ORTHOGONAL POLYNOMIALS
AND
SINGULAR STURM-LIOUVILLE SYSTEMS, I

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CONTENTS

ABSTRACT. The classic orthogonal polynomials which satisfy
a differential equation of second order, the Jacobi (which in-
cludes the Legendre and Tchebycheff), the Laguerre, the Her-
mite and the Bessel polynomials, are examined closely. Their
weight functions are found distributionally, and classically
when possible. The boundary value problems they come from
are rigorously developed as examples of the singular Sturm-
Liouville problem. Finally the indefinite boundary value pro-
blems they satisfy, so far as we know them, are also examined.

I.1. Introduction. In 1976, the second author was presented with the
following problem: find a weight function for the Bessel polynomials.
Although these polynomials were not named nor their properties de-
veloped until H. L. Krall and O. Frink wrote about them in their memor-
able 1949 paper [23], the Bessel polynomials have appeared in the litera-
ture for quite some time. Indeed, in 1873, Hermite used a sequence of
polynomials in his proof of the transcendency of e; these polynomials
turn out to be what Krall and Frink call the Bessel polynomials (see
[34]). Also, in 1880, Hertz essentially produced the formula

$$\left(\frac{\pi x}{2}\right)^{-1/2} e^{1/x} K_{n+1/2}\left(\frac{1}{x}\right) = \sum_{j=0}^{n} \frac{(n+j)!x^j}{2^j j!(n-j)!},$$

where $K_{n+1/2}(z)$ is Macdonald's function; the right side of this equation is
precisely the nth degree Bessel polynomial $y_n(x)$ (see [34]). In 1929, S.
Bochner [4] classified, up to a linear change of variable, all second order
differential equations admitting a sequence of orthogonal polynomial
solutions. In his classification, Bochner rediscovered the Bessel poly-
nomials but neglected to study their properties. Most of these properties
were developed by Krall and Frink, except one: a real valued function
$\psi(x)$ of bounded variation on $\mathbf{R}$ that generates the orthogonality of the
Bessel polynomials is still unknown. That is, find $\psi \in BV(\mathbf{R})$ such that

$$\int_{-\infty}^{\infty} y_n(x)y_m(x)d\psi(x) = K_n\delta_{nm}, \quad K_n \neq 0, n, m = 0, 1, \ldots$$

Like the classic orthogonal polynomials, a real orthogonality property is

essential to their use. A moment theorem due to R. P. Boas [3] guarantees the existence of such a real valued weight. But how is it found? If we proceed along the same lines as we do when we prove the orthogonality of the classical orthogonal polynomials, we see immediately the problem of finding a real weight. Indeed, since the ordinary Bessel polynomials satisfy the differential equation

$$x^2 y'' + 2(x + 1)y' - n(n + 1)y = 0,$$

we arrive, after some manipulations, with the formula

$$(x^2 e^{-2/x}(y_n' y_m - y_n y_m'))' = (n - m)(n + m + 1)e^{-2/x} y_n y_m.$$

There is no interval on the real line for which the function $x^2 e^{-2/x}$ is zero at both ends. In fact, $x^2 e^{-2/x}$ approaches 0 only at $x = 0$ and only from the right side. Hence, since there is no true interval of orthogonality, $e^{-2/x}$ is not a suitable weight for the Bessel polynomials.

It did occur to the second author, however, that the support of ψ should be very near $x = 0$, an idea that has also occurred, for various reasons, to a number of others as well. Hence, ψ might be related to the Heaviside function or Dirac's delta function. It has been shown [27] that the delta series

$$w(x) = \sum_{n=0}^{\infty} \frac{-2^{n+1} \delta^{(n)}(x)}{n!(n + 1)!}$$

serves as a weight for the Bessel polynomials. Is it true that

$$w(x) = \frac{d\psi(x)}{dx},$$

where $\psi(x)$ has bounded variation on **R**? The delta series concept, developed in [27], works for other polynomial sets as well, even when classical weights are known.

This raises a number of questions. How is the delta series related to the classical weight function? What really is the delta series? On what functions does it act continuously? Can it be used in other circumstances? Can it be expressed in more familiar terms? During the past eight years many of these questions have been answered, but by no means is the entire situation completely understood.

Some of the results which answer in part the questions just raised led in another direction. Several new orthogonal polynomial sets were found, which turned out to be excellent examples of singular Sturm-Liouville problems of fourth and sixth order. While most people believe that the Sturm-Liouville problem was completely solved in the period 1910–1950, in fact work still goes on today, especially for higher order problems.

The orthogonal polynomial systems provide some of the few real examples researchers can look at.

One basic problem is how to extend the minimal operator, associated with the Sturm-Liouville differential equation, so that the extension is self-adjoint, for only then will the polynomial's expansion of arbitrary functions be the eigenfunction expansion or spectral resolution of the extension. There are a number of domain problems, which are in the process of being solved by a close examination of singular boundary conditions. Work is continuing, even at present. Surprisingly, confusion was found in the classical cases as well, those thought long solved. The confusion has been cleared up for second order problems and the general situation is yielding slowly to increased pressure applied to it.

