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INDECOMPOSABLE MODULES CONSTRUCTED FROM LIOUVILLE NUMBERS.

FRANK OKOH

ABSTRACT. The submodules of the polynomial Kronecker module are investigated. A pair of vector spaces (V, W) over an algebraically closed field K is called a Kronecker module if there is a K - bilinear map form $K^2 \times V$ to W. Every module over $K[\xi]$ - the polynomial ring in one variable over K - may be viewed as a Kronecker module. The polynomial Kronecker module **P**, is $K[\xi]$ so viewed. Every infinite-dimensional submodule of **P** of finite rank has a unique infinite-dimensional indecomposable direct summand. So attention is focussed on indecomposable submodules. In that direction the main result is: For each positive integer $n > 1$, there is a family $\{V_s: s \in S\}$, Card $S = 2^{\aleph_0}$, of indecomposable submodules of **P** of rank n with the following properties:

(a) $\mathrm{Hom}\,(V_{s_1}, V_{s_2}) = 0$ if $s_1 \neq s_2$;
(b) $\mathrm{End}\,(V_s) = K$ for every s in S;
(c) $\dim \mathrm{Ext}\,(V_{s_1}, V_{s_2}) \geqq 2^{\aleph_0}$ for any s_1, s_2 in S.

This result is proved by constructing extensions of finite-dimensional modules by **P** using Liouville numbers. Each extension, **V**, is shown to share with **P** a common submodule which reflects properties of **V**. A consequence of this is that, for each positive integer $n > 1$, **P** contains a nonterminating descending chain of nonisomorphic indecomposable submodules of rank n.

1. Completely decomposable submodules of P. Throughout the paper K is a fixed algebraically closed field and (a, b) is a fixed basis of the two-dimensional K-vector space K^2. Since the map from $K^2 \times V$ to W is bilinear it is enough to specify it on (a, b) and a basis of V. In $P = (K[\xi], K[\xi])$ the bilinear map is given by $af = f$, $bf = \xi f$ for all polynomials f.

Each $e \in K^2$ gives rise to a linear transformation T_e: $V \to W$ defined by $T_e(v) = ev$, the image of (e, v) under the bilinear map from $K^2 \times V$ to W. If T_e is one-to-one for every nonzero e in K^2, **V** is said to be torsion-free. So P is torsion-free. Observe that P is an ascending union, $\bigcup_{k=1}^{\infty} \mathbf{V}_k$, of finite-dimensional submodules where $\mathbf{V}_1 = (0, [1])$; and, for $k \geqq 2$,

$$V_k = [1, \xi, \ldots, \xi^{k-2}],\ W_k = [1, \ldots, \xi^{k-2}, \xi^{k-1}]. \tag{1}$$

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All references to $\mathbf{V}_k \subset \mathbf{P}$ are to $\mathbf{V}_k$ in (1).

Here as elsewhere $[S]$ denotes the subspace spanned by S. (We are following the practice in [**14**] of using **V**, **X**, and **U** for the respective Kronecker modules (V, W), (X, Y) and (U, Z).) The dimension of $\mathbf{V} =$ the sum of the dimensions of the vector spaces V and W. $\mathbf{V}_k$ above is an example of a finite-dimensional module of type III^k. A Kronecker module is torsion-free if and only if every finite-dimensional submodule is a direct sum of modules of type III^m for various positive integers m. This latter definition generalizes readily to modules over tame hereditary finite-dimensional algebras. For details see [**10**]. There it is shown that torsion-free Kronecker modules may be viewed as flat, **Z**-graded $K[X, Y]$-modules, see [**10**, Proposition 2.4 and Remark 4.8]. See also [**9**, Proposition 2.2].

If **X** is a submodule of **V**, i.e., $X \subset V$, $Y \subset W$ and $T_e(X) \subset Y$ for every $e \in K^2$, then $\mathbf{V}/\mathbf{X}$ is a module with $e(v + X) = ev + Y$. Let $\mathbf{V}_k$ be the submodule of **P** described above. Then, for $k \geqq 2$, $\mathbf{V}_1 \subset \mathbf{V}_k$ and $\mathbf{V}_k/\mathbf{V}_1$ is of type II_∞^{k-1}; $\mathbf{V}_k/(\mathbf{V}_1 \oplus (0, [\xi^{k-1}]))$ is of type I^{k-1}. If $k \geqq 2$, then, for any $\theta \in K$, $V_k = [1, \xi - \theta, \ldots, (\xi - \theta)^{k-2}]$, $W_k = [1, \xi - \theta, \ldots, (\xi - \theta)^{k-1}]$. $\mathbf{V}_k/(0, [(\xi - \theta)^{k-1}])$ is of type II_θ^{k-1}. As m runs over the positive integers the types III^m, II_∞^m, II_θ^m, I^m exhaust the finite-dimensional indecomposable isomorphism types, see [**1**] for details. We have recalled only as much as we need in the sequel. If **V** is a Kronecker module then there is a smallest submodule $\mathbf{X} \subset \mathbf{V}$ such that $\mathbf{V}/\mathbf{X}$ is torsion-free. **V** is torsion if $\mathbf{V} = \mathbf{X}$, e.g., the modules of type II_∞^m, II_θ^m or I^m are torsion. A module **V** is divisible if $\mathbf{T}_e$: $V \to W$ is onto for every nonzero e in K^2. Divisible Kronecker modules have a finished structure, while reduced torsion Kronecker modules are essentially torsion $K[\xi]$-modules. For a systematic treatment of these matters in a setting that includes Kronecker modules as a special case, we refer to [**10**], where many of the results in [**1**] and [**16**] are given a unified treatment. [**10**, Corollary 2.3] is particularly pertinent to us because it gives the results of this paper a free ride to the category of modules over any tame hereditary finite-dimensional algebra.

Even though the modules we deal with here have no analogues in abelian group theory our main result still bears a formal resemblance to several results on rigid families of abelian groups, see [**5**, p. 401], [**8**, §88], and [**17**]. We should point out that some of the results in [**17**] are beachheads of set theory.

While the rank of a Kronecker module can be defined in a manner analogous to rank for modules over domains we shall need the complicated version of [**6**] which we now recall. A submodule $\mathbf{X} \subset \mathbf{V}$ is said to be torsion-closed in **V** if $\mathbf{V}/\mathbf{X}$ is torsion-free. Let **V** be a torsion-free module and X, Y respective subsets of V and W. Then there is a smallest submodule $\mathbf{V}^1$ of **V** with $X \subset V^1$, $Y \subset W^1$ such that $\mathbf{V}/\mathbf{V}^1$ is torsion-free. $\mathbf{V}^1$ is called the torsion-closure of (X, Y) in **V** and is denoted by $\mathrm{tc}_{\mathbf{V}}(X, Y)$.

A subset $\{w_i\}_{i \in I} \subset W$ is said to generate $\mathbf{V}$ if $\mathbf{V} = \text{tc}_{\mathbf{V}}(\varnothing, \{w_i\}_{i \in I})$. It is linearly independent with respect to generation if, for every $i_0 \in I$, $w_{i_0} \notin Y$ where $\mathbf{X} = (X, Y) = \text{tc}_{\mathbf{V}}(\varnothing, \{w_i\}_{i \in I \setminus i_0})$. If $\{w_i\}_{i \in I}$ has both of the above properties it is called a basis of $\mathbf{V}$ with respect to generation and card (I) is called the rank of $\mathbf{V}$. As shown in [**6**, p. 431 ff], any subset of W that generates $\mathbf{V}$ contains a basis of $\mathbf{V}$ with respect to generation and a subset of W linearly independent with respect to generation can be extended to a basis of $\mathbf{V}$ with respect to generation. That is all we need to prove the following additive property of rank.

THEOREM A ([**6**, Theorem 2.4]). *Let* $\mathbf{X}$ *be a submodule of* $\mathbf{V}$. *Then rank* $\mathbf{V} \leqq$ Rank $\mathbf{X}$ + Rank $\mathbf{V}/\mathbf{X}$, *with equality, if* $\mathbf{X}$ *is torsion-closed in* $\mathbf{V}$.

