(A, C)DF_p(B) + D^2h_p(A, B)DF_p(C) + D^3h_p(A, B, C)F(p).$$

PROOF. Use Leibnitz' Rule.

COROLLARY. *If F and G are M-functions, with $G = h\cdot F$ as above, then $II^G = h\cdot \mathrm{II}^F$. In particular, if II^F_p: $T_pM \times T_pM \to \mathbf{R}$ is a nondegenerate bilinear form on T_pM, then so is II^G_p.*

(Recall that a continuous bilinear form Q: $V \times V \to \mathbf{R}$ is said to be nondegenerate if the induced map $\tilde{Q}$: $V \to V^*$, defined by $\tilde{Q}(v)\ (w) = Q(v, w)$, is a linear homeomorphism.)

DEFINITION. A hypersurface M is nondegenerate if there is an M-function F such that II^F is nondegenerate at each point of M.

REMARK. A nondegenerate hypersurface M has a well-defined conformal class of pseudo-Riemannian structures, namely, $\{\mathrm{II}^F\colon F \text{ is an } M\text{-function}\}$.

From now on we assume that M is nondegenerate. Then we can associate a "normal-vector field" ν^F with any M-function F, as follows: for p in M, ν^F_p is the unique vector in T_pV such that

(4) (a) $D^2F_p(\nu^F_p, X) = 0$ for all X in T_pM, and (b) $DF_p(\nu^F_p) = 1$.

The existence and uniqueness of ν^F_p follows easily from the nondegeneracy of II^F. Using the identification $T_pV \simeq V$ (p in M), we think of ν^F as a vector-valued map ν^F: $M \to V$, the F-normal map; it is smooth. (See Lemma 4 of the Appendix.) The lines $\{p + t\nu^F_p\colon t \in \mathbf{R}\}$, for p in M, are the F-normal lines. If we set $\mu^F = D^2F(\nu^F, \nu^F)$, then $D^2F_p(\nu^F_p, A) = \mu^F(p)DF_p(A)$ for all p in M and all A in T_pV.

REMARK. If $G = h\cdot F$ is another M-function, and if $h|_M \equiv 1$, then it is clear from Lemma 1 that $\mathrm{II}^G = \mathrm{II}^F$, $\mathrm{III}^G = \mathrm{III}^F$, and $\nu^G = \nu^F$.

The F-normal ν^F determines in the usual way a symmetric connection ∇^F on M. That is, let π^F_p: $T_pV \to T_pM$ be the projection along ν^F_p; then $(\nabla^F_X Y)_p = \pi^F_p((D_X Y)_p)$.

PROPOSITION 1. *If F is an M-function and X, Y are vector fields on M, then*

$$\nabla^F_X Y = D_X Y + \nu^F \mathrm{II}^F(X, Y).$$

PROOF. If $p \in M$ and $A \in T_pV$, then $A = \pi^F_p(A) + DF_p(A)\nu^F_p$ (because $DF_p = 0$ on T_pM and $DF_p(\nu^F_p) = 1$); thus, $D_X Y = \nabla_X Y + DF(D_X Y)\nu^F_p$. But

$$DF(D_X Y) = D_X(DF(Y)) - D^2F(X, Y) = 0 - \mathrm{II}^F(X, Y),$$

since X and Y are tangent to M.

The connection ∇^F extends, in the standard way, to tensor fields of arbitrary type. We need only the following special case, whose trivial proof we omit.

PROPOSITION 2. *Suppose that η is a covariant tensor field on M, of type r, such that η is the restriction to M of a tensor field $\bar{\eta}$ on V* (i.e., $\eta = i^*\bar{\eta}$). *If $X, Y_1, \ldots Y_r \in T_pM$, then*

$$(\nabla_X^F\eta)(Y_1, \ldots, Y_r) = (D_X\bar{\eta})(Y_1, \ldots, Y_r) - \sum_{\alpha=1}^{r} \mathrm{II}^F(X, Y_\alpha)\bar{\eta}(Y_1, \ldots, Y_{\alpha-1}, \nu_p^F, \ldots, Y_r).$$

EXAMPLE. Suppose $\eta = \mathrm{II}^F$ and $\bar{\eta} = D^2F$; then Proposition 2 implies that $(\nabla_X^F\mathrm{II}^F)(Y, Z) = \mathrm{III}^F(X, Y, Z)$ for all X, Y, Z in T_pM. (Recall that $D^2F_p(\nu_p^F, \cdot) = 0$ on T_pM and that $\mathrm{III}^F = i^*D^3F$.) This implies, since ∇^F is symmetric, that $\mathrm{III}^F = 0$ if and only if ∇^F is the Levi-Civita connection for the pseudo-Riemannian structure II^F.