The purpose of this paper is to survey known results, to tie them into some sort of general framework, to indicate what we believe occurs in cases still pending, and to list some still unsolved problems. This, part I, is restricted to considering polynomials whose differential equations are of second order. Part II will examine polynomials whose differential equation is of order four or six.

I.2. Orthogonal polynomials. Traditionally orthogonal polynomials are introduced through the use of moments: Let $\{\mu_n\}_{n=0}^{\infty}$ be an infinite collection of real numbers with the property

$$\Delta_n = \begin{vmatrix} \mu_0 & \cdots & \mu_n \\ \vdots & & \vdots \\ \mu_n & & \mu_{2n} \end{vmatrix} \neq 0, \quad n = 0, 1, \ldots .$$

Then the Tchebycheff polynomials p_n are defined by setting $p_0 = 1$,

$$p_n(x) = (1/\Delta_{n-1}) \begin{vmatrix} \mu_0 & \cdots & \mu_n \\ \vdots & & \vdots \\ \mu_{n-1} & \cdots & \mu_{2n-1} \\ 1 & & x^n \end{vmatrix},$$

$n = 1, 2, \ldots .$ These polynomials are orthogonal with respect to any linear functional w satisfying

$$\langle w, x^n \rangle = \mu_n, \quad n = 0, 1, \ldots .$$

Since

$$\langle w, x^m p_n(x) \rangle = (1/\Delta_{n-1}) \begin{vmatrix} \mu_0 & \cdots & \mu_n \\ \vdots & & \vdots \\ \mu_{n-1} & & \mu_{2n-1} \\ \mu_m & & \mu_{m+n} \end{vmatrix},$$

for $m = 0, 1, \ldots, n - 1$, the determinant has two equal rows and is 0. Hence p_n is orthogonal to any polynomial of degree less than n, including $p_0, \ldots, p_{n-1}$.

By use of such a linear functional [21] it is possible to show that $\{p_n\}_{n=0}^{\infty}$ satisfies a three term recurrence relation of the form

$$p_{n-1} = (x + B_n)p_n - C_n p_{n-1},$$

where, if $p_n = x^n - S_n x^{n-1} + \ldots$, then

$$B_n = -S_{n+1} + S_n \text{ and } C_n = \Delta_n \Delta_{n-2}/\Delta_{n-1}^2.$$

There is certainly a linear functional to assure this. In fact,

$$w = \sum_{n=0}^{\infty} (-1)^n \mu_n \delta^{(n)}(x)/n!$$

is an orthogonalizing weight distribution for $\{p_n(x)\}_{n=0}^{\infty}$ [27].

Since the classic examples of mathematical physics, the Legendre, Laguerre and Hermite polynomials, all satisfy a collection of differential equations, it is natural to ask when does this happen in general? More specifically, when does a collection of orthogonal polynomials $\{p_n\}_{n=0}^{\infty}$ satisfy a linear differential equation in which all the coefficients except that of the 0th derivative are independent of the indexing parameter n?

This was answered in 1938 by H. L. Krall [22] who showed that the order of such a differential equation has to be even. An elementary examination shows that the coefficient of the ith derivative can be at most a polynomial of degree i, so such a differential equation must have the form

$$\sum_{i=0}^{2r} \left(\sum_{j=0}^{i} \ell_{ij} x^j \right) y^{(i)}(x) = \lambda_n y(x),$$

and λ_n must be given by

$$\lambda_n = \ell_{00} + n\ell_{11} + n(n-1)\ell_{22} + \cdots + n(n-1)\ldots(n-2r+1)\ell_{2r,2r}.$$

Krall [22] showed that the polynomials $\{p_n\}_{n=0}^{\infty}$ satisfy the equation above if and only if the moments $\{\mu_n\}_{n=0}^{\infty}$ satisfy the recurrence relations

$$S_k(m) = \sum_{i=2k+1}^{2r} \sum_{u=0}^{i} \binom{i-k-1}{k}(m - 2k - 1)^{(i-2k-1)}\ell_{i,i-u}\mu_{m-u} = 0,$$

where $(p)^{(q)} = p(p-1) \cdots (p-q+1)$, $(p)^{(0)} = 1$, for $2k + 1 \leq 2r$ and $m = 2k+1, 2k+2, \ldots$, and $\Delta_k \neq 0$, $k = 0, 1, \ldots$.

Further, since, for the classic examples, the differential equations can be multiplied by the weight function w to become symmetric, this should be expected in general. Therefore, in order to find w, Littlejohn [25] showed this

THEOREM. Let $L_{2r}(y) = \sum_{k=0}^{2r} b_k(x) y^{(k)}(x)$. Then $w(x) L_{2r}(y)$ is symmetric if and only if w satisfies the r homogeneous differential equations

$$\sum_{s=k}^{r} \sum_{j=0}^{2s-2k+1} \binom{2s}{2k-1} \binom{2s-2k+1}{j} \frac{2^{2s+2k+2}-1}{s-k+1} B_{2s-2k+2} \, b_{2s}^{(2s-2k+1-j)} w^{(j)}(x)$$
$$- b_{2k-1} w(x) = 0,$$

$k = 1, 2, \ldots r$, where B_{2k} is the Bernoulli number defined by $x/(e^x - 1) = 1 - x/2 + \sum_{k=1}^{\infty} (B_{2k} x^{2k})/(2k)!$.

