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STURM-LIOUVILLE DIFFERENTIAL OPERATORS IN DIRECT SUM SPACES

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ABSTRACT. Sturm-Liouville boundary value problems on two intervals are studied in the setting of the direct sum of the L^2 spaces of functions defined on each of the separate intervals. The interplay between these two L^2 spaces is of critical importance. This study is partly motivated by the occurrence of S-L problems with coefficients that have a singularity in the interior of the basic interval. Such problems are not uncommon in the applied mathematics and mathematical physics literature.

1. Introduction. Sturm-Liouville (S-L) problems with coefficients which have a singularity in the interior of the basic interval under consideration have recently been studied in the Physics literature [**2**, **5**]. Here the interior singular point is viewed as a left end point of one interval and a right end point of another. In effect, then, we have two differential expressions: one for functions defined on interval I_1, the other for functions defined on I_2. For the general theory developed below whether the right end point of I_1 is the same as the left end point of I_2 is of no importance. Indeed the intervals I_1 and I_2 are to be taken as arbitrary; they may be disjoint, overlap, or even be identical and with the same or different differential expressions.

The purpose of this paper is to provide an operator theoretic framework for the study of two differential operators together: M_1 defined on an interval I_1 and M_2 defined on I_2. In particular we define a minimal and a maximal operator each associated with both expressions and characterize all self-adjoint extensions of the minimal operator in terms of "boundary conditions". These conditions involve both expressions on both intervals.

In the regular case they can be interpreted in terms of the values of the unknown function f and its quasi-derivative at all four end points. These conditions include the so called "interface" conditions obtained by other methods (see [**8**]). A special case of these interface conditions is the so called condition for a "point interaction of strength α". (see [**5**, pp. 20, 21]).

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In the singular case our conditions are given, just as in the one interval case, in terms of bilinear forms associated with the differential expressions.

A simple way of getting self-adjoint operators in the direct sum space $L^2(I_1) \oplus L^2(I_2)$ is by taking direct sums of self-adjoint operators from $L^2(I_1)$ and $L^2(I_2)$. In particular, if A_1 is a self-adjoint realization of M_1 in $L^2(I_1)$ and A_2 is a self-adjoint realization of M_2 in $L^2(I_2)$, then $A_1 \oplus A_2$ is a self-adjoint operator associated with both expressions M_1 and M_2. We stress that our development below yields all such self-adjoint operators and in general, many more. Thus some of the self-adjoint operators S generated by both differential expressions M_1 and M_1, obtained below, are such that $P_1\ S$ is not self-adjoint where P_1 is the natural projection "down" to $L^2(I_1)$. Some of these ideas and methods are also to be found in the important paper [5] by Gesztesy and Kirsch. We comment on some results of [5] in this paper at appropriate places in the text.

Notation and basic assumptions. Let $-\infty \leqq a_r < b_r \leqq \infty$; let I_r denote an interval with left end point a_r and right end point b_r, $r = 1, 2$. We use $[a$ to indicate a closed end point a and $(a$ to indicate an open end point a; use of the square bracket $[a$ implies that $a \in R$, the set of real numbers.

Consider Lebesgue measurable functions p_r, q_r, w_r from I_r into R satisfying the following basic conditions:

$$1/p_r,\ q_r,\ w_r \in L_{\text{loc}}(I_r), \quad w_r(t) > 0, \text{ a.e.}, \quad r = 1, 2, \tag{1.1}$$

which are taken to hold throughout this paper. Differential expressions M_1 and M_2 are defined by

$$M_r y = -(p_r y')' + q_r y \text{ on } I_r, \quad r = 1, 2. \tag{1.2}$$

Let $H_r = L^2_{w_r}(I_r)$ denote, for $r = 1, 2$, the set of (equivalence classes) of Lebesgue measurable functions f defined on I_r satisfying

$$\int_{I_r} |f(t)|^2\, w_r(t)dt < \infty, \quad r = 1, 2. \tag{1.3}$$

Let

$$D_r = \{f \in H_r | f,\ p_r f' \in AC_{\text{loc}}(I_r) \text{ and } w_r^{-1} M_r f \in H_r\}, \quad r = 1, 2.$$

Below we will denote $p_r f'$ by $f_r^{[1]}$ and call it the quasi-derivative of f. The subscript r will be omitted in most cases since it is clear from the context.

The operator T_r defined by

$$T_r f = w_r^{-1} M_r f, \quad f \in D_r \tag{1.4}$$

is called the maximal operator of M_r on I_r, $r = 1, 2$. It is well known (see [7, p. 68] that D_r is dense in H_r. Hence T_r has a uniquely defined adjoint. Let

$$T_{0,r} = T_r^* \text{ and } D_{0,r} = \text{domain of } T_r^*, \quad r = 1, 2.$$

The operator $T_{0,r}$ is called the minimal operator of M_r on I_r. Let

(1.5) $$[f, g]_r = f\overline{g^{[1]}} - f^{[1]}\bar{g}, \quad f, g \in D_r, \; r = 1, 2,$$

where $y^{[1]}$ denotes $p_1 y'$ when $r = 1$ and $p_2 y'$ when $r = 2$. Observe that Green's formula holds:

(1.6) $$\int_\alpha^\beta M_r[f]\bar{g} - \int_\alpha^\beta \overline{f\overline{M_r[g]}} = [f, g]_r(\beta) - [f, g]_r(\alpha),$$
$$f, g \in D_r, \quad \alpha, \beta \in I_r, \quad r = 1, 2.$$

For f, $g \in D_r$, the limits $\lim_{\beta \to b_r} [f, g]_r(\beta)$ and $\lim_{\alpha \to a_r} [f, g]_r(\alpha)$ exist and are infinite. These are denoted by $[f, g]_r(b_r)$ and $[f, g]_r(a_r)$, respectively, $r = 1, 2$.

Let

(1.7) $$H = H_1 \oplus H_2 = L^2_{w_1}(I_1) \oplus L^2_{w_2}(I_2).$$

Elements of H will be denoted by $\underset{\sim}{f} = \{f_1, f_2\}$ with $f_1 \in H_1$, $f_2 \in H_2$.

When $I_1 \cap I_2 = \phi$, the direct sum space $L^2_{w_1}(I_1) \oplus L^2_{w_2}(I_2)$ can be naturally identified with the space $L^2_w(I_1 \cup I_2)$, where $w = w_r$ on I_r, $r = 1, 2$. This remark is of particular significance when I_1 and I_2 are abutting intervals, i.e., when $I_1 \cup I_2$ may be taken as *a* single interval.

We now establish some further notation.

(1.8) $$D_0 = D_{0,1} \oplus D_{0,2}, \quad D = D_1 \oplus D_2,$$

(1.9) $$T_0 \underset{\sim}{f} = \{T_{0,1} f_1, T_{0,2} f_2\}, \quad f_1 \in D_{0,1}, f_2 \in D_{0,2},$$
$$\text{where } \underset{\sim}{f} = \{f_1, f_2\}.$$

Also,

(1.10) $$Tf = \{T_1 f_1, T_2 f_2\}, \quad \underset{\sim}{f} = \{f_1, f_2\}, f_1 \in D_1, f_2 \in D_2,$$

(1.11) $$[\underset{\sim}{f}, \underset{\sim}{g}] = [f_1, g_1]_1(b_1) - [f_1, g_1]_1(a_1)$$
$$+ [f_2, g_2]_2(b_2) - [f_2, g_2]_2(a_2), \quad \underset{\sim}{f}, \underset{\sim}{g} \in D,$$

(1.12) $$(f, g) = (f_1, g_1)_1 + (f_2, g_2)_2$$

where, as usual,

$$(y, z)_r = \int_{I_r} y(t)\bar{z}(t) w_r(t) dt, \quad r = 1, 2.$$

Note that T_0 is a closed symmetric operator in H.