The F-normal determines a "Weingarten map" in the following sense.

PROPOSITION 3. *If p is any point of M and $X \in T_pM$, then $(D_X\nu^F)_p \in T_pM$.*

PROOF. Differentiate both sides of the equation $DF(\nu^F) \equiv 1$ along the vector X, obtaining $D^2F_p(\nu_p^F, X) + DF_p(D_X\nu_p^F) = 0$. Hence, $DF_p(D_X\nu_p^F) = 0$, so $D_X\nu_p^F \in T_pM$.

The corresponding notions of F-mean curvatures, F-minimality, etc., are easy to define if $\dim M < \infty$, but we shall not discuss them here.

For the remainder of this section assume that V is of finite dimension $n + 1$ and that it has a prescribed volume form Φ; also, assume that M is oriented. We use this extra structure to single out one of the II^F's as a "preferred" pseudo-Riemannian structure on M.

NOTATION. For p in M, Ψ_p^F denotes the volume form in T_pM associated with II_p^F, and Ω_p^F denotes the n-form $\nu_p^F \lrcorner \Phi$(an element of $\Lambda_p^n(V)$), while $\lambda^F: M \to \mathbf{R}$ is the function defined by $i^*(\Omega^F) = \lambda^F\Psi^F$.

DEFINITION. An M-function F is normalized on M (with respect to Φ) if $\lambda^F \equiv 1$; it is almost-normalized if λ^F is constant.

REMARKS. (a) It is clear that λ^F is never zero (because ν^F is not tangent to M).

(b) F is almost-normalized if and only if $C \cdot F$ is normalized for some constant C. In particular, the property of being almost-normalized is independent of the choice of Φ.

(c) In the Introduction we promised to single out a class of "preferred forms" for the equation of M. These preferred equations are, in fact, just those of the form $F = 0$ with F an almost-normalized M-function.

THEOREM A. *There exist normalized M-functions. More precisely, let H be any M-function such that $\lambda^H > 0$. (Such H exists because $\lambda^{(-H)} = -\lambda^H$.) Then $F = \rho H$ is normalized on M if and only if $\rho > 0$ and $\rho^{(n+2)/2} = \lambda^H$.*

PROOF. Since $\nu^H \lrcorner (\nu^H \lrcorner \Phi) = 0$, it is clear that $\nu^H \lrcorner (dH \wedge \Omega^H) = \Omega^H = \nu^H \lrcorner \Phi$. But, $dH \wedge \Omega^H$ and Φ are $(n+1)$-forms on the $(n+1)$-space V; thus, $dH \wedge \Omega^H = \Phi$. Likewise, if $F = pH$, then

$$\Phi = dF \wedge \Omega^F = (\rho dH + Hd\rho) \wedge \Omega^F = \rho dH \wedge \Omega^F$$

on M, since $H|_M = 0$. Thus,

$$\Omega^H = \nu^H \lrcorner \Phi = \nu^H \lrcorner (\rho dH \wedge \Omega^F) = \rho\Omega^F - \rho dH \wedge (\nu^H \lrcorner \Omega^F).$$

But $i^*dH = 0$, so $\lambda^H\Psi^H = \rho\lambda^F\Psi^F$. Now suppose $\lambda^F > 0$; then $\rho > 0$. Moreover, $\mathrm{II}^F = \rho\mathrm{II}^H$, so $\Psi^F = \rho^{n/2}\Psi^H$. Thus, $\lambda^H = \rho^{(n+2)/2}\,\lambda^F$, which yields the desired result.

COROLLARY. *If F is a normalized M-function and $G = h \cdot F$, then G is normalized on M if and only if $h|_M \equiv 1$. In particular, if F and G are both normalized on M, then $\mathrm{II}^G = \mathrm{II}^F$, $\mathrm{III}^G = \mathrm{III}^F$, and $\nu^G = \nu^F$.*

REMARK. From now on we shall write II, III, and ν in place of II^F, III^F, and ν^F, where F is any normalized M-function. It is clear that these quantities are invariants of the special linear group $SL(V)$. In fact, we shall see (in §3) that II is the Berwald-Blaschke metric on M, ν is the affine normal, and (1/2) III is the Fubini-Pick form (up to sign conventions).