When $k = r$, the differential equation is easily solved to yield the solution

$$w = \left(\exp\left(\frac{1}{r} \int (b_{2r-1}/b_{2r}) \, dt \right) \right) / b_{2r}.$$

Littlejohn also proved the following result [26].

THEOREM. If $w(x) L_{2r}(y)$ is symmetric and w satisfies the equations above in a distributional sense, acting on a space of test functions of slow growth, then w is a distributional weight function for $\{p_n\}_{n=0}^{\infty}$ and $\mu_n = \langle w, x^n \rangle$, $n = 0, 1, \ldots$. Further, if $b_i = \sum_{j=0}^{i} \ell_{ij} x^j$, $i = 1, \ldots, 2r$, then $\langle w, b_{2i-i} \rangle = 0$, $i = 1, 2, \ldots, r$.

Finally it is possible to find the moments rather easily provided the polynomials $\{p_n\}_{n=0}^{\infty}$ are known [19]. If

$$p_n(x) = \sum_{j=0}^{n} a_{nj} x^j, \quad n = 0, 1, \ldots,$$

are orthogonal then

$$\mu_n = (-1)^n \det(a_{i,\, j-1})_n \Big/ \prod_{i=1}^{n} a_{ii},$$

where

$$\det(a_{i,j-1})_n = \begin{vmatrix} a_{1,0} & \cdots & a_{1,n-1} \\ a_{2,0} & \cdots & \vdots \\ \vdots & & \vdots \\ \vdots & & \vdots \\ a_{n,0} & \cdots & a_{n,n-1} \end{vmatrix}.$$

In addition, the same argument used to determine the formula above establishes the following necessary conditions:

$$\sum_{j=0}^{n} a_{nj} \mu_{p-n+j} = 0, \quad (p+1)/2 \leqq n \leqq p, p = 1, 2, \ldots.$$

These are required in order to assure orthogonality.

So, in theory at least, there is a connection from moments to differential equations to polynomials to moments. In practice it is not all that simple.

II.1. The classic orthogonal polynomials. As mentioned earlier, S. Bochner [4] first attacked the problem of classifying all orthogonal polynomial sets which satisfy a second order differential equation of the form

$$(\ell_{22}x^2 + \ell_{21}x + \ell_{20})y'' + (\ell_{11}x + \ell_{10})y' = (\ell_{11}n + \ell_{22}n(n-1))y, \quad n = 0, 1, \dots .$$

Since Bochner there have been a number of papers written on the subject [1], [6], [8], [10], [13], [20], [24], usually from slightly different points of view. There are four different situations, each arising from the zeros of $\ell_{22}x^2 + \ell_{21}x + \ell_{20}$:

(a) If there are two different zeros, then by appropriate translation and scaling, the Jacobi polynomials are discovered;

(b) If there is a single zero, the generalized Laguerre polynomials are the result;

(c) If there are no zeros, then we find the Hermite polynomials; and

(d) If there is a double zero, as Bochner [4] found, and H. L. Krall and O. Frink explored [23], the Bessel polynomials are recovered.

We shall discuss each in turn. We note first that for the second order differential equation above, the moments μ_n must satisfy a three term recurrence relation. Further, the differential equation must be symmetric. We prove this since the original proof, which gives the general result, is quite complicated [22].

THEOREM. *Let* $\{p_n\}_{n=0}^{\infty}$ *be a set of orthogonal polynomials given in* §1.2. *If* p_n *satisfies*

$$(\ell_{22}x^2 + \ell_{21}x + \ell_{20})y'' + (\ell_{11}x + \ell_{10})y' = (\ell_{11}n + \ell_{22}n(n-1))y,$$

then

$$wLy = w((\ell_{22}x^2 + \ell_{21}x + \ell_{20})y'' + (\ell_{11}x + \ell_{10})y')$$

is symmetric and the moments $\{\mu_n\}_{n=0}^{\infty}$ *satisfy*

$$\ell_{11}\mu_n + \ell_{10}\mu_{n-1} + (n-1)(\ell_{22}\mu_n + \ell_{21}\mu_{n-1} + \ell_{20}\mu_{n-2}) = 0, \quad n = 1, 2, \dots .$$

Conversely, if the moments $\{\mu_n\}_{n=0}^{\infty}$ *satisfy*

$$\ell_{11}\mu_n + \ell_{10}\mu_{n-1} + (n-1)(\ell_{22}\mu_n + \ell_{21}\mu_{n-1} + \ell_{20}\mu_{n-2}) = 0;$$

then

$$wLy = w((\ell_{22}x^2 + \ell_{21}x + \ell_{20})y'' + (\ell_{11}x + \ell_{10})y')$$

is symmetric, and p_n *satisfies*

$$(\ell_{22}x^2 + \ell_{21}x + \ell_{20})y'' + (\ell_n x + \ell_{10})y' = (\ell_{11}n + \ell_{22}n(n-1))y.$$