2. Self-adjoint Sturm-Liouville operators in the one interval case. We summarize the characterization of all self-adjoint extensions of the minimal operator $T_{0,1}$ given in Naimark [7, v. II, Ch. V]. See also

Akhiezer and Glazman [**1**]. For definitions and proofs not given here the reader is referred to these two books.

The classification of the self-adjoint extensions of $T_{0,1}$ depends, in an essential way, on the deficiency index of $T_{0,1}$. We briefly recall the definition of this notion for abstract symmetric operators in a separable Hilbert space.

A linear operator A from a Hilbert space H into H is said to be symmetric if its domain $D(A)$ is dense in H and

$$(Af, g) = (f, Ag), \quad f, g \text{ in } D(A).$$

Any such operator has associated with it a pair (d^+, d^-), where each of d^+, d^- is a nonnegative integer or $+\infty$. These extended integers are called the deficiency indices of A and are defined as follows.

For λ in C, the set of complex numbers, let R_λ denote the range of $A - \bar{\lambda}E$, E being the identity operator. Let

$$N_\lambda = \{f \in D(A^*) \mid A^*f = \lambda f\} \tag{2.1}$$

and with

$$N^+ = N_i, \qquad N^- = N_{-i}, \qquad d^+ = \dim N^+,\ d^- = \dim N^-.$$

The spaces N^+, N^- are called the deficiency spaces of A, and the pair (d^+, d^-) are called the deficiency indices of A. For later use we recall the following two results.

For any $\lambda \in C \backslash R$, we have, from the general theory

$$D(A^*) = D(A) \dotplus N_\lambda \dotplus N_{\bar{\lambda}}, \tag{2.2}$$

where $D(A)$, N_λ, $N_{\bar{\lambda}}$ are linearly independent, and the sum is direct (which we indicate with the symbol $\dotplus$).

Any self-adjoint extension S of the symmetric operator A satisfies

$$A \subset S = S^* \subset A^*,$$

and hence is completely determined by specifying its domain $D(S)$, $D(A) \subset D(S) \subset D(A^*)$. This can be proved using formula (2.2).

THEOREM 2.1. *Suppose the symmetric operator A in a Hilbert space H has equal deficiency indices: $d_+ = d_- = d$ and $0 \leqq d < \infty$. Let $\phi_1, \ldots, \phi_d$ be an orthonormal basis of N^+, and let $\theta_1, \ldots, \theta_d$ denote an orthonormal basis of N^-.*

Let $U = (u_{jk})$, $j, k = 1, \ldots, d$ be a $d \times d$ matrix of complex numbers. Define

$$D_U = \{y + \sum_{j=1}^{d} c_j\phi_j + \sum_{j=1}^{d} (\sum_{k=1}^{d} u_{jk}c_k)\theta_j \mid y \in D_0,\ c_j \in \mathbf{C},\ j = 1, \ldots, d\}. \tag{2.3}$$

If U is an unitary matrix, then D_U is the domain of a self-adjoint extension *of A. Conversely, if D(S) is the domain of a self-adjoint extension S of A then* $D(S) = D_U$ *for some* $d \times d$ *unitary matrix U.*

PROOF. See Naimark [7; §14.8, p. 36].

THEOREM 2.2. *The operator* $T_{0,1}$ *is a closed symmetric operator from* H_1 *into* H_1 *and*

$$T_{0,1}^* = T_1, \qquad T_1^* = T_{0,1}. \tag{2.3}$$

PROOF. See [7; §17.4, pp. 68–69].

To relate the deficiency indices of $T_{0,1}$ to the equation

$$M_1 y = \lambda w_1 y \text{ on } I_1 = (a_1, b_1), \tag{2.4}$$

observe that

$$N_\lambda = \{y \in H_1 \mid T_{0,1}^* y = T_1 y = w_1^{-1} M_1 y = \lambda y\}.$$

From this we can conclude that N_1^+, N_1^- consist of the solutions of the equation

$$M_1 y = \lambda w y \tag{2.5}$$

which are in the space $L^2_{w_1}(I_1)$, for $\lambda = +i$ and $\lambda = -i$, respectively. Thus d_1^+, d_1^- are the number of linearly independent solutions of (2.5) which are in the space H_1 for $\lambda = +i$ and $\lambda = -i$, respectively. It is well known that $d_1^+ = d_1^-$ under conditions (1.1) (see [**4**; §9]). The common value is denoted by d_1. From the above discussion we see that there are only three possibilities: $d_1 = 0, 1, 2$.

The end point a_1 is regular if it is finite and

$$p_1^{-1}, q_1, w_1 \in L[a_1, a_1 + \varepsilon], \quad \text{for some } \varepsilon > 0. \tag{2.6}$$

Similarly, the end point b_1 is regular if it is finite and (2.6) holds with the interval $[a_1, a_1 + \varepsilon]$ replaced by $[b_1 - \varepsilon, b_1]$. As mentioned earlier, when we speak of M_1 on $[a_1, b_1)$, it is implied that a_1 is regular. Similarly for b_1.

We say that the end point a_1 or b_1 is singular if it is not regular. Thus a_1 is singular if $a_1 = -\infty$ or if one or more of the functions p_1^{-1}, q_1, w_1 are not integrable in any right neighborhood of a_1. An important distinction between the regular and singular cases is due to the fact that at a regular end point c all initial value problems of equation (2.4) with initial conditions $y(c) = c_1$, $y^{[1]}(c) = c_2$, $c_1, c_2 \in C$ have a unique solution. This is not true if c is singular (see [**3**]).

If one end point is regular, then $d = 1$ or $d = 2$, [**4**]. For historical resons the former is called the limit point case, LP for short, and the latter is known as the limit circle or LC case. Both the LP and LC cases refer to a given singular end point.

Some of the basic facts in the one interval case are summarized in

THEOREM 2.3.
(a) $D_{0,1} = \{f \in D_1 | [f, g](b_1) - [f, g](a_1) = 0, \text{ for all } g \in D_1\}$.
(b) *If* M_1 *is in the limit point case at an end point* c, *then* $[f, g](c) = 0$, *for all* $f, g \in D_1$, $c = a_1$ *or* $c = b_1$.
(c) *If an end point* c *is regular, then, for any solution* y, y *and* $y^{[1]}$ *are continuous at* c.
(d) *If* a_1 *and* b_1 *are both regular, then, for any* $\gamma_1, \gamma_2, \delta_1, \delta_2$ *in* C, *there exists a function* f *in* D_1 *such that* $f(a_1) = \gamma_1, f^{[1]}(a_1) = \gamma_2, f(b_1) = \delta_1$, $f^{[1]}{}_1(b) = \delta_2$.
(e) *If* a_1 *is regular and* b_1 *singular, then* a *function* f *from* D_1 *is in* $D_{0,1}$ *if and only if the following conditions are satisfied*:
(i) $f(a_1) = 0$ *and* $f^{[1]}(a_1) = 0$; *and*
(ii) $[f, g](b_1) = 0$, *for all* g *in* D_1.
The analogous results holds when a_1 *is singular and* b_1 *is regular.*

Since $T_{0,1}$ is symmetric, it follows that if S_1 is any self-adjoint extension of $T_{0,1}$ we have

$$T_{0,1} \subset S_1 = S_1^* \subset T_{0,1}^* = T_1. \tag{2.7}$$

Thus such a self-adjoint operator S_1 is completely determined by its domain $D(S_1)$. From (2.7) we have

$$D_{0,1} \subset D(S_1) \subset D_1. \tag{2.8}$$

To specify $D(S_1)$, we start with formula (2.2) applied to $T_{0,1}$:

$$D_1 = D_{0,1} \dot{+} N_1^+ \dot{+} N_1^-. \tag{2.9}$$

The next result describes those restrictions of D_1 which are self-adjoint domains.