There is a simple geometric criterion for an M-function to be almost-normalized.

THEOREM B. *The M-function F is almost-normalized if and only if $\nabla^F\Psi^F \equiv 0$; i.e., the connection ∇^F is volume-preserving with respect to II^F.*

PROOF. Apply Proposition 2 when $\eta = \lambda^F\Psi^F$ and $\bar{\eta} = \Omega^F$ to obtain

$$\begin{aligned}(\nabla^F_X(\lambda^F\Psi^F))\,&(Y_1,\ldots,Y_n)\\ &= (D_X\Omega^F)(Y_1,\ldots,Y_n) - \sum_{\alpha=1}^{n}\mathrm{II}^F(x, Y_\alpha)\Omega^F(Y_1,\ldots,Y_{\alpha-1},\nu^F,\ldots,Y_n)\end{aligned}$$

for all $Y_1,\ldots,Y_n$ tangent to M. But $\nu^F \lrcorner \Omega^F = 0$, and $i^*(D_X\Omega^F) = i^*((D_X\nu^F) \lrcorner \Phi) = 0$ because $D_X\Phi = 0$ and $D_X\nu^F$ is tangent to M; therefore $\nabla^F_X(\lambda^F\Psi^F) = 0$. Thus, $\nabla^F\Psi^F \equiv 0$ if and only if λ^F is constant, i.e., if and only if F is almost-normalized on M.

COROLLARY. *If* $\mathrm{III}^F \equiv 0$, *then* F *is almost-normalized on* M. *When* dim $V = 2$, *the converse is also true.*

PROOF. We know that $\mathrm{III}^F \equiv 0$ implies ν^F is the Levi-Civita connection for II^F; but the Levi-Civita connection preserves the volume form $\mathscr{V}^F$ of II^F. When V is a plane and M is a convex curve in V, the volume form $\mathscr{V}^F$ determines the metric II^F.

REMARKS. (a) The equation $\nabla^F \mathscr{V}^F = 0$ is (essentially) the "apolarity condition" in affine geometry.

(b) Keep in mind that the space V is of finite dimension in Theorems A and B and their Corollaries. In particular, we have defined "normalized" and "almost-normalized" only in the finite-dimensional case.

2. Nondegenerate quadratic hypersurfaces in a Banach space. A function $F: V \to \mathbf{R}$ on the Banach space V is said to be quadratic if it is of the form $F(p) = (1/2)A(p, p) + B(p) + C$ (p in V), where A is a continuous (nonzero) symmetric bilinear function, B is a continuous linear function, and C is a real number. A nondegenerate hypersurface M in V is said to be a hyperquadric if it lies in the zero-set of some quadratic M-function.

REMARK. If F is a quadratic function, then $D^3F = 0$; in particular, if the zero-set of F is the hyperquadric M, then $\mathrm{III}^F = 0$. Our main result states that the converse is "usually" true.

THEOREM C. *Suppose that* M *is a (connected) nondegenerate hypersurface in the Banach space* V *such that there exists an* M*-function* F *satisfying* $\mathrm{III}^F \equiv 0$. *If, in addition,* dim $V \geqq 3$, *then* M *is a quadratic hypersurface.*

REMARKS. (a) The hypothesis $\mathrm{III}^F = 0$ does not imply $D^3F = 0$ at points of M; indeed, the quadratic Taylor polynomials for F, centered at points of M, can vary from point to point. The solution is to replace F by a "better" M-function G such that $D^2G_p: V \times V \to \mathbf{T}$ is independent of p in M.

(b) The corollary to Theorem B explains the hypothesis "dim $V \geqq 3$"; then cylinders over plane curves explain the hypothesis "M is nondegenerate."

(c) If V is finite-dimensional, then Theorem C is just a reformulation of Berwald's Theorem in affine geometry [**4**, p. 379]. Indeed, F is almost-normalized (since $\mathrm{III}^F \equiv 0$), so $\mathrm{III} = C \cdot \mathrm{III}^F$ for some constant C. In particular, M has vanishing Fubini-Pick form, hence Berwald's theorem implies that M is a hyperquadric. It is just as easy to show that Theorem C implies Berwald's Theorem.

The proof of Theorem C involves the following notion.

DEFINITION. An M-function F has the proper sphere-property on M if all the F-normal lines pass through a single point of V; it has the improper sphere-property on M if, instead, these lines are parallel (i.e., "meet at infinity").