PROOF. Let p_n satisfy the differential equation. Then, since $\{p_n\}_{n=0}^\infty$ is an orthogonal set with respect to w,

$$\begin{aligned}
0 &= (\lambda_n - \lambda_m)\langle w, p_n p_m \rangle \\
&= \langle w, \lambda_n p_n p_m \rangle - \langle w, p_n \lambda_m p_m \rangle \\
&= \langle w, L p_n p_m \rangle - \langle w, p_n L p_m \rangle \\
&= (wLp_n, p_m) - (p_n, wLp_m),
\end{aligned}$$

where $\lambda_n = \ell_{11}n + \ell_{22}n(n-1)$, and $(\cdot, \cdot)$, is the standard notation for inner product.

It is well known that wL is symmetric if and only if

$$- (w(\ell_{22}x^2 + \ell_{21}x + \ell_{20}))' + w(\ell_{11}x + \ell_{10}) = 0$$

If this is applied distributionally to x^m, $m = 0, 1, \ldots$, then

$$\begin{aligned}
0 &= \langle -(w(\ell_{22}x^2 + \ell_{21}x + \ell_{20}))' + w(\ell_{11}x + \ell_{10}), x^m \rangle \\
&= \langle w(\ell_{22}x^2 + \ell_{21}x + \ell_{20}), mx^{m-1} \rangle + \langle w(\ell_{11}x + \ell_{10}), x^m \rangle \\
&= \ell_{22}m\langle w, x^{m+1} \rangle + \ell_{21}m\langle w, x^m \rangle + \ell_{20}m\langle w, x^{m-1} \rangle \\
&\quad + \ell_{11}\langle w, x^{m+1} \rangle + \ell_{10}\langle w, x^m \rangle.
\end{aligned}$$

Since $\mu_m = \langle w, x^m \rangle$, this is the recurrence relation.

Conversely, if the recurrence relation holds, then

$$-(w(\ell_{22}x^2 + \ell_{21}x + \ell_{20}))' + w(\ell_{11}x + \ell_{10}) = 0.$$

As a consequence wL is symmetric. Let $b_2 = \ell_{22}x^2 + \ell_{21}x + \ell_{20}$, $b_1 = \ell_{11}x + \ell_{10}$. Symmetry implies

$$0 = \langle w, b_2(p_n'' p_m - p_n p_m'') + b_1(p_n' p_m - p_n p_m') \rangle.$$

If $m = 0$,

$$0 = \langle w, (b_2 p_n'' + b_1 p_n') \rangle.$$

This states that $(b_2 p_n'' + b_1 p_n')$ is orthogonal to p_0.

If $m = 1$,

$$0 = \langle w, (b_2 p_n'' + b_1 p_n')p_1 - (b_1 p_n)p_1' \rangle.$$

Now

$$\langle w, (b_1 p_n)p_1' \rangle = \langle w, p_n(b_1 p_1') \rangle = 0$$

since p_n is orthogonal to polynomials of the first degree. Thus

$$0 = \langle w, (b_2 p_n'' + b_1 p_n')p_1 \rangle,$$

and $(b_2 p_n'' + b_1 p_n')$ is orthogonal to p_1.

If this is continued, we find $(b_2 p_n'' + b_1 p_n')$ is orthogonal to p_0, p_1, ..., p_{n-1}. But by inspection it is a polynomial of degree n. Therefore it must be a multiple, λ_n, of p_n. If the coefficients of x^n are compared in

$$(\ell_{22}x^2 + \ell_{21}x + \ell_{20})p_n'' + (\ell_{11}x + \ell_{10})p_n' = \lambda_n p_n,$$

it follows immediately that $\lambda_n = \ell_{11}n + \ell_{22}n(n-1)$. The proof is complete.

As we have noted,

$$w = \sum_{n=0}^{\infty} (-1)^n \mu_n \delta^{(n)}(x)/n!$$

is a solution to the equation

$$-(w(\ell_{22}x^2 + \ell_{21}x + \ell_{20}))' + w(\ell_{11}x + \ell_{10}) = 0$$

in a distributional sense. It is, in fact, a distributional Taylor's series. For, if ϕ is analytic,

$$\langle w, \phi \rangle = \left\langle w, \sum_{n=0}^{\infty} \phi^{(n)}(0)x^n/n! \right\rangle$$

$$= \sum_{n=0}^{\infty} \phi^{(n)}(0)\langle w, x^n \rangle/n!$$

$$= \sum_{n=0}^{\infty} (-1)^n \langle \delta^{(n)}(x), \phi \rangle \mu_n/n!$$

$$= \left\langle \sum_{n=0}^{\infty} (-1)^n \mu_n \delta^{(n)}(x)/n!, \phi \right\rangle,$$

i.e.,

$$w = \sum_{n=0}^{\infty} (-1)^n \mu_n \delta^{(n)}(x)/n!.$$

II.2.a. The Jacobi polynomials. If the coefficient of the second derivative has two distinct zeros, they may be located at ± 1 by a suitable transformation. Then, by choosing the remaining coefficients appropriately, the result is the Jacobi differential equation

$$(1 - x^2)y'' + ((\beta - \alpha) - (2 + \alpha + \beta)x)y' + n(n + \alpha + \beta + 1)y = 0.$$