PROOF. Let $\{y_i\}_{i \in I}$ be a basis of $\mathbf{X}$ with respect to generation and let $\{w_j\}_{j \in J}$ be coset representatives of $\{\bar{w}_j\}_{j \in J}$ a basis of $\mathbf{V}/\mathbf{X}$ with respect to generation. Since $\{y_i\}_{i \in I} \cup \{w_j\}_{j \in J}$ clearly generates $\mathbf{V}$, it is enough to show that if $\mathbf{X}$ is torsion-closed in $\mathbf{V}$, then the set is linearly independent with respect to generation. Let $\mathbf{V}_1 = \text{tc}_{\mathbf{V}}(\varnothing, \{y_i\}_{i \in I} \cup \{w_j\}_{j \in J})$. It is immediate that $\mathbf{X} \subset \mathbf{V}_1$. Also, $\mathbf{V}_1/\mathbf{X}$ is a torsion-closed submodule of $\mathbf{V}/\mathbf{X}$. W_1/Y contains $\{\bar{w}_j\}_{j \in J}$. Therefore, $\mathbf{V} = \mathbf{V}_1$. From the set $\{y_i\}_{i \in I} \cup \{w_j\}_{j \in J}$ extract a basis B of $\mathbf{V}$ with respect to generation that includes $\{y_i\}_{i \in I}$. Since $\{\bar{w}_j\}_{j \in J}$ is a basis of $\mathbf{V}/\mathbf{X}$ with respect to generation, no w_j can be omitted. Hence $B = \{y_i\}_{i \in I} \cup \{w_j\}_{j \in J}$ and we are done.

As a result of Theorem A, a torsion-closed submodule, $\mathbf{X}$, of a torsion-free module of finite rank is a proper submodule if and only if rank $\mathbf{X} <$ rank $\mathbf{V}$.

The rank one torsion-free Kronecker modules, like rank one torsion-free abelian groups, are characterized by height functions, [**6**, §3]. Let $\tilde{K} = K \cup \{\infty\}$. Let $\mathbf{V}$ be a torsion-free module and let $w \in W$. Let $\mathbf{V}_k$ be the submodule of $\mathbf{P}$ described in (1). We shall define $H^{\mathbf{V}}(w)_\theta$-the height of w in $\mathbf{V}$ at θ in terms of homomorphisms from $\mathbf{V}_k$ to $\mathbf{V}$. Recall that a homomorphism (φ, ψ): $\mathbf{V}_1 \to \mathbf{V}_2$ is a pair of linear maps φ: $V_1 \to V_2$ and ψ: $W_1 \to W_2$ such that, for all e in K^2 and all v in V_1,

$$e\varphi(v) = \psi(ev). \tag{2}$$

$H^{\mathbf{V}}(w)_\infty \geqq k - 1$ if and only if there is a homomorphism (φ, ψ) from $\mathbf{V}_k$ to $\mathbf{V}$ with $\psi(1) = w$. If $\theta \neq \infty$ then $H^{\mathbf{V}}(w)_\theta \geqq k - 1$ if there is a homomorphism from $\mathbf{V}_k$ to $\mathbf{V}$ with $\psi(\xi - \theta)^{k-1} = w$. For $\theta \in \tilde{K}$, $H^{\mathbf{V}}(w)_\theta = \infty$ if $H^{\mathbf{V}}(w)_\theta > k$ for all positive integers k. If $H^{\mathbf{V}}(w)_\theta \geqq m$ we shall say that at θ, w generates a submodule of $\mathbf{V}$ of type III^k, $k \geqq m$. In §3 we shall repeatedly use the fact that if (φ, ψ) is a homomorphism from $\mathbf{V}_1$ to $\mathbf{V}_2$, both assumed torsion-free, and at θ, w in W_1 generates a submodule of $\mathbf{V}_1$ of type III^k, $k \geqq m$, then $\psi(w)$ does the same in $\mathbf{V}_2$ unless $\psi(w) = 0$. So,

$$\mathrm{Hom}(\mathrm{III}^{m_1}, \mathrm{III}^{m_2}) = 0 \text{ if } m_1 > m_2. \tag{3}$$

We shall now see, mostly by quoting results from [**15**], why the study of submodules of **P** of finite rank may be restricted to studying extensions of $(n-1)\mathrm{III}^1$ by **P**, $n \geqq 2$. ($n\mathbf{V}$ stands for $\mathbf{V} \oplus \ldots \oplus \mathbf{V}$ (n copies).) The next proposition disposes of the rank one submodules.

PROPOSITION B. (a) *A rank one torsion-free module* **V** *is isomorphic to* **P** *if and only if any nonzero element w in W has the property that*

$$H^{\mathbf{V}}(w)_\theta = \infty \textit{ if and only if } \theta = \infty \textit{ and } H^{\mathbf{V}}(w)_\theta = 0 \quad \textit{for all but finitely many } \theta \textit{ in } \tilde{K}. \tag{4}$$

(b) *Every infinite-dimensional submodule of* **P** *of rank one is isomorphic to* **P**.

(c) *Every endomorphism* (φ, ψ) *of* **P** *is given by multiplication by some polynomial f*, i.e., $\varphi(p) = \psi(p) = pf$ *for all polynomials p.*

PROOF. (a). This follows from [**6**, Theorem 3.7] or [**8**, Section 85].

(b). If $\mathbf{X} \subset \mathbf{P}$, then, for any y in Y, $H^{\mathbf{X}}(y)_\theta \leqq H^{\mathbf{P}}(y)_\theta$ for all $\theta \in \tilde{K}$. If **X** is infinite-dimensional and of rank one, then $H^X(y)_\theta = \infty$ for some θ. By (4), $\theta = \infty$. So (b) follows from (a).

(c). This follows from (2) with $f = \varphi(1)$.

The next result gives us some structure for infinite-dimensional submodules of **P** of finite rank > 1.

THEOREM C. (a) [**15**, Corollary 1.6, Proposition 1.11 and **11**, Lemma 1.11]. *Let* **X** *be an infinite-dimensional submodule of* **P** *of finite rank n. Then* **P**/**X** *is finite-dimensional. Moreover,* **X** *is isomorphic to an extension of a module of type* $(n-1)\ \mathrm{III}^1$ *by* **P**.

(b) [**15**, Theorem 1.14]. *An extension of a finite-dimensional torsion-free module by* **P** *is isomorphic to a submodule of* **P**.

(c) [**15**, Corollary 1.15]. *An extension of a module of type* III^m *by* **P** *is isomorphic to a submodule* **X** *of* **P** *where X is of codimension one in* $K[\xi]$ *and* $Y = K[\xi]$.

Theorem *D* below justifies zeroing in on the indecomposable submodules of **P**. In **P** a submodule of finite rank is indecomposable if and only if it is purely simple, [**15**, Theorem 1.8] or [**13**, Theorem 4].

THEOREM D. [**15**, Corollary 1.9]. *An infinite-dimensional submodule* **X** *of* **P** *of finite rank is of the form*

$$\mathbf{X} = \mathbf{X}_1 \dot{+} \mathbf{X}_2,$$

where $\mathbf{X}_1$ *is finite-dimensional and* $\mathbf{X}_2$ *is a unique infinite-dimensional indecomposable submodule of* $\mathbf{X}$. *Moreover any infinite-dimensional indecomposable submodule of* $\mathbf{X}$ *is contained in* $\mathbf{X}_2$.

The last sentence in Theorem D is not in **[15]** but it is proved in the same manner as the uniqueness of $\mathbf{X}_2$.