It is easy to verify that nondegenerate quadratics have a sphere property however, so do many nonquadratics. In fact, if $\dim V < \infty$ and if F is an almost-normalized M-function, then F has a sphere-property if and only if M is an "affine hypersphere" (see Blaschke [**2**, p. 290]).

LEMMA 2. *Let F be an M-function on a nondegenerate hypersurface M of V. If* $\dim V \geqq 3$, *then the following statements are equivalent.*

(a) *F has a sphere property on M.*

(b) *The tensor field ∇^F III^F is symmetric in all four places.*

(c) *The symmetric tensor $H^F = i^*(\nu^F \lrcorner D^3F)$ is of the form $\alpha^F \cdot \mathrm{II}^F$ for some smooth function $\alpha^F\colon M \to \mathbf{R}$.*

(d) *If $R\colon M \to V$ is the position vector, then $D\nu^F = \sigma^F DR$ for some constant σ^F.*

In addition, if these statements are true, then $\sigma^F = \mu^F - \alpha^F$.

PROOF OF LEMMA 2. It is trivial to prove that (d) implies (c) and is equivalent to (a), so we shall verify only that (b) is equivalent to (c) and that (c) implies (d).

((b) $\Leftrightarrow$ (c)). For X, Y, Z, W tangent to M, set $T^F(X, Y, Z, W) = (\nabla_W^F \mathrm{III})(X, Y, Z)$. Then Proposition 2 implies

$$\begin{aligned} T^F(X, Y, Z, W) &= (D_W(D^3F))(X, Y, Z) - (\mathrm{II}^F(W, X)D^3F(\nu^F, Y, Z) \\ &\quad + \mathrm{II}^F(W, Y)D^2F(X, \nu^F, Z) + \mathrm{II}^F(W, Z)D^3F(X, Y, \nu^F)). \end{aligned} \tag{5}$$

We rewrite (5) in terms of $H^F = i^*(\nu^F \lrcorner D^3F)$ as follows.

$$\begin{aligned} \mathrm{II}^F(W, X)H^F(Y, Z) + \mathrm{II}^F(W, Y)H^F(Z, X) + \mathrm{II}^F(W, Z)H^F(X, Y) \\ = D^4F(X, Y, Z, W) - T^F(X, Y, Z, W). \end{aligned} \tag{6}$$

It is obvious from (6) that (c) implies (b). Conversely, suppose (b) is true, i.e., T^F is symmetric; then the left side of (6) is also symmetric in X, Y, Z, W. Cyclically permuting X, Y, Z, W and comparing the resulting expressions leads to

$$\mathrm{II}^F(Z, W)H^F(X, Y) - \mathrm{II}^F(X, Y)H^F(Z, W) = 0,$$

which implies (since II^F is nondegenerate) that H^F is proportional to II^F. Thus, (b) implies (c).

((c) $\Rightarrow$ (d)). The equation (D^2F) $(\nu^F, A) = \mu^F DF(A)$, valid on M for each A in V, implies for each Y tangent to M,

$$D^3F(\nu^F, A, Y) + D^2F(D_Y\nu^F, A) = (D_Y\mu^F)DF(A) + \mu^F D^2F(A, Y). \tag{7}$$

Since $D_Y\nu^F$ is tangent to M, replacing A in (7) by X tangent to M yields

$$H^F(X, Y) + \mathrm{II}^F(D_Y\nu^F, X) = \mu^F\mathrm{II}^F(X, Y). \tag{8}$$

Assume (c); then (8) becomes

$$\mathrm{II}^F(D_Y\nu^F, X) = (\mu^F - \alpha^F)\mathrm{II}^F(X, Y).$$

But II^F is nondegenerate, so $D_Y\nu^F = \sigma^F Y$, where $\sigma^F = \mu^F - \alpha^F$. The position vector $R\colon M \to V$ satisfies $D_Y R = Y$, hence

$$D\nu^F = \sigma^F DR.$$

The constancy of σ^F follows, in the usual fashion, from the equality of "mixed partials" for V-valued functions on M; that is, from the symmetry of ∇^F. (This is where we use the hypothesis dim $V \geqq 3$, i.e., dim $M \geqq 2$.)