The moments, μ_n, satisfy

$$(\alpha + \beta + n + 1)\mu_n + (\alpha - \beta)\mu_{n-1} - (n - 1)\mu_{n-2} = 0.$$

While a bit of a bother, they can be found [27]:

$$\mu_n = \sum_{j=0}^{n} \binom{n}{j}(-1)^j 2^j(\alpha + 1)_j/(\alpha + \beta + 2)_j$$

$$= \sum_{j=0}^{n} \binom{n}{j}(-1)^{n-j}2^j(\beta + 1)_j/(\alpha + \beta + 2)_j,$$

where $(a)_j = a(a + 1)\ldots(a + j - 1)$, $n = 0, 1, \ldots$.
  The weight equation is

$$(1 - x^2)w' - ((\beta - \alpha) - (\alpha + \beta)x)w = 0,$$

which, so long as $\alpha, \beta > -1$, has the solution

$$w = (1 - x)^{\alpha}(1 + x)^{\beta}(AH(1 - x) + BH(1 + x) + C)$$

where

$$H(x) = 0, \quad x < 0$$
$$= 1, \quad x \geqq 0.$$

When $A = 1$, $B = 1$, $C = -1$, this, of course, is the classic Jacobi weight function

$$w = (1 - x)^{\alpha}(1 + x)^{\beta}, \quad -1 \leqq x \leqq 1, \quad \alpha, \beta > -1$$
$$= 0 \qquad\qquad\qquad , \quad \text{otherwise.}$$

When either α or β or both are less than -1, a regularization of $(1 - x)^{\alpha}(1 + x)^{\beta}$ is required [27]. If $-N - 1 < \alpha < -N$, $-M - 1 < \beta < -M$, where M and N are positive integers, we replace

$$\langle w, \phi \rangle = \int_{-1}^{1} (1 - x)^{\alpha}(1 + x)^{\beta}\phi(x)dx,$$

which holds when $\alpha, \beta > -1$. By [27],

$$\langle w, \phi \rangle = \left(\frac{\Gamma(\alpha + \beta + 2)}{\Gamma(\alpha + 1)\Gamma(\beta + 1)2^{\alpha+\beta+1}}\right)\left(\int_{0}^{1}(1 - x)^{\alpha}((1 + x)^{\beta}\phi(x)\right.$$

$$\left. - \sum_{j=0}^{N-1}\frac{((1 + x)^{\beta}\phi(x)^{(j)}}{j!}\bigg|_{x=1}(-1)^j(1 - x)^j\right)dx$$

$$+ \int_{-1}^{0}(1+x)^{\beta}\left((1-x)^{\alpha}\phi(x) - \sum_{k=0}^{M-1}\frac{((1 - x)^{\alpha}\phi(x)^{(k)}}{k!}\bigg|_{x=-1}(1+x)^k\right)dx$$

$$+ \sum_{j=0}^{N-1}\frac{((1 + x)^{\beta}\phi(x))^j}{j!}\bigg|_{x=1}\frac{(-1)^j}{(\alpha + 1 + j)}$$

$$+ \sum_{k=0}^{M-1}\frac{((1 - x)^{\alpha}\phi(x))^{(k)}}{k!}\bigg|_{x=-1}\frac{1}{(\beta + 1 + k)}.$$

The regularization generates an indefinite inner product space through

the inner product $(f, g) = \langle w, f\bar{g}\rangle$, in contrast to the Hilbert space generated by

$$(f, g) = \int_{-1}^{1} f(x)\overline{g(x)} \, (1 - x)^{\alpha}(1 + x)^{\beta} dx$$

when α, $\beta > -1$. To the authors' knowledge, the inner product space has not been carefully examined.

II.2.b *The Legendre polynomials.* As a special case, when $\alpha = \beta = 0$, the Legendre problem is found. The differential equation is

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0.$$

The moments, μ_n, satisfy $(n + 1)\mu_n - (n - 1)\mu_{n-2} = 0$, and are $\mu_{2n} = 2/(2n + 1)$, $\mu_{2n+1} = 0$, $n = 0, 1, \ldots$.

The weight function w satisfies $(1 - x^2)w' = 0$. Its distributional solution is

$$w = AH(1 - x) + BH(1 + x) + C.$$

The choice $A = 1$, $B = 1$, $C = -1$ gives the classic

$$w = 1, \quad -1 \leqq x \leqq 1$$
$$= 0, \quad \text{otherwise.}$$

II.2.c. *The Tchebycheff polynomials of the first kind.* When $\alpha = \beta = -1/2$, we find the differential equation satisfied by the Techebycheff polynomials of the first kind,

$$(1 - x^2)y'' - xy' + n^2y = 0.$$

The moments, μ_n, satisfy $n\mu_n - (n - 1)\mu_{n-2} = 0$, and are $\mu_{2n} = (2n)!/2^{2n}(n!)^2$, $\mu_{2n+1} = 0$, $n = 0, 1, \ldots$.