THEOREM 2.4. *If the operator* S_1 *with domain* $D(S_1)$ *is a self-adjoint extension of the minimal operator* $T_{0,1}$ *with deficiency index* d, *then there exist* $\psi_1, \ldots, \psi_d$ *in* $D(S_1) \subset D_1$ *satisfying the following conditions*:
(i) $\psi_1, \ldots, \psi_d$ *are linearly dependent modulo* $D_{0,1}$;
(ii) $[\psi_j, \psi_k](b_1) - [\psi_j, \psi_k](a_1) = 0, j, k = 1, \ldots, d$; *and*
(iii) $D(S_1)$ *consists of the set of all* f *in* D_1 *satisfying*

$$[f, \psi_j](b_1) - [f, \psi_j](a_1) = 0, \quad j = 1, \ldots, d. \tag{2.10}$$

Conversely, given $\psi_1, \ldots, \psi_d$ *in* D_1 *which satisfy conditions* (i) *and* (ii), *the set* $D(S_1)$ *defined by* (iii) *is the domain of a self-adjoint extension of* $T_{0,1}$.

PROOF. See Naimark [7, Theorem 4, pp. 75–76].

REMARK. When $d = 0$ conditions (i), (ii), (iii) are vacuous. In this case it follows directly from formula (2.2) that the minimal operator $T_{0,1}$ is itself self-adjoint and has no proper self-adjoint extensions. When $d > 0$, conditions (iii) are "boundary conditions" and (i) and (ii) are the conditions on the "boundary conditions" which determine self-adjoint operators.

To illuminate conditions (ii) and (iii) we consider some special cases. These will be convenient to use for comparison purposes in §3 when we discuss the corresponding "two interval" cases.

Case 1. Both end points a_1 and b_1 are regular. From [7, Lemma 2, p. 63], given any α_1, α_2, β_1, β_2 in C, there exists a $\phi \in D_1$ such that $\phi(a_1) = \alpha_1$. $\phi^{[1]}(a_1) = \alpha_2$, $\phi(b_1) = \beta_1$, $\phi^{[1]}(b_1) = \beta_2$. Using this it is not difficult to show that (iii) is equivalent to the equations

$$(2.11)\qquad \begin{aligned} a_{11}f(a_1) + a_{12}f^{[1]}(a_1) + b_{11}f(b_1) + b_{12}f^{[1]}(b_1) = 0 \\ a_{12}f(a_1) + a_{22}f^{[1]}(a_1) + b_{21}f(b_1) + b_{22}f^{[1]}(b_1) = 0. \end{aligned}$$

Condition (i) is equivalent to the linear independence of the two equations (2. 11) and (ii) can be reduced to the following three conditions

$$(2.12)\qquad a_{11}\bar{a}_{22} - a_{12}\bar{a}_{21} = b_{11}\bar{b}_{22} - b_{12}\bar{b}_{21}$$

$$(2.13)\qquad a_{11}\bar{a}_{12} - \bar{a}_{11}a_{12} = b_{11}\bar{b}_{12} - \bar{b}_{11}b_{12}$$

$$(2.14)\qquad a_{21}\bar{a}_{22} - \bar{a}_{21}a_{22} = b_{21}\bar{b}_{22} - \bar{b}_{21}b_{22}.$$

REMARK. Note that (2. 13) and (2. 14) hold whenever the matrices A $= (a_{ij})$, $B = (b_{ij})$, $i, j = 1, 2$, are both real and (2. 12), in this case, reduces to

$$(2.15)\qquad \det A = \det B.$$

The special case $\det A = 0 = \det B$ of (2.15) contains the separated boundary conditions case:

$$(2.16)\qquad \begin{aligned} a_{11}f(a_1) + a_{12}f^{[1]}(a_1) = 0 \\ b_{21}f(b_1) + b_{22}f^{[1]}(b_1) = 0 \end{aligned}$$

Case 2. Assume a_1 is LP and b_1 is regular. In this case $d = 1$. Recall that, by part b of Theorem 2.3, $[f, g]\,(a_1) = 0$, for any $f, g \in D_1$. Hence (2. 10) reduces to

$$(2.\ 17)\qquad [f, \phi_j]\,(b_1) = 0, \quad j = 1.$$

Proceeding as in Case 1 above, (2.17) can be replaced by

$$(2.18)\qquad b_{11}f(b_1) + b_{12}f^{[1]}(b_1) = 0,$$

Condition (i) means that not both of b_{11}, b_{12} are zero and (ii) becomes

$$(2.\ 19)\qquad b_{11}\bar{b}_{12} - \bar{b}_{11}b_{12} = 0.$$

Since b_{11} can be taken to be real (2.19) just means that both b_{11}, b_{12} must be real.

Of course the case when a_1 is regular and b_1 is LP is entirely similar.

If one end point is regular and the other LP then only a condition at the regular end point is needed to determine a self-adjoint extension. If both end points are LP, then $d = 0$ and the minimal operator $T_{0,1}$ is itself self-adjoint with no proper self-adjoint extensions. At each LC or regular end point a condition is needed to determine a self-adjoint extension according to (2.10). In the case of a regular end point these conditions can be interpreted in terms of the values of the function f and its quasi-derivative $f^{[1]}$. This cannot be done at a singular end point c, say, since only in rather exceptional cases will the limits $f(t)$, $f^{[1]}(t)$ as $t \to c$ both exist and be finite for $f \in D_1$ or even f a solution of $M_1 f = w_1 f$. This holds even though, as we have seen, $[f, g](c) = \lim_{t\to c}[f, g](t)$ exists, for all $f, g \in D_1$. Thus $[f, g](c) = f(c)g^{[1]}(c) - f^{[1]}(c)g(c)$ is meaningless, in general, at an LC end point c.

3. The two interval case. In this section we characterize the self-adjoint extensions of the symmetric operator T_0 which was defined in §1 and illustrate (and hopefully illuminate) this characterization in a number of special cases. A critical role is played by an extension of Theorem 2.4 to the two interval case involving the extended sesquilinear form $[\underset{\sim}{f}, \underset{\sim}{g}]$ introduced in §1.

We have seen that T_0 is a closed symmetric operator in the direct sum Hilbert space $H = H_1 \oplus H_2$. We summarize a few additional properties of T_0 in the form of a lemma.

LEMMA 3. 1. *We have*

(a) $T_0^* = T_{0,1}^* \oplus T_{0,2}^* = T_1 \oplus T_2$. *In particular*, $D(T_0^*) = D = D_1 \oplus D_2$.

(b) $N^+ = N_1^+ \oplus N_2^+$, $N^- = N_1^- \oplus N_2^-$.