PROOF OF THEOREM C. It is clear that $\nabla^F\,\mathrm{III}^F$ is symmetric; indeed, $\nabla^F\,\mathrm{III}^F = 0$ because $\mathrm{III}^F = 0$. Thus there is a constant σ^F such that $D\nu^F = \sigma^F DR$. The key step is to replace F by a "better" M-function $G = h \cdot F$ for which $h|_M = 1$ and $2Dh(\nu^F) = \sigma^F - \mu^F$. (For example, define h at $q = p + t\nu^F(p)$ by $h(q) = 1 + t/2(\sigma^F - \mu^F(p))$.) Then $\nu^G = \nu^F$ so (2) implies $\mu^G = \mu^F + 2Dh(\nu^F) = \sigma^F$. In addition, $\mathrm{III}^G = \mathrm{III}^F = 0$ so Lemma 2 also applies to G, and we have $D\nu^G = \sigma^G DR$, with $\sigma^G = \mu^G - \alpha^G$. Now, $\nu^G = \nu^F$ implies $\sigma^G = \sigma^F$; hence, $\alpha^G = 0$, i.e., $H^G = 0$. From this and $\mathrm{III}^G = 0$ we get

$$D^3G_p(A, X, Y) = 0 \tag{9}$$

for all X, Y in T_pM and all A in T_pV (p in M). Next, replace F by G in (7); since μ^G equals the constant σ^G, we obtain

$$D^3G_p(A, \nu_p^G, Y) = 0. \tag{10}$$

Equations (9) and (10) imply (for A, B in T_pV and Y in T_pM)

$$D^3G_p(A, B, Y) = 0. \tag{11}$$

If we identify T_pV with V, we can give (11) a simple interpretation: for any fixed A, B in V, the number $D^2G_p(A, B)$ is independent of p in M. Indeed, if p_0, $p \in M$ and $\gamma\colon [0, 1] \to M$ is a smooth path in M from p_0, to p, then $\gamma'(t) \in T_{\gamma(t)}M$ and

$$\begin{aligned} &D^2G_p(A, B) - D^2G_{p_0}(A, B) \\ &\quad = \int_0^1 \frac{d}{dt} D^2\,G_{\gamma(t)}(A, B)dt = \int_0^1 D^3G_{\gamma(t)}(A, B, \gamma'(t))dt = 0. \end{aligned} \tag{12}$$

Now let $Q\colon V \to \mathbf{R}$ be the quadratic Taylor polynomial for G at p_0, so $Q(p_0) = G(p_0) = 0$, $DQ_{p_0}(A) = DG_{p_0}(A)$, and $D^2Q_{p_0}(A, B) = D^2G_{p_0}(A, B)$. Since Q is quadratic, $D^2Q_q(A, B)$ is independent of q in V, so we have

$$\begin{aligned} DQ_p(A) - DQ_{p_0}(A) &= \int_0^1 \frac{d}{dt} DQ_{\gamma(t)}(A)dt = \int_0^1 D^2Q_{\gamma(t)}(A, \gamma'(t))dt \\ &= \int_0^1 D^2Q_{p_0}(A, \gamma'(t))dt = \int_0^1 D^2G_{p_0}(A, \gamma'(t))dt \\ &= \int_0^1 D^2G_{\gamma(t)}(A, \gamma'(t))dt = DG_p(A) - DG_{p_0}(A), \end{aligned}$$

where we use (12) to justify the fifth equality. Thus, $DQ_p = DG_p$ for all p in M. A similar argument shows that $Q(p) = G(p)$, for all p in M. Since G vanishes on M, so does Q; thus, M is a hyperquadric.

3. Computations in $\mathbf{R}^{n+1}$. In this section we write down explicit formulas when V is finite-dimensional. Without loss of generality we assume $V = \mathbf{R}^{n+1}$ and $\Phi = dx^1 \wedge \ldots \wedge dx^{n+1}$, where $x = (x^1, \ldots, x^{n+1})$ is the standard coordinate system on $\mathbf{R}^{n+1}$. We adopt the following index conventions:

$$1 \leqq i, j, k, \ell \leqq n + 1, \quad 1 \leqq \alpha, \beta, \gamma \leqq n.$$

Our previous discussions involved only zero-sets of functions, but it is trivial to extend these ideas to arbitrary level sets; we leave the details to the reader.

Recall that the classical adjoint of a matrix (A_{ij}) is a matrix (α^{ij}) such that $\sum_k \alpha^{ik} A_{kj} = \det(A_{k\ell}) \cdot \delta^i_j$. Its entries are polynomials in the A_{ij}'s.