The weight function, w, satisfies $[1 - x^2]w' - xw = 0$. Its distributional solution is

$$w = (1 - x^2)^{-1/2}(AH(1 - x) + BH(1 + x) + C).$$

Again, if $A = 1$, $B = 1$, $C = -1$, the classic weight function

$$w = (1 - x^2)^{-1/2}, \quad -1 \leqq x \leqq 1$$
$$= 0 \quad\quad\quad, \quad \text{otherwise}$$

is recovered.

II.2.d. *The Tchebycheff polynomials of the second kind.* When $\alpha = \beta = 1/2$, we find the differential equation satisfied by the Tchebycheff polynomials of the second kind:

$$(1 - x^2)y'' - 3xy' + n(n + 2)y = 0.$$

The moments, μ_n, satisfy $(n + 2)\mu_n - (n - 1)\mu_{n-2} = 0$, and are $\mu_{2n} = (2n)/2^{2n}(n)!(n + 1)!$, $\mu_{2n+1} = 0$, $n = 0, 1, \ldots$.

The weight function, w, satisfies $(1 - x^2)w' + xw = 0$. Its distributional soulution is

$$w = (1 - x^2)^{1/2}(AH(1 - x) + BH(1 + x) + C).$$

If $A = 1, B = 1, C = -1$, the classic weight function

$$w = (1 - x^2)^{1/2}, \quad -1 \leqq x \leqq 1$$
$$= 0 \quad \quad , \quad \text{otherwise}$$

is recovered.

II.3.a. The generalized Laguerre polynomials. If the coefficient of the second derivative has only one zero, it may be located at 0 by a suitable translation. By choosing the remaining coefficients appropriately, the generalized Laguerre equation

$$xy'' + (\alpha + 1 - x)y' + ny = 0$$

emerges. The moments, μ_n, satisfy $\mu_n - (n + \alpha)\mu_{n-1} = 0$. An easy induction shows $\mu_n = \Gamma(n + \alpha + 1)/\Gamma(\alpha + 1)$.

The weight equation is $xw' + (x - \alpha)w = 0$ This may be rewritten as

$$x^{\alpha+1}e^{-x}(x^{-\alpha}e^x w)' = 0.$$

which, so long as $\alpha > -1$, has the solution

$$w = x^\alpha e^{-x}(AH(x) + B).$$

When $A = 1, B = 0$, the result is the classic Laguerre weight function

$$w = x^\alpha e^{-x}, \quad 0 \leqq x < \infty$$
$$= 0 \quad , \quad x < 0$$

When $\alpha \leqq -1$, difficulties arise. If α is a negative integer, the polynomial set degenerates. But if $-j - 1 < \alpha < -j$, where j is a positive integer, then the weight function is a regularization of the classic function above, [27],

$$\langle w, \phi \rangle = \frac{1}{\Gamma(\alpha + 1)} \int_0^\infty x^\alpha \Big(e^{-x}\phi(x)$$
$$- \sum_{\ell=0}^{j-1}\Big(\sum_{k=\ell}^{j-1}\frac{(-1)^k x^k}{\ell!(k - \ell)!}\Big)(-1)^\ell \phi^{(\ell)}(0)\Big)dx.$$

If integration by parts is performed j times, this can be expressed by

$$\langle w, \phi \rangle = \frac{(-1)^j}{\Gamma(\alpha + j + 1)} \int_0^\infty x^{\alpha+j}(e^{-x}\phi(x))^{(j)}dx.$$

Just as in the Jacobi case $\alpha < -1$ or $\beta < -1$, when $\alpha < -1$ here, the regularization leads to an indefinite inner product space through the inner product

$$(f, g) = \langle w, f\bar{g} \rangle.$$

This situation has been explored extensively [16], [17] and will be described later.

II.3.b. *The ordinary Laguerre polynomals.* When $\alpha = 0$, the ordinary Laguerre polynomials result. The differential equation is

$$xy'' + (1 - x)y' + ny = 0.$$

The moments satisfy $\mu_n - n\,\mu_{n-1} = 0$ and are $\mu_n = n!$

The weight function satisfies $xw' + xw = 0$. As before, this may be written as

$$xe^{-x}(e^x w)' = 0.$$

The distributional solution is $w = e^{-x}(AH(x) + B)$, and the classic Laguerre weight function

$$w = e^{-x}, \quad 0 \leqq x < \infty$$
$$= 0 \quad , \quad x < 0$$

is attained by letting $A = 1$, $B = 0$.

II.4. **The Hermite polynomials.** If the coefficient of y'' is never zero, then, by appropriately choosing the coefficients, the Hermite equation

$$y'' - 2xy' + 2ny = 0$$

is found. The moments, μ_n, satisfy $2\mu_n - (n - 1)\mu_{n-2} = 0$ and are $\mu_{2n} = \sqrt{\pi}\,(2n)!/(4^n n!)$, $\mu_{2n+1} = 0$, $n = 0, 1, \ldots$.