(c) *The deficiency indices* (d^+, d^-) of T_0 *are given by*:

$$d^+ = d_1^+ + d_2^+, \qquad d^- = d_1^- + d_2^-.$$

(d) $D = D_0 \dotplus N^+ \dotplus N^-$.

Proof. Part (a) follows immediately from the definition of the operator T_0 and from the general definition of an adjoint operator. The other parts are either direct consequences of part (a) or follow immediately from the definitions.

Since $d_j^+ = d_j^-$, $j = 1, 2$ we have $d^+ = d^- = d$. Also, the only possible values of d are 0, 1, 2, 3, and 4.

Applying Theorem 2.1 to the symmetric operator T_0 with equal and finite deficiency indices d we get

THEOREM 3. 2. *Let* $\phi_1, \ldots, \phi_d$ *be an orthonormal basis of* N^+ *and* $\theta_1, \ldots,$

θ_d be an orthonormal basis of N^-. For $U = (u_{jk}), j, k = 1, \ldots, d$, a *$d \times d$ matrix, define*

(3.1) $$D_U = \{y + \sum_{j=1}^{d} c_j\phi_j + \sum_{j=1}^{d} c_j \sum_{k=1}^{d} u_{kj}\theta_k \mid y \in D_0, c_j \in \mathbf{C}, j = 1, \ldots, d\}.$$

If U is unitary, then D_U is the domain of a self-adjoint extension of T_0. Conversely, if S is a self-adjoint entension of T_0 with domain $D(S)$, then there exists a $d \times d$ unitary matrix U such that $D(S) = D_U$.

REMARK. If U_j is a unitary matrix of dimension d_j, $j = 1, 2$, then the "block diagonal matrix"

$$U = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}$$

is unitary of dimension d. Such a U_1 determines a self-adjoint extension S_1 of $T_{0,1}$ in H_1, and U_2 determines a self-adjoint extension S_2 of $T_{0,2}$ in the space H_2. So some self-adjoint extensions S of T_0 in the space $H = H_1 + H_2$ are generated by pairs of self-adjoint extensions, one from H_1, the other from H_2. Note, however, that there are many self-adjoint extensions of T_0 in H which are not generated by a unitary matrix of such block diagonal form, i.e., which do not correspond to pairs of self-adjoint operators in this way.

The next result is fundamental to our work here. It is a straightforward extension of Theorem 4, pp. 75–76 in [7].

THEOREM 3.3. *If the operator S with domain $D(S)$ is a self-adjoint extension of T_0, then there exist $\underset{\sim}{\phi}_j \subset D(S) \subset D, j = 1, \ldots, d$ satisfying the following conditions*:

(3.2) (i) $\underset{\sim}{\phi}_1, \ldots, \underset{\sim}{\phi}_d$ *are linearly independent modulo D_0*;

(3.3) (ii) $[\underset{\sim}{\phi}_j, \underset{\sim}{\phi}_k] = 0, j, k = 1, \ldots, d$; *and*

(iii) *$D(S)$ consists precisely of those f in D which satisfy*

(3.4) $$[\underset{\sim}{f}, \underset{\sim}{\phi}_j] = 0, \quad j = 1, \ldots, d.$$

Conversely, given $\underset{\sim}{\phi}_j \in D$, $j = 1, \ldots, d$ which satisfy (i) *and* (ii), *the set $D(S)$ defined by* (iii) *is the domain of a self-adjoint extension of T_0.*

PROOF. The proof is entirely similar to that of Theorem 4, pp. 75–76 in Naimark [7] and therefore omitted.

REMARK 1. Let the vectors $\underset{\sim}{f} = \{f_1, f_2\}$ and $\underset{\sim}{g} = \{g_1, g_2\}$ be in D. From (1.11) we have

(3.5) $$[\underset{\sim}{f}, \underset{\sim}{g}] = [f_1, g_1]_1(b_1) - [f_1, g_1]_1(a_1) + [f_2, g_2]_2(b_2) - [f_2, g_2]_2(a_2).$$

Conditions (3.4) can be viewed as general "boundary conditions" for the equation

$$(3.6) \qquad -(py')' + qy = \lambda wy$$

on both intervals I_1 and I_2, with $p = p_1$ on I_1, or $p = p_2$ on I_2, etc. Conditions (i) and (ii) can be interpreted as conditions on the "boundary conditions" (iii) which determine self-adjoint domains.

These criteria depend on the coefficient functions, since the ψ_j's depend on D which depends on the coefficients. In some special cases this dependence can be eliminated as we will show below.

If S_1 is a self-adjoint extension of $T_{0,1}$ and S_2 is a self-adjoint extension of $T_{0,2}$, then

$$(3.7) \qquad S = S_1 \oplus S_2$$

is a self-adjoint extension of T_0. Are there others? Below we will refer to self-adjoint extensions of T_0 which do not arise as in (3.7) as "new".

The conditions (2.4) stated in terms of the form [,] depend on the sequilinear forms $[\ ,\ ]_1$ and $[\ ,\ ]_2$. From Theorem 2.3 part (b), it follows that, at any LP end point, the term in (3.4) which involves that end point is zero.

Case 1. $d = 0$. This can only occur when all four end points are LP. In this case $T^0 \mid$ is itself self-adjoint and has no proper self-adjoint extensions.

Case 2. $d = 1$. In this case we must have three LP end points and one LC or regular. There are no new self-adjoint extensions, i.e., all self-adjoint extensions of T_0 can be obtained by forming direct sums of self-adjoint extensions of $T_{0,1}$ and $T_{0,2}$. These are obtained as in the "one interval" case. In other words the conditions of Theorem 3.3 reduce to the known self-adjointness conditions on the interval with the LC or regular end point.

Case 3. $d = 2$. There must be two LP end points. Each of the other two may be LC or regular.

(i) If both LP end points are from the same interval, say I_1, then

$$S = T_{0,1} \oplus S_2,$$

where S_2 is a self-adjoint extension of $T_{0,2}$, generates all s.a. extensions of T_0. The conditions of Theorem 3.3 reduce to those for determining the extensions of $T_{0,2}$ on I_2.

(ii) If there is one LP and one LC or regular end point from each interval, then "mixing" can occur and we get new self-adjoint extensions of T_0. For the sake of definiteness assume that the end points a_1 and b_2 are LP and a_2, b_1 are LC or regular. The other cases are entirely similar.

For $\underset{\sim}{f}$, $\underset{\sim}{\psi}_j \in D$, with $\underset{\sim}{f} = \{f_1, f_2\}$, $\underset{\sim}{\psi}_j = \{\psi_{j1}, \psi_{j2}\}$, condition (3.4) reads

$$\begin{aligned} 0 = [\underset{\sim}{f}, \underset{\sim}{\psi}_j] = [f_1, \psi_{j1}]_1(b_1) - [f_1, \psi_{j1}]_1(a_1) \\ + [f_2, \psi_{j2}]_2(b_2) - [f_2, \psi_{j2}]_2(a_2), \quad j = 1, 2. \end{aligned} \tag{3.8}$$

By Theorem 2.3, part (b), the terms involving the LP end points a_2 and b_1 are zero so that (3.8) reduces to

$$[f_1, \psi_{j1}]_1(b_1) - [f_2, \psi_{j2}]_2(a_2) = 0, \quad j = 1, 2. \tag{3.9}$$

Similarly, in this case, (3.3) reduces to

$$[\psi_{j1}, \psi_{k1}]_1(b_1) = [\psi_{j2}, \psi_{k2}]_2(a_2) = 0, \quad j, k = 1, 2. \tag{3.10}$$

Conditions (3.9) and (3.10) depend on the coefficient functions p_r, q_r, w_r, $r = 1, 2$ since the functions ψ_{rs} depend on D which depends on these coefficients. In general this dependence cannot be removed except in certain special cases including those cases of regular end-points.