A consequence of Theorem D and Proposition $B\,(b)$ is that a completely decomposable submodule of $\mathbf{P}$ of finite rank n is isomorphic to a direct sum of a finite-dimensional submodule of rank $n-1$ and a module isomorphic to $\mathbf{P}$. Moreover by the last sentence in Theorem D, its isomorphism type is determined by the isomorphism type of the finite-dimensional component. Since by Kronecker's theorem [**6**, Theorem 4.3], a torsion-free finite-dimensional module of rank $n-1$ is of type $\mathrm{III}^{m_1} \oplus \cdots \oplus \mathrm{III}^{m_{n-1}}$, we have

THEOREM 1.1. *The set* IN^{n-1} *of unordered* $(n-1)$*-tuples of natural numbers is a parametrisation of the isomorphism classes of completely decomposable submodules of* $\mathbf{P}$ *of rank* n.

2. Indecomposable submodules of P of finite rank. By Theorem C (a), an infinite-dimensional submodule $\mathbf{V}$ of $\mathbf{P}$ of rank n is isomorphic to an extension of a module of type $(n-1)\ \mathrm{III}^1$ by $\mathbf{P}$. So we may assume that we have the extension

$$E: 0 \to \mathbf{X}' \to \mathbf{V} \to \mathbf{P} \to 0, \tag{5}$$

where

$$\mathbf{X}' = (0, [w_2, \ldots, w_n]),\ V = K[\xi],\ W = V \oplus [w_2, \ldots, w_n].$$

The bilinear map from $K^2 \times V$ to W is given by $af = f + y_f$, $bf = \xi f + y_f'$, where y_f and y_f' are elements of Y' depending on the polynomial f. Fortunately there is no loss of generality if this unwieldy bilinear map is replaced by

$$\begin{aligned} af &= f \\ bf &= \xi f + \sum_{j=2}^{n} \ell_j(f) w_j, \end{aligned} \tag{6}$$

where $\ell_2, \ldots, \ell_n$ are linear functionals on $K[\xi]$, see [**11**, Theorem 1.8]. Since we may, by Theorem 1.1, restrict ourselves to indecomposable submodules of $\mathbf{P}$ we shall assume throughout that $\{\ell_2, \ldots, \ell_n\}$ is a linearly independent set of linear functionals [**14**, Lemma 2.2].

By setting $\ell_i(\xi^k) = \alpha_{ik}$, ℓ_i may be identified with $\sum_{k=0}^{\infty} \alpha_{ik}\xi^k$. Hence L may be considered an element of $K[[\xi]]^{n-1}$, where $K[[\xi]]$ is the ring of formal power series over K. Conversely any element of $K[[\xi]]^{n-1}$ gives some L which can be used to get a bilinear map from $K^2 \times V$ to W. Denote the corresponding extension by E_L and the middle module by $\mathbf{V}_L$.

Throughout the paper all extensions of $(n-1)\,\mathrm{III}^1$ by $\mathbf{P}$ will be constructed from the vector spaces in (5) using some L.

PROPOSITION 2.1. Ext($\mathbf{P}$, $(n-1)\mathrm{III}^1$) *is isomorphic to* $K[[\xi]]^{n-1}$ *as* $K[\xi]$-*modules.*

PROOF. We have the following short exact sequence $0 \to (n-1)\mathrm{III}^1 \to (n-1)\mathrm{II}^1_\infty \to (n-1)\mathrm{I}^1 \to 0$. This induces an isomorphism

$$\mathrm{Hom}(\mathbf{P}, (n-1)\mathrm{I}^1) \to \mathrm{Ext}(\mathbf{P}, (n-1)\mathrm{III}^1) \tag{7}$$

because $\mathrm{Hom}(\mathbf{P}, \mathrm{II}^1_\infty) = 0 = \mathrm{Ext}(\mathbf{P}, \mathrm{II}^1_\infty)$ see, e.g., [7]. Given a power series $\sum_{k=0}^{\infty}\alpha_k\xi^k$ one gets an element of $\mathrm{Hom}(\mathbf{P}, \mathrm{I}^1)$ by setting $\varphi(\xi^k) = \alpha_k\, v$, $\psi(\xi^k) = 0$, where $\{v\}$ is a basis of the top space of I^1. Conversely if (φ, ψ) is an element in $\mathrm{Hom}(\mathbf{P}, \mathrm{I}^1)$, then $\varphi(\xi^k) = \alpha_k\, v$ and $\sum_{k=0}^{\infty}\alpha_k\xi^k \in K[[\xi]]$. Hence $\mathrm{Hom}(\mathbf{P}, (n-1)\mathrm{I}^1)$, and so $\mathrm{Ext}(\mathbf{P}, (n-1)\mathrm{III}^1)$ is isomorphic to $K[[\xi]]^{n-1}$. Since both $\mathrm{Hom}(\mathbf{P}, (n-1)\mathrm{I}^1)$ and $\mathrm{Ext}(\mathbf{P}, (n-1)\mathrm{III}^1)$ are modules over $\mathrm{End}(\mathbf{P}) = K[\xi]$ it follows that the isomorphism in (7) is a $K[\xi]$-module isomorphism.

REMARK 2.2. If $q \in K[\xi]$ and $q(0) \neq 0$, then $qK[[\xi]] = K[[\xi]]$. So q acts as a unit on $\mathrm{Ext}(\mathbf{P}, (n-1)\mathrm{III}^1)$. This implies that the modules in (7) are modules over the discrete valuation ring $R = \{p/q \mid p, q \in K[\xi], q(0) \neq 0\}$. However $\mathbf{V}_L$ need not be isomorphic to $\mathbf{V}_{qL}$, even when q is a unit in R. We illustrate this with an example when $n = 2$. Let $L = (1, 0, 0, \ldots)$ and $q = 1/(1-\xi^2) = 1 + \xi^2 + \xi^4 \ldots \in K[[\xi]]$. So in $\mathbf{V}_L$ (see (6) with $n = 2$)

$$\begin{aligned} a \cdot \xi^k &= \xi^k, \; k = 0, 1, \ldots, \\ \mathrm{b}.1 &= \xi + w_2, \\ \mathrm{b}\xi^k &= 0, \; k = 1, 2, \ldots. \end{aligned}$$

So $\mathbf{V}_L = ([\mathbf{1}], [1, \xi + w_2]) \dotplus (\xi K[\xi], \xi K[\xi])$, $a1 = 1$, $\mathrm{b}.1 = \xi + w_2$; $(\xi K[\xi], \xi K[\xi]) \subset \mathbf{P}$. That is, $\mathbf{V}_L$ is a module of type $\mathrm{III}^2 \oplus \mathbf{P}$.

Let $L' = qL$. In $\mathbf{V}_{L'}$

$$\begin{aligned} a\xi^k &= \xi^k, \; k = 0, 1, \ldots, \\ b\xi^k &= \xi^{k+1} + w_2, \text{ if } k \text{ is even}, \\ b\xi^k &= \xi^{k+1}, \text{ if } k \text{ is odd}, \end{aligned}$$

$$\mathbf{V}_{L'} = ([\xi, \xi^2], [\xi, \xi^2, \xi^3 + w_2]) \dotplus ((1-\xi^2)K[\xi], (1-\xi^2)K[\xi]),$$

$$a\xi = \xi, b\xi = \xi^2, a\xi^2 = \xi^2, b\xi^2 = \xi^3 + w_2,$$

and

$$(1 - \xi^2)K[\xi], (1 - \xi^2)K[\xi] \subset \mathbf{P}.$$

That is, $\mathbf{V}_L$ is a module of type $\mathrm{III}^3 \oplus \mathbf{P}$. Hence $\mathbf{V}_L$ is not isomorphic to $\mathbf{V}_{L'}$, by Theorem 1.1.