PROPOSITION 4. *Suppose that $F\colon U \to \mathbf{R}$ is a smooth function on a domain U in $\mathbf{R}^{n+1}$. Let (α_F^{ij}) denote the classical adjoint of the Hessian matrix $(\partial^2 F/\partial x^i \partial x^j)$. Suppose $M = \{x \in U\colon F(x) = c\}$ is a (nonempty) level set of F. Then*

(a) *A necessary and sufficient condition for F to be an M-function and M to be nondegenerate is that the following hold at each point of M:*

$$\sum_{i,j} \alpha_F^{ij} \frac{\partial F}{\partial x^i} \frac{\partial F}{\partial x^j} \neq 0. \tag{13}$$

(b) *If (13) holds on M then the components $(\nu^F)^i$ of the F-normal are given by*

$$(\nu^F)^i = \Big(\sum_j \alpha_F^{ij} \frac{\partial F}{\partial x^j}\Big) \Big/ \Big(\sum_{k,\ell} \alpha_F^{k\ell} \frac{\partial F}{\partial x^k} \frac{\partial F}{\partial x^\ell}\Big). \tag{14}$$

(c) *If (13) holds on M and λ^F is positive, then*

$$\lambda^F = \Big|\sum_{i,j} \alpha_F^{ij} \frac{\partial F}{\partial x^i} \frac{\partial F}{\partial x^j}\Big|^{-1/2}; \tag{15}$$

thus, $|\sum_{i,j} \alpha_F^{ij} (\partial F/\partial x^i)(\partial F/\partial x^j)|^{-1/(n+2)} \cdot (F - c)$ is normalized on M. In particular, a necessary and sufficient condition for F itself to be normalized (*resp., almost-normalized*) *is that the right side of* (15) *equal* 1 *on* M (*resp., be constant on* M).

PROOF. Observe that the conditions are not affected by linear changes of coordinates in $\mathbf{R}^{n+1}$ preserving Φ; then check what happens where $\partial F/\partial x^\alpha = 0$, $1 \leqq \alpha \leqq n$.

EXAMPLE 1. Suppose that M is the graph in $\mathbf{R}^{n+1}$ of a smooth function $f\colon \mathscr{D} \to \mathbf{R}$ defined on some domain $\mathscr{D}$ in $\mathbf{R}^n$. Denote the partial derivatives of f by f_α, $f_{\alpha\beta}$, etc. Define $F\colon \mathscr{D} \times \mathbf{R} \to \mathbf{R}$ by $F = x^{n+1} - f(x^1, \ldots, x^n)$; then M is the zero-set of F. Since $(\partial F/\partial x^1 \ldots, \partial F/\partial x^n, \partial F/\partial x^{n+1}) = (-f_1, \ldots, -f_n, 1)$, it is clear that F is an M-function. Set $\Delta = \det(-f_{\alpha\beta})$. It is trivial to check that

$$\left(\frac{\partial^2 F}{\partial x^i \partial x^j}\right) = \begin{pmatrix} -f_{\alpha\beta} & 0 \\ \vdots & \\ 0 \cdots 0 & \end{pmatrix} \text{and } (\alpha_F^{ij}) = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \Delta \end{pmatrix},$$

so $\sum_{i,j} \alpha_F^{ij} (\partial F/\partial x^i)(\partial F/\partial x^j) = \Delta$. Assume $\Delta \neq 0$ (so M is nondegenerate), and set $h = |\Delta|^{-1/(n+2)}$; then Proposition 4 says that $G = h \cdot F$ is normalized. Let $z^i = x^i|_M$, so $z^{n+1} = f(z^1, \ldots, z^n)$. And think of $\mathrm{II} = \mathrm{II}^G$ and $\mathrm{III} = \mathrm{III}^G$ as quadratic and cubic forms (resp.) in the differentials $dz^1, \ldots, dz^n$; then Lemma 1 yields

$$\mathrm{II} = -h \sum_{\alpha,\beta} f_{\alpha\beta} dz^\alpha dz^\beta$$

and

$$\mathrm{III} = -3 \sum_{\alpha,\beta,\gamma} \left(\frac{\partial h}{\partial z^\gamma} dz^\gamma\right)(f_{\alpha\beta} dz^\alpha dz^\beta) - h \sum_{\alpha,\beta,\gamma} f_{\alpha\beta\gamma}\, dz^\alpha dz^\beta dz^\gamma.$$

Computing $\nu = (\nu^1, \ldots, \nu^n, \nu^{n+1})$ takes more work, but it is not hard to verify that $\nu^1, \ldots, \nu^n$ are determined by the nonsingular system

$$\sum_\alpha h^2 f_{\alpha\beta} \nu^\alpha = \frac{\partial h}{\partial x^\beta}, \quad 1 \leqq \beta \leqq n,$$

while the condition $DG(\nu) = 1$ yields

$$\nu^{n+1} = h^{-1} + \sum_\alpha f_\alpha \nu^\alpha.$$

REMARK. These formulas show that up to sign, II is the Berwald-Blaschke metric on M, ν is the affine normal, and (1/2) III is the Fubini-Pick form. (Compare with the formulas for these quantities in, say, Flanders [4].)