The weight equation is $w' + 2xw = 0$. Rewritten as

$$e^{-x^2}(e^{x^2} w)' = 0,$$

its only solution is $w = Ae^{-x^2}$, $-\infty < x < \infty$, which when $A = 1$ yields the classic weight function

$$w = e^{-x^2}, \quad -\infty < x < \infty.$$

II.5.a. **The generalized Bessel Polynomials.** If the initial coefficient has a double zero, it may be located at $x = 0$. The equation that results is

$$x^2 y'' + (ax + b)y' - n(n + a - 1)y = 0.$$

Since this equation is related to the Bessel (function) equation [23], it, as well as its polynomial solutions, bear the same name: Bessel.

The moments satisfy $(n + a - 1)\mu_n + b\mu_{n-1} = 0$; so if $\mu_0 = -b$, then

$$\mu_n = (-b)^{n+1}/(a)_n, \quad n = 0, 1, \ldots,$$

where $(a)_n = a(a + 1) \ldots (a + n - 1)$.

The weight equation is $x^2w' + ((2 - a)x - b)w = 0$. This may be rewritten as

$$x^a e^{-b/x}(x^{2-a}e^{b/x}w)' = 0.$$

For the past 40 years or so, however, no one has been able to find a satisfactory solution of these equations. The obvious solution

$$w = x^{a-2}e^{-b/x}$$

fails to vanish at ∞, or any other point other than zero, rapidly enough to serve as a weight function. The distributional solution

$$w = -\sum_{n=0}^{\infty} \frac{b^{n+1}\delta^{(n)}(x)}{(a)_n n!}$$

satisfies the equation [26], and does indeed serve as a distributional weight (see [27]), but is apparently not the weight generated by a function of bounded variation guaranteed by Boas [3]. Hence it is not fully accepted.

There are variations on the formula above. In [18] it is shown that w, defined by

$$\langle w, \phi \rangle = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_\alpha^\beta \phi(x) \operatorname{Im}\{(-b/z)_1F_1(1, a, -b/z)\} dx$$

where $z = x + i\varepsilon$ and $\alpha < 0$, $\beta > 0$, also works. Indeed, this is the Stieltjes-Perron inversion for the Cauchy representation of w (see [29; pp. 369–372]), but again it does not seem to be what has so long been sought; the problem is open.

II.5.b. *The ordinary Bessel polynomials.* When $a = b = 2$, the ordinary Bessel polynomials, first discussed by H. L. Krall and O. Frink [23] are found. The differential equation is

$$x^2y'' + (2x + 2)y' - n(n + 1)y = 0.$$

The moments satisfy $(n + 1)\mu_n - 2\mu_{n-1} = 0$; if $\mu_0 = -2$, then

$$\mu_n = (-2)^{n+1}/(n + 1)!, \quad n = 0, 1, \ldots.$$

The weight equation is $x^2w' - 2w = 0$, which, as mentioned earlier, defies solution. And $e^{-2/x}$ is not the correct solution. The distributional solution

$$w = - \sum_{n=0}^{\infty} \frac{2^{n+1}\delta^{(n)}(x)}{(n+1)!n!}$$

does generate orthogonality [27], but no connection with anything more familiar is known.

The formula

$$\langle w, \phi \rangle = \lim_{\varepsilon \to 0} - \frac{1}{\pi} \int_{\alpha}^{\beta} \phi(x) \exp\left(\frac{-2x}{x^2 + \varepsilon^2}\right) \sin\left(\frac{2\varepsilon}{x^2 + \varepsilon^2}\right) dx$$

seems to be another variant on the distributional series. It likewise generates orthogonality [18].

II.5.c. *Bessel polynomials and the Laplace transform.* In 1955, 1958 and 1961 H. E. Salzer established a connection between the Bessel polynomials with $a = b = 1$ and the inversion of the Laplace transform [31, 32, 33]. The book by Grosswald [12] contains an excellent summary of his results. Rather than reproduce it here, let us merely cite that in this case the differential equation is

$$x^2 y'' + (x + 1)y' - n^2 y = 0.$$

The moments satisfy $n\mu_n + \mu_{n-1} = 0$; if $\mu_0 = -1$, then

$$\mu_n = (-1)^{n+1}/n!, \quad n = 0, 1, \ldots.$$

The weight equation is $x^2 w' + (x - 1)w = 0$ and is just as enigmatic as ever. Weights given by

$$w = - \sum_{n=0}^{\infty} \frac{\delta^{(n)}(x)}{(n!)^2}$$

or by

$$\langle w, \phi \rangle = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\alpha}^{\beta} \phi(x) \exp\left(\frac{-x}{x^2 + \varepsilon^2}\right) \left(\frac{-x}{x^2 + \varepsilon^2} \sin\left(\frac{\varepsilon}{x^2 + \varepsilon^2}\right)\right.$$
$$\left. + \frac{\varepsilon}{x^2 + \varepsilon^2} \cos\left(\frac{\varepsilon}{x^2 + \varepsilon^2}\right)\right) dx$$

are available.