Our indecomposable modules will come from judicious choices of L. As in the proof of Proposition 2.1, L may be considered a homomorphism from $\mathbf{P}$ to $(n - 1)\mathrm{I}^1$. So we get the exact sequence

$$0 \to \mathbf{X}_L \to \mathbf{P} \to (n - 1)\mathrm{I}^1 \to 0$$

where

$$X_L = \bigcap_{j=2}^{n} \operatorname{Ker} \ell_j, \qquad Y = K[\xi].$$

Since $X_L \subset V_L$, $Y \subset W$, W as in (5), the module $\mathbf{X}_L = (X_L, Y)$ is also a submodule of $\mathbf{V}_L$. It shares many properties with $\mathbf{V}_L$. In particular $\mathbf{V}_L$ is indecomposable if and only if $\mathbf{X}_L$ it indecomposable; see Corollary 2.6.

PROPOSITION 2.3. Rank $\mathbf{X}_L$ = Rank $\mathbf{V}_L$.

PROOF. Since $\{\ell_2, \ldots, \ell_n\}$ is a linearly independent set of functionals on $K[\xi]$, X_L is of codimension $n - 1$ in $K[\xi]$. Let $f_1, f_2, \ldots, f_{n-1}$ be representatives of a basis of V/X_L. In $\mathbf{V}/\mathbf{X}_L$, $af_1 = af_2 = \ldots = af_{n-1} = 0$ while $bf_i = \xi f_i + w_i' \neq 0$ from (6) and the fact that $f_i \notin X_L$. So we have the exact sequence:

$$0 \to \mathbf{X}_L \to \mathbf{V} \to (n - 1)\mathrm{II}^1_\infty \to 0. \tag{8}$$

There is a surjective map (φ, ψ) from $(n - 1)\mathrm{III}^2$ to $(n - 1)\mathrm{II}^1_\infty$ with kernel of type $(n - 1)\mathrm{III}^1$. Using (8) and pullback we get

$$\begin{array}{ccccccccc} 0 & \to & \mathbf{X}_L & \to & \mathbf{V}_L & \to & (n-1)\mathrm{II}^1_\infty & \to & 0 \\ & & \| & & \uparrow (\varphi', \psi') & & \uparrow (\varphi, \psi) & & \\ 0 & \to & \mathbf{X}_L & \to & \mathbf{V}' & \to & (n-1)\mathrm{III}^2 & \to & 0. \end{array} \tag{9}$$

From the middle part of (9) we get

$$0 \to (n - 1)\mathrm{III}^1 \to \mathbf{V}' \to \mathbf{V}_L \to 0. \tag{10}$$

From Theorem A, (9), and (10) we get that

$$\text{Rank } \mathbf{V}' = (n - 1) + \text{Rank } \mathbf{X}_L,$$
$$\text{Rank } \mathbf{V}' = (n - 1) + \text{Rank } \mathbf{V}_L.$$

Hence Rank $\mathbf{X}_L$ = Rank $\mathbf{V}_L$.

Let L, L_1 be two elements of $K[[\xi]]^{n-1}$ and let $(\varphi, \psi) \in \mathrm{Hom}(\mathbf{V}_L, \mathbf{V}_{L_1})$. Since from (2), $a\varphi(f) = \psi(af)$, we deduce from (6) that

(11) $\varphi = \psi$ on $K[\xi]$. Moreover, φ determines ψ.

See (12) below for the justification of the last sentence in (11). $f \in X_L$ if and only if $bf = \xi f$, by (6). Since $av = v$ for all $v \in V$, this implies that $f \in X_L$ if and only if there is a nonzero homomorphism (μ, ν) from $\mathbf{V}_3 \subset \mathbf{P}$ in (1) to $\mathbf{V}_L$ with $\nu(1) = f$. From this and (2) we deduce that $\varphi(\mathbf{X}_L) \subset \mathbf{X}_{L_1}$. So we have the homomorphism

$$\chi : \mathrm{Hom}\,(\mathbf{V}_L, \mathbf{V}_{L_1}) \to \mathrm{Hom}(\mathbf{X}_L, \mathbf{X}_{L_1})$$
$$(\varphi, \psi) \to (\varphi', \varphi)$$

where φ' denotes the restriction of φ to X_L. The injectivity of χ follows from (11). χ is, in fact, bijective. Let $(\varphi', \varphi) \in \mathrm{Hom}(\mathbf{X}_L, \mathbf{X}_{L_1})$. We want to extend this to (φ, ψ), an element of $\mathrm{Hom}(\mathbf{V}_L, \mathbf{V}_{L_1})$. On $K[\xi]$, put $\varphi = \psi$. So we need only define ψ on $[w_2, \ldots, w_n]$. Choose $f_2, \ldots, f_n$ in $K[\xi]$ such that for each $j = 2, \ldots, n$, $\ell_j(f_j) \neq 0$ but $\ell_i(f_j) = 0$ for $i \neq j$. This is possible because $\{\ell_2, \ldots, \ell_n\}$ is a linearly independent set of functionals on $K[\xi]$. Set

$$\psi(w_j) = \frac{\xi\varphi(f_j) + L_1(\varphi(f_j))\omega - \varphi(\xi f_j)}{\ell_j(f_j)} \tag{12}$$

where $\omega = [w_2, \ldots, w_n]^t$ an $(n - 1) \times 1$ matrix and $L(\varphi_1(f_j)) = (\ell_2'(\varphi(f_j)), \ldots, \ell_n'(\varphi(f_j)))$ – a $1 \times (n - 1)$ matrix. This proves.

THEOREM 2.4. *The restriction map*

$$\chi : \mathrm{Hom}(\mathbf{V}_L, \mathbf{V}_{L_1}) \to \mathrm{Hom}(\mathbf{X}_L, \mathbf{X}_{L_1})$$

is bijective.

From Theorem 2.4 we obtain the following corollaries.

COROLLARY 2.5. $\mathbf{X}_L$ *is isomorphic to* $\mathbf{X}_{L_1}$ *if and only if* $\mathbf{V}_L$ *is isomorphic to* $\mathbf{V}_{L_1}$.

COROLLARY 2.6. $\mathbf{X}_L$ *is indecomposable if and only if* $\mathbf{V}_L$ *is indecomposable.*

REMARK 2.7. If $\mathbf{V}_L$ is completely decomposable then by Theorem D, $\mathbf{V}_L = \mathbf{X}_1 \dotplus \mathbf{X}_2$ where $\mathbf{X}_1$ is finite-dimensional and $\mathbf{X}_2$ is isormorphic to $\mathbf{P}$. From Proposition B (a) and (6) we deduce that $\mathbf{X}_2 \subset \mathbf{X}_L$. Hence $\mathbf{X}_L = \mathbf{X}_2 \dotplus \mathbf{X}_3$ where $\mathbf{X}_3$ is finite-dimensional. Since $\mathbf{X}_L \neq \mathbf{V}_L$, $\dim \mathbf{X}_1 \neq \dim \mathbf{X}_3$. Hence from Theorem 1.1, $\mathbf{X}_L$ is not isomorphic to $\mathbf{V}_L$. Also if $\mathbf{V}_L$ is a module in the set $\{\mathbf{V}_s : s \in S\}$ of Theorem 3.6 then $\mathbf{V}_L$ is not isomorphic to $\mathbf{X}_L \subset \mathbf{V}_L$ because $\mathrm{End}(\mathbf{V}_L) = K$. This implies that the modules in the chain in Proposition 2.8 are not isomorphic when $n > 1$.

PROPOSITION 2.8. *For any positive integer n,* **P** *contains a nonterminating descending chain of indecomposable submodules of rank n.*

PROOF. If I is a nonzero ideal in $K[\xi]$ then (I, I) is a submodule of **P** isomorphic to **P**. So if $n = 1$ any nonterminating descending chain of ideals of $K[\xi]$, $I_1 \supset I_2 \supset \ldots$, gives rise to a similar chain of submodules of **P** of rank one.