EXAMPLE 2. Suppose F is the quadratic polynomial $1/2 \sum_{i,j} A_{ij} x^i x^j + \sum_k B_k x^k + C$ ($A_{ij} = A_{ji}$, B_k, and C all constants), and that M is the zero-set of F. Let (α^{ij}) be the classical adjoint of (A_{ij}). Then at each point of M we have

$$\sum_{i,j} \alpha^{ij} \frac{\partial F}{\partial x^i} \frac{\partial F}{\partial x^j} = \sum_{i,j} \alpha^{ij}\Big(\sum_k A_{ik}x^k + B_i\Big)\Big(\sum_\ell A_{j\ell}x^\ell + B_j\Big) \tag{16}$$
$$= \Big(\sum_{i,j} \alpha^{ij}B_iB_j\Big) - 2C \cdot \det(A_{k\ell}).$$

Equation (16) says that F is an M-function and M is nondegenerate provided $\sum_{i,j} \alpha^{ij} B_iB_j \neq 2C\det(A_{k\ell})$; in that case, F is almost-normalized. Equation (14) then becomes

$$(\nu^F)^i = \Big(\det(A_{k\ell})x^i + \sum_j \alpha^{ij}B_j\Big)\Big/\Big(\sum_{k,\ell} \alpha^{k\ell}B_kB_\ell - 2C\det(A_{k\ell})\Big).$$

Case 1. Suppose $\det(A_{k\ell}) \neq 0$. Then $1/\det(A_{k\ell}) \cdot (\alpha^{ij})$ is the inverse (A^{ij}) of (A_{ij}), so

$$(\nu^F)^i = (x^i - p_0^i)\Big/\Big(-\sum_i B_ip_0^i - 2C\Big),$$

where $p_0^i = -\sum_j A^{ij}B_j$. Thus, the F-normal lines all pass through the "center" p_0. That is, F has the proper sphere-property on M.

Case 2. Suppose $\det(A_{k\ell}) = 0$. Then

$$(\nu^F)^i = \Big(\sum_j \alpha^{ij}B_j\Big)\Big/\Big(\sum_{k,\ell} \alpha^k_\ell B_kB_\ell\Big),$$

so all the F-normal lines are parallel; thus F has the improper sphere-property.

REMARK. Proposition 4 implies that the necessary and sufficient condition for a function F to be normalized on each of its level hypersurfaces is that F satisfy the following nonlinear partial differential equation on U:

$$\sum_{i,j} \alpha_F^{ij} \frac{\partial F}{\partial x^i} \frac{\partial F}{\partial x^j} = 1. \tag{17}$$

The condition for F to be almost-normalized on each level hypersurface is

$$\sum_{i,j} \alpha_F^{ij} \frac{\partial F}{\partial x^i} \frac{\partial F}{\partial x^j} = g(F)$$

for some nonzero function g. Example 2 shows that every quadratic polynomial $F = (1/2) \sum_{i,j} A_{ij}x^ix^j + \sum_k Bx^k + C$ is almost-normalized on each of its nondegenerate level hypersurfaces; hence the usual equation of a (nondegenerate) hyperquadric M, which describes M as the zero-set of a quadratic polynomial, is a "preferred" equation for M (in the sense of the Introduction). Note, however, that the only quadratics which are normalized on more than one level hypersurface are those satisfying $\det(A_{ij}) = 0$ and $\sum_{i,j} \alpha^{ij} B_i B_j = 1$.

Equation (17) is reminiscent of the eikonal equation in geometric optics.

$$\sum_i \left(\frac{\partial F}{\partial x^i} \right)^2 = 1. \tag{18}$$

It is well-known that if F is a solution of (18) on all of $\mathbf{R}^{n+1}$, then F is linear.

CONJECTURE. *If F is a solution of the "affine eikonal equation" (17) on all of $\mathbf{R}^{n+1}$, and the level hypersurfaces of F are convex (i.e., II^F is definite), then F is a quadratic polynomial (necessarily of the type in case 2 above).*

Appendix. We prove two technical lemmas which we used in §1.