Suppose b_1 and a_2 are regular. Then (3.9) is equivalent to the two equations

$$\begin{aligned} a_{11}f_2(a_2) + a_{12}f_2^{[1]}(a_2) + b_{11}f_1(b_1) + b_{12}f_1^{[1]}(b_1) = 0 \\ a_{21}f_2(a_2) + a_{22}f_2^{[1]}(a_2) + b_{21}f_1(b_1) + b_{22}f_1^{[1]}(b_1) = 0, \end{aligned} \tag{3.11}$$

where a_{rs}, $b_{rs} \in C$, r, $s = 1, 2$. This follows from Theorem 2.3, part (d). Given a_{rs}, $b_{rs} \in \mathbf{C}$, choose $\psi_{12} \in D_2$ and $\psi_{11} \in D_1$ such that

$$\begin{aligned} \psi_{12}(a_2) = \bar{a}_{12}, \quad \psi_{12}^{[1]}(a_2) = -\bar{a}_{11}, \\ \psi_{11}(b_1) = -\bar{b}_{12}, \quad \psi_{11}^{[1]}(b_1) = \bar{b}_{11}. \end{aligned}$$

Then (3.9) with $j = 1$ becomes the first equation in (3.11). Similarly the values of $\psi_{21} \in D$, and $\psi_{22} \in D_2$ can be chosen so that (3.9) with $j = 2$ becomes the second equation in (3.11).

Now (3.10) becomes a set of conditions on the two equations in (3.11). There are three of these: one for $j = 1$, $k = 2$ (the case $j = 2$, $k = 1$ is equivalent to this one), one for $j = k = 1$ and one for $j = k = 2$. These are as follows:

$$a_{11}\bar{a}_{22} - a_{12}\bar{a}_{21} = b_{11}\bar{b}_{22} - b_{12}\bar{b}_{21} \tag{3.12}$$

$$a_{11}\bar{a}_{12} - a_{12}\bar{a}_{11} = b_{11}\bar{b}_{12} - b_{12}\bar{b}_{11} \tag{3.13}$$

$$a_{21}\bar{a}_{22} - \bar{a}_{21}a_{22} = a_{21}\bar{b}_{22} - \bar{b}_{21}b_{22}. \tag{3.14}$$

Condition (3.2) is equivalent to requiring the linear independence of the two equations in (3.11), i.e., the two four-vectors

(3.15) $(a_{11}, a_{12}, b_{11}, b_{12})$ and $(a_{21}, a_{22}, b_{21}, b_{22})$

are linearly independent.

In particular (3.15) implies that both equations in (3.11) must be present, i.e., not all four coefficients of either equation can be zero.

Next we list a number of examples to illustrate the type of boundary conditions that determine self-adjoint domains in this two interval case.

EXAMPLE 1.

(3.16) $$f_1(b_1) = f_2(a_2) \text{ and } f_1^{[1]}(b_1) = f_2^{[1]}(a_2).$$

This is the case $a_{11} = -1$, $a_{12} = 0$, $b_{11} = 1$, $b_{12} = 0$, $a_{21} = 0$, $a_{22} = -1$, $b_{21} = 0$, $b_{22} = 1$.

If $b_1 = a_2$ so that the two intervals are adjacent, then the vector $\underset{\sim}{f} = \{f_1, f_2\}$ can be identified with a function f which, together with its quasi-derivative $f^{[1]}$, is continuous on the interval (a_1, b_2), including the point $b_1 = a_2$.

Note that the self-adjoint operator determined by (3.16) when $b_1 = a_2$ is equivalent to the unique self-adjoint operator obtained in the one interval theory on (a_1, b_2), i.e., the minimal operator in $L_w^2(a_1, b_2)$; recall that in this Case 3 we have assumed the LP condition holds at a_1 and b_2. This equivalence is based on identifying the space $L_w^2(a_1, b_2)$ with the direct sum space $L_{w_2}^2(a_1, b_1) \oplus L_{w_2}^2(a_2, b_2)$. Here w is identified with the function defined on (a_1, b_2) whose restriction to (a_1, b_1) is w_1 and whose restriction to (a_2, b_2) is w_2.

It is interesting to observe that while the one interval theory in $L_w^2(a_1, b_2)$ yields only one self-adjoint operator, since a_1 and b_2 are both LP, the two interval theory on (a_1, b_1) and (a_2, b_2) yields infinitely many self-adjoint operators. However, only one of these self-adjoint extensions is unitarily equivalent to the unique self-adjoint operator obtained in the adjoint operators. However, only one of these self-adjoint extensions is single interval theory on (a_1, b_2), i.e., that described by the special choice of (3.11) given by (3.16).

EXAMPLE 2.

(3.17) $$f_1(b_1) = 0 = f_2(a_2).$$

This is the case $b_{11} = 1 = a_{21}$ and all other coefficients zero.

If $b_1 = a_2$ and the vector $\underset{\sim}{f} = \{f_1, f_2\}$ is identified with the function ft defined on (a_1, b_2) by $f(t) = f_1(t)$, for t in (a_1, b_1), and $f(t) = f_2(t)$, for t in (a_2, b_2), then (3.17) is simply the continuity requirement for f at $b_1 = a_2$. Of course $f^{[1]}$ might not be continuous at b_1.

EXAMPLE 3.

(3.18) $$f_1^{[1]}(b_1) = 0 = f_2^{[1]}(a_2).$$

Just as in Example 2, the vector $\underset{\sim}{f}$ can be identified with a function f defined on (a_1, b_2), if $b_1 = a_2$. Then (3.18) requires $f^{[1]}$ but not f to be continuous at b_1.

EXAMPLE 4. Let $0 = b_{11} = b_{12} = a_{21} = a_{22}$. Then equations (3.11) become separated

$$(3.19)\qquad \begin{aligned} a_{11}f_2(a_2) + a_{12}f_2^{[1]}(a_2) &= 0 \\ b_{21}f_1(b_1) + b_{22}f_1^{[1]}(b_1) &= 0. \end{aligned}$$

Observe that the self-adjointness condition (3.12) is automatically satisfied since both sides of (3.12) are zero and (3.13) and (3.14) reduce to

$$(3.20)\qquad a_{11}\bar{a}_{12} - \bar{a}_{11}a_{12} = 0$$

$$(3.21)\qquad b_{21}\bar{b}_{22} - \bar{b}_{21}b_{22} = 0,$$

respectively.

Since b_2 is LP and the first equation in (3.19) is a separated boundary condition at the regular end point a_2, this equation with condition (3.20) determines a self-adjoint operator S_2 in H_2. Similarly, the second equation in (3.19) with (3.21) determines a self-adjoint operator S_1 in H_1. The operator of Example 4 is simply $S_1 \oplus S_2$ in H.