Let **V** be an indecomposable module of rank $n \geqq 2$ constructed as in Theorem 3.1. By Theorem C (b), for some $\mathbf{X}_0 \subset \mathbf{P}$, $\mathbf{V} \cong \mathbf{X}_0$. We now show that every indecomposable submodule, $\mathbf{X}_k$, of **P** of rank $n \geqq 2$ contains an indecomposable submodule also of rank n. By Theorem C (a), $\mathbf{X}_k$ is isomorphic to an extension of $(n - 1)\mathrm{III}^1$ by **P**. So for some $(n - 1)$-tuple of linearly independent functionals L on $K[\xi]$ we have an isomorphism $(\varphi, \phi)\colon \mathbf{V}_L \to \mathbf{X}_k$. By Proposition 2.3 and Corollary 2.6, $\mathbf{X}_L$ is a proper indecomposable submodule of $\mathbf{V}_L$ of rank n. Let $\mathbf{X}_{k+1} = (\varphi, \phi)(\mathbf{X}_L)$. The required nonterminating chain is $\mathbf{X}_0 \supset \mathbf{X}_1 \supset \ldots$.

The next proposition is an isomorphism criterion which will be used a lot in §3.

PROPOSITION 2.9. *If (φ, ϕ) is an isomorphism from $\mathbf{V}_L$ onto $\mathbf{V}_{L_1}$ then there exists a positive integer M such that* $\deg p(\xi) = \deg \varphi(p(\xi))$ *whenever $p(\xi)$ is a polynomial of degree not less than M.*

PROOF. Let $(\varphi, \phi)\colon \mathbf{V}_L \to \mathbf{V}_{L_1}$ be an isomorphism onto $\mathbf{V}_{L_1}$. From (11) we know that $\varphi = \phi$ on $K[\xi]$. Let $\varphi(\xi^k) = p_k = \phi(\xi^k)$. Using the notation in (12), we get from (6) that $b\xi^{k-1} = \xi^k + L(\xi^{k-1})\omega$. So $\phi(b\xi^{k-1}) = \phi(\xi^k) + L(\xi^{k-1})\phi(\omega) = p_k + L(\xi^{k-1})\phi(\omega)$. On the other hand, from (2) and (6), we have

$$\phi(b\xi^{k-1}) = b\varphi(\xi^{k-1}) = \xi p_{k-1} + L_1(\varphi(\xi^{k-1}))\omega,$$

$$\xi p_{k-1} + L_1(\varphi(\xi^{k-1}))\omega = p_k + L(\xi^{k-1})\phi(\omega).$$

The components of both sides in $[w_2, \ldots, w_n]$ are equal. So the equations below implicitly ignore them, because p_k is a polynomial.

$$p_k = \xi p_{k-1} - L(\xi^{k-1})\phi(\omega). \tag{13}$$

Hence,

$$p_k = \xi^k p_0 - \xi^{k-1} L(1)\phi(\omega) - \xi^{k-2} L(\xi)\phi(\omega) \cdots - \xi^L(\xi^{k-2})\phi(\omega) - L(\xi^{k-1})\phi(\omega).$$

Since φ is an automorphism of $K[\xi]$, $[p_0, p_1, \ldots] = K[\xi]$. So there exists an integer $m \geqq 0$ such that

$$\deg p_m \geqq \max\{\deg \phi(w_2), \ldots, \deg \phi(w_n)\}. \tag{14}$$

Since $p_{m+1} = \xi p_m - L(\xi^m)\psi(\omega)$ it follows that $\deg p_{m+1} = \deg p_m + 1$. Similarly, for $k = 1, 2, \ldots,$

$$\deg p_{m+k} = \deg p_m + k. \tag{15}$$

Since φ is an automorphism of $K[\xi]$, $[p_{m+1}, p_{m+2}, \ldots]$, like $[\xi^{m+1}, \xi^{m+2}, \ldots]$, is of codimension $m + 1$ in $K[\xi]$. From that we deduce that $\deg p_m = m$. Let $m' = \max\{\deg p_j : j = 1, \ldots, m - 1\}$. The required M of the proposition is $m + m'$.

REMARK 2.10. If (φ, ψ) in Proposition 2.9 is only one-to-one then (14), hence (15), is still valid. So we can conclude that there are integers $M \geqq 0$ and k_0 such that for all polynomials of degree exceeding M, $\deg p(\xi)$ and $\deg(\varphi(p(\xi)))$ differ by at most $|k_0|$. In fact, $k_0 = m - \deg p_m$, m as in (15). We shall use this form of the proposition in the proof of Lemma 3.3.

COROLLARY 2.11. *Let* $\mathbf{V}_L$ *be an indecomposable module. Then the group of automorphisms of* $\mathbf{V}_L$ *is isomorphic to the group of units of* K.

PROOF. Let (φ, ψ) be an automorphism of $\mathbf{V}_L$. Let M be the integer in Proposition 2.9. Then φ maps the finite-dimensional subspace $V' = [1, \xi, \ldots, \xi^M]$ into itself. Since K is algebraically closed, $\varphi|_{V'}$ has an eigenvalue α with corresponding eigenvector $v \neq 0$. Therefore the endormorphism $(\varphi, \psi) - \alpha\mathrm{I}$, I the identity map on $\mathbf{V}_L$, is not one-to-one. Since $\mathbf{V}_L$ is purely simple by [**15**, Theorem 1.8]; $(\varphi, \psi) = \alpha\mathrm{I}$ by Proposition 1.3 of [**14**].

Immediate from Corollary 2.11 is

COROLLARY 2.12. *Let* $\mathbf{V}_L$, $\mathbf{V}_{L_1}$ *be two indecomposable modules. Then there is at most one isomorphism from* $\mathbf{V}_L$ *onto* $\mathbf{V}_{L_1}$ *up to a scalar multiple.*

Corollary 2.12 was first proved in [**3**] for the case $n = 2$.

We conclude this section with an example showing that for any positive integer M there exist $\mathbf{V}_L$, $\mathbf{V}_{L_1}$ and an isomorphism from $\mathbf{V}_L$ onto $\mathbf{V}_{L_1}$ that does not preserve degree before M. If $\mathbf{V}_L$ is indecomposable, then Corollary 2.12 implies that no other isomorphism can do any better.

EXAMPLE 2.13. In this example, $n = 2$. So L is a single sequence. Let $L = (a_0, a_1, a_2, \ldots, a_{M-1}, a_M, \ldots)$ with $a_0 = 1, a_1 = \ldots = a_{M-1} = 0$, $a_M = 1$. a_{M+k} can be arbitrary for $k = 1, 2, \ldots$. $(L(\xi^k) = a_k)$. In particular we can start with a sequence L' that gives an indecomposable module $\mathbf{V}_{L'}$ and then perturb the first M entries as above. The new sequence still gives an indecomposable module, [**15**, Proposition 2.3]. Choose a basis $\{p_0, p_1, \ldots\}$ for $K[\xi]$ with $p_0 = \xi^M$ and

$$p_k = \xi p_{k-1} - a_{k-1} q, \tag{16}$$

where $q = \xi^{M+1} - 1$. q plays the role of $\psi(\omega)$ in (13). So, $p_0 = \xi^M$, $p_1 =$

1, $p_2 = \xi$, . . . , $p_M = \xi^{M-1}$, $p_{M+1} = \xi^M - \xi^{M+1} + 1$, etc. Let $L_1(p_k) = L(\xi^k)$. Since $\{p_0, p_1, \ldots\}$ is a basis of $K[\xi]$, L_1 extends to a linear functional on $K[\xi]$. So we get the module $\mathbf{V}_{L_1}$. Now define (φ, ψ): $\mathbf{V}_L \to \mathbf{V}_{L_1}$ as follows: $\varphi(\xi^k) = \psi(\xi^k) = p_k$ and $\psi(w_2) = q + w_2$. φ is a vector space automorphism of $K[\xi]$ and ψ is onto $W = \mathbf{V} \oplus [w_2]$. So it remains only to check that (φ, ψ) is a Kronecker module map, i.e., for the fixed basis (a, b) of K^2,

$$a\varphi(f) = \psi(af),$$
$$b\varphi(f) = \psi(bf) \quad \text{for all } f \in K[\xi].$$

It is enough to check this on $\{\xi^k\colon k = 0, 1, 2 \ldots\}$.

$$\begin{aligned} a\varphi(\xi^k) &= a \cdot p_k = p_k = \psi(a\xi^k), \\ b\varphi(\xi^k) &= bp_k = \xi p_k + L_1(p_k)w_2 \text{ in } \mathbf{V}_{L_1}, \\ b\xi^k &= \xi^{k+1} + L(\xi^k)w_2 \text{ in } V_L, \\ \psi(b\xi^k) &= \psi(\xi^{k+1}) + L(\xi^k)\psi(w_2) = p_{k+1} + L(\xi^k)(q + w_2). \end{aligned} \tag{17}$$

From (16), $p_{k+1} = \xi p_k - L(\xi^k)q$. So $\psi(b\xi^k) = \xi p_k - L(\xi^k)q + L(\xi^k)q + L(\xi^k)w_2 = \xi p_k + L(\xi^k)w_2$. Since $L(\xi^k) = L_1(p_k)$, we get from (17) that $\psi(b\xi^k) = b\varphi(\xi^k)$ as required. Since φ does not preserve degree till $M + 1$, we are done.