LEMMA 3. *Suppose that F and G are smooth M-functions. Then there is a smooth nonvanishing function h, on some neighborhood of M in V, such that $G = h \cdot F$. That is, $h = G/F$, which is defined and smooth off M, has a smooth nonzero exiension to M.*

PROOF. It suffices to prove this locally. Using the Implicit-function Theorem [3, Chapter 10] we reduce to the case in which $V = H_0 \oplus \mathbf{R}$, for some hyperplane H_0, and $F(u, \tau) = \tau$ for (u, τ) near the origin of $H_0 \oplus \mathbf{R}$. Then

$$G(u, \tau) = \int_0^1 \frac{d}{dt} G(u, t\tau) dt = h(u, \tau) \cdot F(u, \tau),$$

where $h(u, \tau) = \int_0^1 D_2 G(u, t\tau) dt$.

LEMMA 4. *Let M be a hypersurface in the Banach space V. Suppose that $P: M \to V^*$ and $Q: M \to V^* \oplus V^*$ are smooth maps such that for all p in M, $P_p: V \to \mathbf{R}$ is not the zero map and $Q_p: V \times V \to \mathbf{R}$ is nondegenerate. Let $\nu: M \to V$ be the map characterized at each p in M by*

(a) $Q_p(\nu_p, X) = 0$ *for all X in $T_p M$;*

(b) $P_p(\nu_p) = 1$.

Then ν is a smooth map.

PROOF. Fix $\bar{p}$ in M and set $\bar{\nu} = \nu_{\bar{p}}$. The vector $\bar{\nu}$ determines a splitting $V \simeq H \oplus R$, where $H = T_{\bar{p}} M$. Near $\bar{p}$ identify TM and T^*M with the product bundles $M \times H$ and $M \times H^*$ (resp.). Define $\tilde{Q}: M \times V \to H^*$ by $\tilde{Q}(p, v)(X) = Q_p(v, X)$ (p near $\bar{p}$, v in V, X in $T_p M \approx H$), and define $\tilde{P}: M \times V \to \mathbf{R}$ by $\tilde{P}(p, v) = P_p(v)$. Finally, define $L: M \times V \to T^*M \times \mathbf{R} = M \times H^* \times \mathbf{R}$ by $L(p, v) = (p, \tilde{Q}(p, v), \tilde{P}(p, v))$. Then L is smooth and ν_p is characterized by the equation $L(p, \nu_p) = (p, 0, 1)$. Thus it suffices to prove that L^{-1} is smooth; by the Implicit-function Theorem [3, Chapter 10], it is enough to prove that the Jacobian map $DL_{(\bar{p}, \bar{\nu})}: T_{(\bar{p}, \bar{\nu})}$

$(M \times V) \to T_{L(\bar{p},\bar{\nu})}(T^*M \times \mathbf{R})$ is a linear homeomorphism. Identify $T_{(\bar{p},\bar{\nu})}(M \times V)$ with $H \oplus H \oplus \mathbf{R}$ and $T_{L(\bar{p},\bar{\nu})}(T^*M \times \mathbf{R})$ with $H \oplus H^* \oplus \mathbf{R}$. Then, for (X, Y, c) in $H \oplus H \oplus \mathbf{R}$, we have (using the fact that $Q_{\bar{p}}(\bar{\nu}, \cdot) = 0$ on $T_{\bar{p}}M$)

$$DL_{(\bar{p},\bar{\nu})}(X, Y, c) = (X, D_1\tilde{Q}_{(\bar{p},\bar{\nu})}(X) + \tilde{Q}(p, Y), D_1\tilde{P}_{(\bar{p},\bar{\nu})}(X) + c),$$

where D_1 denotes the "partial derivative" along M. The nondegeneracy hypothesis says that the linear map $Y \mapsto \tilde{Q}(\bar{p}, Y)$ is a homeomorphism of $T_{\bar{p}}M$ onto $T^*_{\bar{p}}M$, so it is now obvious that $(DL)_{(\bar{p},\bar{\nu})}$ is also a linear homeomorphism.

COROLLARY. *If F is an M-function for a nondegenerate hypersurface M in V, then the F-normal $\nu^F: M \to V$ is smooth.*

REMARK. If V is of finite dimension, then the results in §3 give an explicit formula for ν^F which shows directly that ν^F is smooth.

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