EXAMPLE 5. Choose $a_{11} = 1$, $a_{12} = 0$, $b_{11} = -1$, $b_{12} = 0$. Then the first equation in (3.11) becomes

$$(3.22)\qquad f_2(a_2) = f_1(b_1).$$

When (3.22) holds, then, under conditions (3.12), (3.13), and (3.14), the second equation in (3.11) reduces to

$$(3.23)\qquad f_1^{[1]}(b_1) - f_2^{[1]}(a_2) = cf_1(b_1), \quad c \text{ real}.$$

To see this, note that (3.12) reduces to $a_{22} = -b_{22}$. Thus we get

$$(3.24)\qquad a_{22}f_2^{[1]}(a_2) - a_{22}f_1^{[1]}(b_1) = -a_{21}f_2(a_2) - b_{21}f_1(b_1).$$

If $a_{22} = 0$, then $b_{22} = 0$ and $a_{21} = -b_{21}$. But this would make equations (3.19) linearly dependent. Hence $a_{22} \neq 0$. Diving (3.24) by a_{22} we get

$$(3.25)\qquad f_1^{[1]}(b_1) - f_2^{[1]}(a_2) = cf_1(b_1), \quad c = (a_{21} + b_{21})/a_{22}.$$

Now (3.13) is equivalent to $c = \bar{c}$, giving (3.23).

In case $b_1 = a_2$, (3.25) can be interpreted as an interface condition. We identify the vector $\underset{\sim}{f} = \{f_1, f_2\}$ with the function f defined on (a_1, b_2) whose restriction to (a_1, b_1) is f_1 and whose restriction to (a_2, b_2) is f_2. Then

(3.26) $$f_1^{[1]}(b_1) = \lim_{t\to b_1^-} p(t)f'(t), \quad f_2^{[1]}(a_2) = \lim_{t\to a_2^-} p(t)f'(t),$$

and equation (3.22) can be interpreted as

(3.27) $$\lim_{t\to b_1^-} f(t) = f_1(b_1) = f_2(a_2) = \lim_{t\to a_2^+} f(t).$$

With these interpretations, equations (3.22) and (3.23) are well known self-adjoint interface conditions [5, 8]. In [5], (3.25), and (3.22) with the interpretations (3.26), (3.27) are referred to as a point interaction of strength c.

In (3.25), $c = 0$ is allowed, but then Example 5 reduces to Example 1.

EXAMPLE 6. $$f_2(a_2) = -f_1(b_1)$$

$$f_1^{[1]}(b_1) + f_2^{[1]}(a_2) = c\, f_1(b_1), \quad c \text{ real}.$$

To verify this, take $a_{11} = 1$, $a_{12} = 0$, $b_{11} = 1$, $b_{12} = 0$, $a_{22} = 1$, $b_{22} = 1$. Then the first equation in (3.11) becomes $f_2(a_2) + f_1(b_1) = 0$ and the second reduces to

$$f_2^{[1]}(a_2) + f_1^{[1]}(b_1) = -a_{21}f_2(a_2) - b_{21}f_1(b_1).$$

Conditions (3.12), (3.13) hold for arbitrary a_{21}, b_{21}, and (3.14) gives

$$a_{21} - \bar{a}_{21} = b_{21} - \bar{b}_{21} \text{ or } a_{21} - b_{21} = \bar{a}_{21} - \bar{b}_{21}.$$

Now, substituting the first condition into the second, we get

$$f_2^{[1]}(a_2) + f_1^{[1]}(b_1) = (a_{21} - b_{21})f_1(b_1) = cf_1(b_1)$$

with $\bar{c} = c$, i.e., c real.

More generally, we get

EXAMPLE 7. $f_2(a_2) = rf_1(b_1)$, where r is real, $r \neq 0$, and $f_2^{[1]}(a_2) - r^{-1}f_1^{[1]}(b_1) = cf_1(b_1)$, where c is real.

Choosing $a_{11} = 1$, $a_{12} = 0$, $b_{11} = -r$, $b_{12} = 0$, we get the first equation. The choice $a_{22} = 1$, $b_{22} = -r^{-1}$ gives the second equation with $c = -(ra_{21} + b_{21})$. Conditions (3.12) - (3.14) are satisfied if c is real. Clearly any real number c can be realized with an appropriate choice of a_{21} and b_{21}.

Case 4. $d = 3$. Here we must have either $d_1 = 2$, $d_2 = 1$ or $d_1 = 1$, $d_2 = 2$. We assume the former holds. The latter is entirely similar. Thus, we must have either a_1, b_1, a_2 are LC or regular and b_2 is LP, or a_1, b_1, b_2 are LC or regular and a_2 is LP. Again, for definiteness, we assume the former holds. In this case only the term involving a_2 (which is LP) in (3.4), equivalently (3.5), is zero for all $\underset{\sim}{f}$ in D. Using the notation from Case 3 the "boundary condition" (3.4) becomes

$$(3.28)\qquad 0 = [f, \phi_j] = [f_1, \phi_{j1}]_1(b_1) - [f_1, \phi_{j1}]_1(a_1) - [f_2, \phi_{j2}]_2(a_2), \quad j = 1, 2, 3,$$

and the "conditions on the boundary condition" (3.3) become

$$(3.29)\qquad [\phi_{j1}, \phi_{k1}]_1(b_1) - [\phi_{j1}, \phi_{k1}]_1(a_1) - [\phi_{j2}, \phi_{k2}]_2(a_2) = 0, \quad j, k = 1, 2, 3.$$

Since the conditions (3.28) involve both intervals (a_1, b_2), (a_1, b_2), there is "mixing" and we obtain self-adjoint operators which are not direct sums of self-adjoint operators from the one interval case (as well as all those which are).

If all three end points a_1, b_1, a_2 are LC, then condition (3.28) cannot be simplified except for some special cases. But, just as in Case 3, if one or more of a_1, b_1, a_2 is regular, then the term in (3.28) and (3.29) which involves that point can be simplified.

Case 3*a*. All three points a_1, b_1, a_2 are regular. In this case the values of the ϕ_j's at each of these three end points can be determined arbitrarily. So, proceeding as we did in Case 3, we can show that each of the conditions (3.28) is equivalent to one of the three equations

$$(3.30)\qquad a_{11}f_1(a_1) + a_{12}f_1^{[1]}(a_1) + b_{11}f_1(b_1) + b_{12}f_1^{[1]}(b_1) + c_{11}f_2(a_2) + c_{12}f_2^{[1]}(a_2) = 0,$$

$$(3.31)\qquad a_{21}f_1(a_1) + a_{22}f_1^{[1]}(a_1) + b_{21}f_1(b_1) + b_{22}f_1^{[1]}(b_1) + c_{21}f_2(a_2) + c_{22}f_2^{[1]}(a_2) = 0,$$

$$(3.22)\qquad a_{31}f_1(a_1) + a_{32}f_1^{[1]}(a_1) + b_{31}f_1(b_1) + b_{32}f_1^{[1]}(b_1) + c_{31}f_2(a_2) + c_{32}f_2^{[1]}(a_2) = 0.$$

The linear independence condition (i) of Theorem 3.3 is equivalent to the linear independence of these three equations.

Two special cases of equations (3.30), (3.31), (3.32) are mentioned. In the first the boundary conditions from the interval I_1 are not linked with those of interval I_2. The second is a special case of the first in which the boundary conditions at a_1 and b_1 are separated.