REMARK 2.14. An important advance towards classifying rank two submodules of $\mathbf{P}$ would be a technique for constructing modules isomorphic to a given $\mathbf{V}_L$ that did not depend on computing recursively with the components of L, see (16). If $\alpha = (\alpha_2, \ldots, \alpha_n)$, $\alpha_2\alpha_3 \ldots \alpha_n \neq 0$, then, with $L' = (\alpha_2 \ell_2, \ldots, \alpha_n \ell_n)$, $\mathbf{V}_{L'} \cong \mathbf{V}_L$, by Corollary 2.5.

3. Modules constructed from Liouville sequences. In this section the main results on indecomposable submodules of $\mathbf{P}$ are proved. We recall that the vector spaces remain as in (5). All we have to do is specify the sequence of linear functionals $L = (\ell_2, \ldots, \ell_n)$ on $K[\xi]$.

To that end let $A = (a_i)_{i=0}^{\infty}$ be the Liouville sequence (1, 0, 1, 0, 0, 1, 0, . . . , 0, 1, . . .) where the number of zeros between successive 1's is 1!, 2!, 3!, etc. Let $A_1 = (a_{k_1}, a_{k_2}, \ldots)$ be the subsequence of A consisting of the 1's in A, e.g., $k_1 = 0$, $k_2 = 2$, $k_3 = 5$. For $i = 2, \ldots, n$, let

$$A_{1i} = (a_{k_i}, a_{k_{i+n}}, a_{k_{i+2n}}, \ldots). \tag{18}$$

We shall now define $n - 1$ linear functionals $\ell_2, \ldots, \ell_n$ on $K[\xi]$ by

$$\begin{aligned} \ell_i(\xi^{k_{i+jn}}) &= 1, \; j = 0, 1, 2 \ldots, \\ \ell_i(\xi^m) &= 0 \text{ if } m \neq k_{i+jn}. \end{aligned} \tag{19}$$

What (19) says is that, for a fixed i, $\ell_i(\xi^k) = 0$ if the component a_k in the sequence A is 0 or if a_k is a term outside A_{1i}. So for any ξ^k, $\ell_i(\xi^k) = 1$ for at most one element i in $\{2, 3, \ldots, n\}$. Using $L = (\ell_2, \ldots, \ell_n)$ from (19) we construct a module $\mathbf{V}_L$ of rank n as in (6).

THEOREM 3.1. *The module* $\mathbf{V}_L$ *constructed from* (19) *is indecomposable.*

PROOF. By Theorem C (b), V_L is isomorphic to a submodule of $\mathbf{P}$. So, by Theorem D, it has the form $\mathbf{X}_1 \oplus \mathbf{V}'$ where $\mathbf{V}'$ is a unique infinite-dimensional indecomposable submodule of $\mathbf{V}_L$. We shall show that $\mathbf{X}_1 = 0$. Suppose $\mathbf{X}_1 \neq 0$. Then by Kronecker's theorem it is of type $III^{m_1'} \oplus \cdots \oplus III^{m_r'}$ for some positive integers $m_1', m_2', \ldots, m_r'$. Since the length of zeros in the Liouville sequence A keeps on increasing we can find some positive integer j such that the number of zeros m_1 preceding $a_{k_{2+jn}}$ and the number of zeros m_2 following it are respectively greater than $\max\{m_1', m_2', \ldots, m_r'\}$. So there is a homomorphism (φ, ψ) from $\mathbf{V}_{m_1+2} \subset \mathbf{P}$ (see (1)) to $\mathbf{V}_L$ with $\psi(1) = \xi_{k_{2+jn}-m_1}$ and $\psi(\xi^{m_1+1}) = \xi^{k_{2+jn}+1} + w_2$. Since V_{m_1+2} is of type III^{m_1+2} and $m_1 + 2 > \max\{m_1', m_2', \ldots, m_r'\}$, it follows from (3) that the submodule of $\mathbf{V}_L$ generated by $\xi^{k_{2+jn}-m_1}$ is contained in $\mathbf{V}'$. In particular, $\xi^{k_{2+jn}+1} + w_2 \in W'$. Similarly, by the choice of m_2, $\xi^{k_{2+jn}+1} \in W'$. Hence $w_2 \in W'$. Replacing 2 by $i = 3, \ldots, n$ in the above argument gives that $[w_2, \ldots, w_n] \subset W'$. Hence the torsion-closed finite-dimensional submodule $(0, [w_n, \ldots, w_n]) \subset \mathbf{V}'$. Since $\mathbf{V}'$ is infinite-dimensional, the remark after Theorem A gives that $\mathbf{V}' = \mathbf{V}_L$. Hence $\mathbf{X}_1 = 0$ and $\mathbf{V}_L$ is indecomposable.

In order to get many isomorphism classes of $\mathbf{V}_L$'s we shall now, as in [15], construct lots of Liouville sequences. Let F be the field $\mathbf{Z}/2\mathbf{Z}$. Choose a set of S of representatives for a basis of the F-vector space $\prod_{\aleph_0} F / \oplus_{\aleph_0} F$. The set S has the following properties.

LEMMA 3.2. (a) Card $(S) = 2^{\aleph_0}$. (b) *For* $s = (s_j)_{j=0}^{\infty}$ *in* S *the set* $\{j \in N: s_j = 1\}$ *is infinite.* (c) *For two distinct elements* s, t *in* S *the set* $\{j \in N: s_j \neq t_j\}$ *is infinite.*

A typical sequence in S may not have large enough lengths of zeros to qualify as a Liouville sequence. To introduce enough zeros we define a function, g, on nonnegative integers:

$$g(0) = 0 \tag{20}$$
$$g(r) = \sum_{i=1}^{r} i! + r.$$

(For later use we note that $(r + 1)! \geqq g(r)$ for all r.) If $s = (s_j)_{j=0}^{\infty}$ is in S, we construct a new sequence whose n^{th} term (counting from 0) is s_r if $n = g(r)$ and is 0 if $n \neq g(r)$, for any r. So $R_s = (s_0\ 0\ s_1\ 0\ 0\ s_2\ 0 \ldots)$

where the number of zeros between successive s_j's is 1!, 2!, 3!, etc. By Lemma 3.2 (a) there are $2^{\aleph_0}$ distinct elements in $T = \{R_s : s \in S\}$.

Let $R_s^1 = (a_{k_1}\, a_{k_2} \ldots)$ be the subsequence of R_s consisting of 1's. For each $i = 2, \ldots, n$, obtain R_s^{1i} from R_s^1 exactly as A_{1i} was obtained from A_1 in (18). Then, using these R_s^{1i}, $i = 2, \ldots, n$, we define $\ell_2, \ldots, \ell_n$ exactly as in (19). With these linear functionals we construct a module, $\mathbf{V}_s$, as in (6). Like $\mathbf{V}_L$ in Theorem 3.1, $\mathbf{V}_s$ is indecomposable. We shall now prove that if $s \neq s'$ then $\mathbf{V}_s$ is not isomorphic to $\mathbf{V}_{s'}$. This will follow from

LEMMA 3.3. *If s and s' are distinct elements of S, then* $\mathrm{Hom}(\mathbf{V}_s, \mathbf{V}_{s'}) = 0$.

PROOF. Since $\mathbf{V}_s$ and $\mathbf{V}_{s'}$ are indecomposable, hence purely simple by [14, Theorem 1.8], [13, Proposition 1.3] says that any nonzero homomorphism is monic. We shall suppose (φ, ψ) monic and then get a contradiction. (φ, ψ) monic implies the existence of integers $M \geqq 0$, k_0 such that if $k > M$, then deg ξ^k and deg $\varphi(\xi^k)$ differ by at most $|k_0|$, ($|k_0|$ = absolute value of k_0), by Remark 2.10. For an integer $r \geqq 4 + |k_0| + M$,

$$(21)\qquad \begin{aligned} (r+1)! &> g(r) + |k_0| + M \\ g(r) &> M. \end{aligned}$$

Suppose, for some r satisfying (21), we have that

$$(22)\qquad s_{r+1} = 0, \text{ but } s'_{r+1} = 1.$$

$s_{r+1} = 0$ implies the existence of a homomorphism (μ, ν) from $\mathbf{V}_k \subset \mathbf{P}$ to $\mathbf{V}_s$, $k = r! + (r+1)!$, with $\nu(1) = \xi^{g(r)+1}$. By the choice of r, $\psi(\xi^{g(r)+1}) = c_0 + c_1\xi + \ldots + c\xi^{g(r)+|k_0|+1}$. The presence of $s'_{r+1} = 1$ rules out the existence of a nonzero homomorphism (μ', ν') from $\mathbf{V}_k \subset \mathbf{P}$ to $\mathbf{V}_{s'}$ with $\nu'(1) = \psi(\xi^{g(r)+1})$. Hence $\psi(\xi^{g(r)+1}) = 0$.

If (22) is not satisfied, then, for all r satisfying (21), we have

$$(23)\qquad s_{r+1} = 0 \text{ implies that } s'_{r+1} = 0.$$

Since the components of s and s' are either 0 or 1, (23) is equivalent to

$$(24)\qquad s'_{r+1} = 1 \text{ implies that } s_{r+1} = 1.$$

Since $s \neq s'$, Lemma 3.2 (c) and (24) ensure the existence of a triple (r_1, r_2, r_3), $r_1 < r_2 < r_3$, such that each r_i satisfies (21) and $s'_{r_1} = 1$, $s'_{r_2} = 0$, and $s'_{r_3} = 1$ while $s_{r_1} = 1$, $s_{r_2} = 1$, and $s_{r_3} = 1$. Moreover, all entries (in R_s) between s_{r_1} and s_{r_2}, s_{r_2} and s_{r_3} are zero.

There is a homomorphism (μ, ν) from $\mathbf{V}_{k_1} \subset \mathbf{P}$ to $\mathbf{V}_s$ with $\nu(1) = \xi^{g(r_1)+1}$ and $\nu(\xi^{k_1-1}) = \xi^{g(r_2)+1} + w_{i_0}$ for some i_0 in $\{2, 3, \ldots, n\}$, $k_1 = (r_1 + 1)! + 2$. Composing this with (φ, ψ) gives a homomorphism (μ', ν') from $\mathbf{V}_{k_1}$ to $\mathbf{V}_{s'}$ with $\nu'(1) = \psi(\xi^{g(r_1)+1})$. If the latter is 0, then we conclude from (2) that $(\varphi, \psi) = 0$. So let $\psi(\xi^{g(r_1)+1}) = c_0 + c_1\xi + \cdots +$

$c\xi^{g(r_1)+|k_0|+1} \neq 0$. Since $s'_{r_1} = 1$, the existence of $(\mu', \nu',)$ forces $c_t = 0$ for $t \leqq g(r_1)$. Since $|k_0|$ is small relative to $(r_2 + 1)!$ we conclude that $\nu'(W_{k_1}) \subset W' = [\xi^{g(r_1)+1}, \xi^{g(r_1)+2}, \ldots, \xi^{g(r_3)-1}]$. In particular $\psi(\xi^{g(r_2)+1} + w_{i_0}) \in W'$. Now, $\psi(\xi^{g(r_2)+1})$ and $\xi^{g(r_2)+1}$ differ in degree by at most $|k_0|$. So the former is also in W'. So $\psi(w_{i_0}) \in W'$.

Now pick a positive integer j such that the number of zeros m_1 preceding $a_{k_{i_0}+jn}$ and the number of zeros m_2 following it are respectively greater than $g(r_3)$. The same argument as above puts $\psi(w_{i_0})$ in a subspace of $K[\xi]$ that intersects W' trivially. So $\psi(w_{i_0}) = 0$. Hence $(\varphi, \psi) = 0$ as required.

The idea necessary for the proof of the next lemma is already in the proof of the preceding one.

LEMMA 3.4. *For each* $s \in S$, $\mathrm{End}(\mathbf{V}_s) = K$.

PROOF. Let (φ, ψ): $\mathbf{V}_s \to \mathbf{V}_s$ be a nonzero homomorphism. Let r_1, r_2 satisfy (21) with $r_1 < r_2$, $s_{r_1} = s_{r_2} = 1$. With $k_1 = (r_1 + 1)! + 2$, there is a homomorphism (μ, ν) from $V_{k_1} \subset \mathbf{P}$ to $\mathbf{V}_s$, where $\nu(1) = \xi^{g(r_1)+1}$. Let $\psi(\xi^{g(r_1)+1}) = c_0 + c_1\xi + \ldots + c\xi^{g(r_1)+|k_0|+1}$. As in the last lemma, $c_t = 0$ for $t \leqq g(r_1)$. Since $s_{r_2} = 1$, the only way to have a nonzero homomorphism (μ', ν') from $\mathbf{V}_k$ to $\mathbf{V}_s$ with $\nu'(1) = \psi(\xi^{g(r_1)+1})$ is for c_t to be zero for $t \geqq g(r_1) + 2$. Also, $c = 0$. So ψ, hence φ, acts as multiplication by scalars on high powers of ξ. The scalars must be identical, otherwise $\psi(w_i)$ would not be well-defined; see the concluding argument in the proof of the last lemma. With the scalars identical on these high powers of ξ^k we conclude that $\psi([w_2, \ldots, w_n]) = [w_2, \ldots, w_n]$. Therefore, (φ, ψ) induces an endomorphism $(\bar{\varphi}, \bar{\psi})$ of $\mathbf{P}$ (see (5)). But, by Proposition B (c), $(\bar{\varphi}, \bar{\psi})$ is multiplication by a polynomial which must therefore be a constant. So (φ, ψ) is multiplication by a constant α on all of $K[\xi]$. From (2) and (6) we conclude that $\psi(w_i) = \alpha w_i$, $i = 2, 3, \ldots, n$.

Since the set $T = \{R_s: s \in S\}$ is uncountable we can now prove

THEOREM 3.5. *Let n be any positive integer and let $\mathfrak{c}$ be the cardinality of the continuum. Then* (a) *there are at least $\mathfrak{c}$ isomorphism classes of indecomposable extensions of a module of type* $(n - 1)$ III^1 *by* $\mathbf{P}$; (b) *there are at least $\mathfrak{c}$ isomorphism classes of indecomposable submodules of* $\mathbf{P}$ *of rank n.*

PROOF. (a) Each $\mathbf{V}_s$ is an extension of $(n - 1)$ III^1 by $\mathbf{P}$. So (a) follows from Lemma 3.2 (a), Theorem 3.1, and Lemma 3.3. (b) follows from (a) and Theorem C (b).