Case 3a(i). The intervals I_1 and I_2 are decoupled. This can be achieved by choosing $c_{11} = c_{12} = c_{21} = c_{22} = a_{31} = a_{32} = b_{31} = b_{32} = 0$. (One can take $\underline{\phi}_j$'s of the form $\underline{\phi}_1 = \{\phi_{11}, 0\}$, $\underline{\phi}_2 = \{\phi_{21}, 0\}$, $\underline{\phi}_3 = \{0, \phi_{31}\}$.) The three boundary condition equations now reduce to

$$(3.33)\qquad a_{11}f_1(a_1) + a_{12}f_1^{[1]}(a_1) + b_{11}f_1(b_1) + b_{12}f_1^{[1]}(b_1) = 0,$$

$$(3.34)\qquad a_{21}f_1(a_1) + a_{22}f_2^{[1]}(a_1) + b_{21}f_1(b_1) + b_{22}f_1^{[1]}(b_1) = 0,$$

$$c_{31}f_2(a_2) + c_{32}f_2^{[1]}(a_2) = 0. \tag{3.35}$$

Equation (3.35) is independent of (3.33) and (3.34), but these two are coupled.

The self-adjoint conditions (3.3) now reduce to the known one interval two point self-adjoint boundary conditions on (3.33) and (3.34) and the usual one interval one end point self-adjointness condition on (3.35). See Naimark [5, pp. 78, 79]. We state these for the convenience of the reader but omit the straightforward but tedious calculations showing their equivalence with (3.3):

$$a_{11}\bar{a}_{22} - a_{12}\bar{a}_{21} = b_{11}\bar{b}_{22} - b_{12}\bar{b}_{21}, \tag{3.36}$$

$$a_{11}\bar{a}_{12} - a_{12}\bar{a}_{11} = b_{11}\bar{b}_{12} - b_{12}\bar{b}_{11}, \tag{3.37}$$

$$a_{21}\bar{a}_{22} - \bar{a}_{21}a_{22} = b_{21}\bar{b}_{22} - \bar{b}_{21}b_{22}, \tag{3.38}$$

$$c_{31}\bar{c}_{32} - \bar{c}_{31}c_{32} = 0. \tag{3.39}$$

Of course, in this case, the boundary conditions (3.33), (3.34) satisfying the self-adjointness criteria (3.36), (3.37), (3.38) determine a self-adjoint extension S_1 of $T_{0,1}$ and the "boundary condition" (3.35) with coefficients satisfying (3.39) determines a self-adjoint extension S_2 of $T_{0,2}$. The self-adjoint operator determined by (3.33) – (3.35) satisfying (3.36) – (3.39) is simply the operator $S_1 \oplus S_2$ in $H = H_1 + H_2$.

The particular case of this special case mentioned above is obtained by decoupling the equations (3.34) and (3.35). This can be done without violating the linear independence condition by choosing $b_{11} = b_{12} = a_{21} = a_{22} = 0$. Now each of the three equations (3.30), (3.31), (3.32) involves only one end point:

$$a_{11}f_1(a_1) + a_{12}f_1^{[1]}(a_1) = 0, \tag{3.40}$$

$$b_{21}f_1(b_1) + b_{22}f_1^{[1]}(b_1) = 0, \tag{3.41}$$

$$c_{31}f_2(a_2) + c_{32}f_2^{[1]}(a_2) = 0. \tag{3.42}$$

The self-adjointness conditions are

$$a_{11}\bar{a}_{12} - \bar{a}_{11}a_{12} = 0, \tag{3.43}$$

$$b_{21}\bar{b}_{22} - \bar{b}_{21}b_{22} = 0, \tag{3.44}$$

$$c_{31}\bar{c}_{32} - \bar{c}_{31}c_{32} = 0. \tag{3.45}$$

Case 3a(ii). Although a_1 and a_2 are end points of different intervals, they can be coupled in the same way as a_1 and b_1 were coupled in (3.33), (3.34) and b_1 can be decoupled. Choose $b_{11} = b_{12} = b_{21} = b_{22} = a_{31} = a_{32} = c_{31} = c_{32} = 0$ so that (3.30) to (3.32) become

(3.46) $$a_{11}f_1(a_1) + a_{12}f_1^{[1]}(a_1) + c_{11}f_2(a_2) + c_{12}f_2^{[1]}(a_2) = 0,$$

(3.47) $$a_{21}f_1(a_1) + a_{22}f_1^{[1]}(a_1) + c_{21}f_2(a_2) + c_{22}f_2^{[1]}(a_2) = 0,$$

(3.48) $$b_{31}f_1(b_1) + b_{32}f_1^{[1]}(b_1) = 0.$$

The self-adjointness conditions now are

(3.49) $$a_{11}\bar{a}_{22} - a_{12}\bar{a}_{21} = c_{11}\bar{c}_{22} - c_{12}\bar{c}_{21},$$

(3.50) $$a_{11}\bar{a}_{12} - a_{12}\bar{a}_{11} = c_{11}\bar{c}_{12} - c_{12}\bar{c}_{11},$$

(3.51) $$a_{21}\bar{a}_{22} - a_{21}\bar{a}_{22} = c_{21}\bar{c}_{22} - c_{21}\bar{c}_{22},$$

(3.52) $$b_{31}\bar{b}_{32} - b_{31}\bar{b}_{32} = 0.$$

In addition, the three equations (3.46), (3.47), (3.48) must be linearly independent, i.e., the three vectors $(a_{11}, a_{12}, 0, 0, c_{11}, c_{12})$, $(a_{21}, a_{22}, 0, 0, c_{21}, c_{22})$, $(0, 0, b_{31}, b_{32}, 0, 0)$ must be linearly independent.

We now return to the general Case 3a where the boundary conditions are given by equations (3.30), (3.31), (3.32). These boundary conditions determine a self-adjoint extension of the minimal operator T_0 in the space H if and only if the following two criteria are satisfied.

(i) The three equations are linearly independent, i.e., the three six dimensional vectors are linearly independent:

$$(a_{j1}, a_{j2}, b_{j1}, b_{j2}, c_{j1}, c_{j2}),\ j = 1, 2, 3.$$

(ii) The coefficients a_{jk}, b_{jk}, c_{jk} satisfy the following set of conditions:

(3.53) $$b_{11}\bar{b}_{22} - b_{12}\bar{b}_{21} = a_{11}\bar{a}_{22} - a_{12}\bar{a}_{21} + c_{11}\bar{c}_{22} - c_{12}\bar{c}_{21}$$

(3.54) $$b_{11}\bar{b}_{32} - b_{12}\bar{b}_{31} = a_{11}\bar{a}_{32} - a_{12}\bar{a}_{31} + c_{11}\bar{c}_{32} - c_{12}\bar{c}_{31}$$

(3.55) $$b_{21}\bar{b}_{32} - b_{22}\bar{b}_{31} = a_{21}\bar{a}_{32} - a_{22}\bar{a}_{31} + c_{21}\bar{c}_{32} - c_{22}\bar{c}_{31}$$

(3.56) $$b_{11}\bar{b}_{12} - \bar{b}_{11}b_{12} = a_{11}\bar{a}_{12} - \bar{a}_{11}a_{12} + c_{11}\bar{c}_{12} - \bar{c}_{11}c_{12}$$

(3.57) $$b_{21}\bar{b}_{22} - \bar{b}_{21}b_{22} = a_{21}\bar{a}_{22} - \bar{a}_{21}a_{22} + c_{21}\bar{c}_{22} - \bar{c}_{21}c_{22}$$

(3.58) $$b_{31}\bar{b}_{32} - \bar{b}_{31}b_{32} = a_{31}\bar{a}_{32} - \bar{a}_{31}a_{32} + c_{31}\bar{c}_{32} - \bar{c}_{31}c_{32}.$$

The verification of these conditions is quite similar to that of Case 3. We omit the straightforward but tedious details but do point out that, since $[\underset{\sim}{\psi}_j, \underset{\sim}{\psi}_k] = 0$ if and only if $[\underset{\sim}{\psi}_k, \underset{\sim}{\psi}_j] = 0$, (3.3) yields six conditions: $[\underset{\sim}{\psi}_1, \underset{\sim}{\psi}_2] = 0$, $[\underset{\sim}{\psi}_1, \underset{\sim}{\psi}_3] = 0$, $[\underset{\sim}{\psi}_2, \underset{\sim}{\psi}_3] = 0$ and $[\underset{\sim}{\psi}_j, \underset{\sim}{\psi}_j] = 0$, $j = 1, 2, 3$. The first of these is equivalent to (3.53), the second to (3.54), etc.

Case 5. $d = 4$. This means that $d_1 = 2 = d_2$. Therefore each one of the four end points a_1, b_1, a_2, b_2 is either LC or regular. With the notation $\underset{\sim}{f} = \{f_1, f_2\}$, $\underset{\sim}{\psi}_j = \{\psi_{j1}, \psi_{j2}\}$, conditions (3.4) of Theorem 3.3 take the form

$$(3.59)\quad [f_1, \psi_{j1}]_1(b_1) = [f_1, \psi_{j1}]_1(a_1) + [f_2, \psi_{j2}]_2(b_2) - [f_2, \psi_{j2}]_2(a_2) = 0, \quad j = 1, 2, 3, 4.$$

At a singular end point these conditions can be simplied only in special cases.

Case 5(a). All four end points are regular. Just as before, equations (3.59) can be written as

$$(3.60)\quad \begin{aligned} &a_{j1}f_1(a_1) + a_{j2}f_1^{[1]}(a_1) + b_{j1}f_1(b_1) + b_{j2}f_1^{[1]}(b_1) + c_{j1}f_2(a_2) \\ &+ c_{j2}f_2^{[1]}(a_2) + d_{j1}f_2(b_2) + d_{j2}f_2^{[1]}(b_2) = 0, \quad j = 1,2,3,4. \end{aligned}$$

In order for these boundary condition equations (3.60) to determine a self-adjoint extension of T_0 they must be linearly independent and satisfy the following set of 10 conditions:

$$(3.61)\quad a_{j1}\bar{a}_{k2} - a_{j2}\bar{a}_{k1} + c_{j1}\bar{c}_{k2} - c_{j2}\bar{c}_{k1} = b_{j1}\bar{b}_{k2} - b_{j2}\bar{b}_{k1} + b_{j1}\bar{d}_{k2} - b_{j2}\bar{d}_{k1}, \quad j, k = 1, 2, 3, 4.$$

There are only 10 of these conditions since they are symmetric in j and k.

There are many interesting special cases.

Case 5a(i). Any one of the four end point conditions can be "separated out", e.g., to get separated conditions at b_2, choose $0 = d_{j1} = d_{j2}$, $j = 1, 2, 3$ and $0 = a_{41} = a_{42} = b_{41} = b_{42} = c_{41} = c_{42}$. Then equation $j = 4$ in (3.61) becomes

$$d_{41}f_2(b_2) + d_{42}f_2^{[1]}(b_2) = 0$$

and the other three reduce to (3.30), (3.31), (3.32). Thus besides the linear independence condition (i), the self-adjointness conditions are

$$d_{41}\bar{d}_{42} - \bar{d}_{41}d_{42} = 0$$

and (3.53) through (3.58).

The procedure for getting separated conditions at any one of the other end points is entirely similar and so we omit the details.

Case 5a(ii). Separated conditions can be obtained at any two of the four end points. As always, the four equations (3.60) must be linearly independent. To get separated conditions at, say a_1 and b_2, we consider the special case of (3.60) given by (3.11) and

$$(3.62)\quad c_{31}f_1(a_1) + c_{32}f_1^{[1]}(a_1) = 0,$$

$$(3.63)\quad d_{41}f_2(b_2) + d_{42}f_2^{[1]}(b_2) = 0,$$

The self-adjointness conditions now are given by (3.12), (3.13), and (3.14), in addition to

$$c_{31}\bar{c}_{32} - \bar{c}_{31}c_{32} = 0 = d_{41}\bar{d}_{42} - \bar{d}_{41}d_{42}.$$

Similarly, we can obtain separated conditions at any two other end points. The conditions, given that an appropriate change in notation has been made, are the same. Notice that it makes no difference whether or not the two end points with the separated conditions are from the same interval.

Next we simply list a number of self-adjoint boundary conditions, i.e., conditions which determine a self-adjoint extension of T_0 in H. The verification is left to the reader.

I. $f_1(a_1) = f_1(b_1), f_1^{[1]}(a_1) - f_1^{[1]}(b_1) = c_1 f_1(a_1), \quad c_1$ real

$f_2(a_2) = f_2(b_2), f_2^{[1]}(a_2) - f_2^{[1]}(b_2) = c_2 f_2(a_2), \quad c_2$ real

II. $f_1(a_1) = f_2(a_2), f_1^{[1]}(a_1) - f_2^{[1]}(a_2) = c_3 f_1(a_1), \quad c_3$ real

$f_1(b_1) = f_2(b_2), f_1^{[1]}(b_1) - f_2^{[1]}(b_2) = c_4 f_1(b_1), \quad c_4$ real

Note that both I and II include the case $c_j = 0$ so that both the functions and the quasi-derivatives match up.

The four equations

III.
$$f_1(a_1) + f_1(b_1) + f_2(a_2) + f_2(b_2) = 0$$
$$f_1(a_1) + f_1(b_1) - f_2(a_2) - f_2(b_2) = 0$$
$$f_1^{[1]}(a_1) + f_1^{[1]}(b_1) + f_2^{[1]}(a_2) + f_2^{[1]}(b_2)$$
$$= a_{31}f_1(a_1) + b_{31}f_1(b_1) + c_{31}f_2(a_2) + d_{31}f_2(b_2)$$
$$f_1^{[1]}(a_1) + f_1^{[1[}(b_1) + f_2^{[1]}(a_2) + f_2^{[1]}(b_2)$$
$$= a_{41}f_1(a_1) + b_{41}f_1(b_1) + c_{41}f_2(a_2) + d_{41}f_2(b_2),$$

with any real coefficients a_{j1}, b_{j1}, c_{j1}, $d_{j1}, j = 3, 4$, determine the domain of a self-adjoint extension in H provided that

1. the four equations are linearly independent, and
2. $a_{31} - a_{41} + c_{31} - c_{41} = b_{31} - b_{41} + d_{31} - d_{41}$.

Particular examples of coefficients satisfying conditions 1 and 2 are:

(i) $a_{31} = b_{31} = c_{31} = d_{31} = 1, \quad a_{41} = 0 = b_{41}, \quad c_{41} = 1 = d_{41}.$

In this case the third and fourth equations of III become, respectively,

$$f_1^{[1]}(a_1) + f_1^{[1]}(b_1) + f_2^{[1]}(a_2) + f_2^{[1]}(b_2) = 0$$

and

$$f_2(a_2) = -f_2(b_2) \text{ or } f_1(a_1) = -f_1(b_1).$$

Here we have used the first and second equations of III.

In the examples above we have emphasized self-adjoint boundary conditions at regular end points. In a future paper we plan to study the form of the singular LC boundary conditions including some special ones of interest in Mathematical Physics. We also plan to take up the general higher order case as well as the cases of finitely many or countably infinitely many intervals.

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