We now exhibit a set of rank 1 modules $\{\mathbf{V}_i: i \in I\}$, Card (I) = Card (K) and $\mathrm{Hom}(\mathbf{V}_i, \mathbf{V}_j) = 0$ if $i \neq j$. Write the field K as a disjoint union $K = \dot{\bigcup}_{i \in I} K_i$, Card (K_i) = Card (I) = Card (K). Let $V_i = [1/(\xi - \theta): \theta \in K_i]$ and $W_i = V_i \dotplus [1]$.

$\mathbf{V}_i$ is a module with $av_i = v_i$ and $bv_i = \xi v_i$ for all v_i in V_i. Also rank $\mathbf{V}_i = 1$. If $i \neq j$, then V_i and V_j have no poles in common. Hence Hom $(\mathbf{V}_i, \mathbf{V}_j) = 0$. Moreover, by [13, Theorem 1], dim Ext $(\mathbf{V}_i, \mathbf{V}_j) \geqq 2^{\mathrm{Card}(\aleph_0)}$ for any i, j in I. By Proposition B (b) no submodules of $\mathbf{P}$ of rank one can have such properties. It is quite a different story for higher ranks.

THEOREM 3.6. *Let n be a fixed positive integer > 1. The modules in $\{\mathbf{V}_s : s \in S\}$ are all of rank n and have the following properties*
(a) $\mathrm{Hom}(\mathbf{V}_{s_1}, \mathbf{V}_{s_2}) = 0$, *if* $s_1 \neq s_2$
(b) $\mathrm{End}(\mathbf{V}_s) = K$ *for each* $s \in S$.
(c) $\dim \mathrm{Ext}(\mathbf{V}_{s_1}, \mathbf{V}_{s_2}) \geqq \mathfrak{c}$ *for any* s_1, s_2 *in* S.

PROOF. For (a) and (b) see Lemma 3.3 and Lemma 3.4. (c) We have the exact sequence

$$0 \to (n-1)\mathrm{III}^1 \to \mathbf{V}_{s_1} \to \mathbf{P} \to 0. \tag{25}$$

The proof consists of comparing dimensions from several long exact sequences obtained from (25). First we have

$$\mathrm{Ext}(\mathbf{P}, \mathbf{P}) \to \mathrm{Ext}(\mathbf{V}_{s_1}, \mathbf{P}) \to \mathrm{Ext}((n-1)\mathrm{III}^1, \mathbf{P}).$$

$\mathrm{Ext}(\mathbf{P}, \mathbf{P}) = 0$ (see, e.g., the table in [7]) and $\mathrm{Ext}((n-1)\mathrm{III}^1, \mathbf{P}) = 0$ because a module of type III^1 is projective. Therefore

$$\mathrm{Ext}(\mathbf{V}_s, \mathbf{P}) = 0 \text{ for any } s \text{ in } S, \tag{26}$$

We also have the exact sequence

$$\begin{aligned} &0 \to \mathrm{Hom}(\mathbf{V}_{s_2}, (n-1)\mathrm{III}^1) \to \mathrm{Hom}(\mathbf{V}_{s_2}, \mathbf{V}_{s_1}) \\ &\to \mathrm{Hom}(\mathbf{V}_{s_2}, \mathbf{P}) \to \mathrm{Ext}(\mathbf{V}_{s_2}, (n-1)\mathrm{III}^1) \to \mathrm{Ext}(\mathbf{V}_{s_2}, \mathbf{V}_{s_1}) \\ &\to \mathrm{Ext}(\mathbf{V}_{s_2}, \mathbf{P}) \to 0. \end{aligned} \tag{27}$$

By the already-proved part (a) and part (b), $\dim \mathrm{Hom}(\mathbf{V}_{s_2}, \mathbf{V}_{s_1}) \leqq 1$. By (26), $\mathrm{Ext}(\mathbf{V}_{s_2}, \mathbf{P}) = 0$. Therefore, from (27) we obtain

$$\begin{aligned} &\dim \mathrm{Ext}(\mathbf{V}_{s_2}, (n-1)\mathrm{III}^1) \\ &= \dim \mathrm{Ext}(\mathbf{V}_{s_2}, \mathbf{V}_{s_1}) + \dim \mathrm{Hom}(\mathbf{V}_{s_2}, \mathbf{P}) \end{aligned} \tag{28}$$

provided all the cardinal numbers are infinite.

Let us now compute the dimensions of $\mathrm{Ext}(\mathbf{V}_{s_2}, (n-1)\mathrm{III}^1)$ and $\mathrm{Hom}(\mathbf{V}_{s_2}, \mathbf{P})$. From (25) again we obtain the long exact sequence

$$\begin{aligned} &\mathrm{Hom}((n-1)\mathrm{III}^1, (n-1)\mathrm{III}^1) \to \mathrm{Ext}(\mathbf{P}, (n-1)\mathrm{III}^1) \\ &\to \mathrm{Ext}(\mathbf{V}_{s_1}, (n-1)\mathrm{III}^1) \to \mathrm{Ext}((n-1)\mathrm{III}^1, (n-1)\mathrm{III}^1). \end{aligned}$$

$\mathrm{Ext}((n-1)\mathrm{III}^1, (n-1)\mathrm{III}^1) = 0$ and $\mathrm{Hom}((n-1)\mathrm{III}^1, (n-1)\mathrm{III}^1)$ is finite-dimensional. By Theorem 1 of [13], dim Ext $(\mathbf{P}, (n-1)\mathrm{III}^1) \geqq 2^{\aleph_0}$. Therefore for any $s \in S$, in particular s_2,

$$\dim \operatorname{Ext}(\mathbf{V}_{s_2}, (n-1)\mathrm{III}^1) \geqq 2^{\aleph_0}. \tag{29}$$

Finally from (25), with s_2 replacing s_1, we get

$$0 \to \operatorname{Hom}(\mathbf{P}, \mathbf{P}) \to \operatorname{Hom}(\mathbf{V}_{s_2}, \mathbf{P}) \to \operatorname{Hom}((n-1)\mathrm{III}^1, \mathbf{P}) \to \operatorname{Ext}(\mathbf{P}, \mathbf{P}).$$

As already remarked, $\operatorname{Ext}(\mathbf{P}, \mathbf{P}) = 0$. $\operatorname{Hom}(\mathbf{P}, \mathbf{P})$ is countable-dimensional by Proposition *B* (c), as is $\operatorname{Hom}((n-1)\mathrm{III}^1, \mathbf{P})$. Therefore, $\operatorname{Hom}(\mathbf{V}_{s_2}, \mathbf{P})$ is also countable-dimensional. Going back to (28) with all the information gives that

$$\dim \operatorname{Ext}(\mathbf{V}_{s_2}, \mathbf{V}_{s_1}) = \dim \operatorname{Ext}(\mathbf{V}_{s_2}, (n-1)\mathrm{III}^1) \geqq 2^{\aleph_0}$$

by (29).

REMARKS 3.7. (a) Theorem 3.6 is proved in [**16**, Theorem 6.9] for rank one modules over tame finite-dimensional hereditary algebras. In fact the example before Theorem 3.6 is merely the Kronecker module analogue of the modules in the proof of [**16**, Theorem 6.9]. Nevertheless, combining this example with [**10**, Corollary 2.3] gives a slight strengthening of Ringel's result in the rank 1 case.

(b) We conclude with the following observation on submodules of **P** of infinite rank. The module **V** in Lemma 1.3.2 of [**12**] is of infinite rank and has no direct summand of type III^m for any m. However every submodule of **V** of finite rank is finite-dimensional. Therefore any direct summand of **V** is of infinite rank. It can be shown that for any integer $k > 0$, **V** is a direct sum of 2^k submodules each of which has the same decomposition property. In the light of [**4**, Theorem B] it is still possible for **V** to have an indecomposable direct summand. Since we can embed **V** in **P** we can state: either **P** contains a superdecomposable submodule or an indecomposable submodule of infinite rank.

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DEPARTMENT OF MATHEMATICS, WAYNE STATE UNIVERSITY, DETROIT, MI 48202