III.1. Second order boundary value problems. The method of separation of variables has been employed for at least two centuries in the solution of certain partial differential equations. Some of the ordinary differential equations, resulting from the method, have associated with them boundary conditions applied at the ends of the interval involved. Others, especially those which have as solutions the classical orthogonal polynomials, have other constraints placed at the interval's ends in order to uniquely determine the solution. These seemed to be inherently different. Until recently

they were not well understood. Usually, physical intuition was used to choose the right solution.

We now classify these problems as regular and singular. As it turns out, the singular theory can be developed so it includes the regular problems as a special case. We shall, however, discuss the regular case briefly, so it may be contrasted to the singular theory, and also so we may use the results in the singular case as well.

Our setting is $L^2(a, b; w)$, the Hilbert space on the interval from a to b with weight $w > 0$. The differential equation to be considered is of the form

$$(py')' + qy = \lambda w y,$$

which comes from the operator equation $\ell y = \lambda y$, where $\ell y = (1/w)$ $((py')' + qy)$. We assume that w, p, q, $1/p$ are continuous on (a, b) and that $w > 0$. It is easily recognized that ℓ is symmetric, or formally self-adjoint in the language of differential equations. In $L^2(a, b; w)$, however, there is more to the story. Not only must the form of ℓ be given, its domain also must be specified. The same is true of its (Hilbert space) adjoint. For self-adjointness to occur, both form and domain must be the same. This is equally true for both regular and singular problems.

III.2. The differential operator and adjoint operator. In order to specify a differential operator on $L^2(a, b; w)$ we first consider the minimal operator L_0. Operators L defined then by the use of boundary conditions are extensions of L_0. Since $L_0 \subset L$, their adjoints satisfy $L^* \subset L_0^*$. The operator L_0^* is the maximal operator, and since its form is known, the form of L^* is likewise known:

DEFINITION. We denote, by D_0', those elements y in $L^2(a, b; w)$ satisfying:
1. $\ell y = (1/w)((py')' + qy)$ exists a.e. and is in $L^2(a, b; w)$; and
2. Both y and y' vanish in a neighborhood of a and b.

DEFINITION. We define the operator L_0' by setting

$$L_0' y = \ell y,$$

for all y in D_0'.

We define the operator L_0 as the closure of L_0'.

We also define the maximal operator L_M in a similar fashion:

DEFINITION. We denote, by D_M, those elements y in $L^2(a, b; w)$ satisfying
1. $\ell y = (1/w)((py')' + qy)$ exists a.e. and is in $L^2(a, b; w)$.

DEFINITION. We define the operator L_M by setting

$$L_M y = \angle y,$$

for all y in D_M.

This leads us to the well known result, found in Naimark [**28**; pp. 68–70].

THEOREM. *The domain of* L_0^* *is* D_M. $L_0^* = L_M$ *Furthermore,* $L_M^* = L_0$.

As stated earlier, since $L_0 \subset L$, where L is any differential operator given by

$$Ly = (1/w) ((py')' + qy),$$

whose domain D is restricted by either regular or singular boundary conditions at a and b, the adjoints satisfy $L^* \subset L_0^* = L_M$. Hence the form of L^* is given by

$$L^* z = (1/w) ((py')' + qy).$$

The question of domain is still open at this point.

III.3. Regular boundary value problems. We assume that w, p, q, $1/p$ are continuous on $[a, b]$ and that $w > 0$. We define the operator L in the following way.

DEFINITION. We denote, by D_L, those elements y in $L^2(a, b; w)$ satisfying:
1. $\angle y = (1/w) ((py')' + qy)$ exists a.e. and is in $L^2(a, b; w)$; and
2. For $1 \leqq m \leqq 4$,

$$a_{j1} y(a) + a_{j2} y'(a) + b_{j1} y(b) + b_{j2} y'(b) = 0, \quad j = 1, \dots, m,$$

where the augmented matrix $[a_{ij}, b_{ij}]$ has rank m.

DEFINITION. We define the operator L by setting

$$Ly = \angle y$$

for all y in D_L.

The boundary conditions may be written in matrix form as well by setting $A = [a_{ij}]$, $B = [b_{ij}]$, $Y = [\begin{smallmatrix} y \\ y' \end{smallmatrix}]$. Then

$$AY(a) + BY(b) = 0.$$

Since we already know the form of the adjoint operator, all that is necessary is to calculate its domain, i.e., the boundary conditions constraining it. We do so by first looking at Green's formula. Let y be in D_L and z be in the domain of the adjoint. Then

$$0 = (Ly, z) - (y, L^*z)$$

$$= \int_a^b \bar{z}((py')' + qy)\mathrm{dt} - \int_a^b ((\overline{pz')' + qz})y \,\mathrm{dt}$$

$$= \int_a^b (p(y'\bar{z} - y\bar{z}'))' \,dt$$

$$= p(y'\bar{z} - y\bar{z})|_a^b.$$

If we let

$$J = \begin{bmatrix} 0 & p \\ -p & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} y \\ y' \end{bmatrix}, \quad Z = \begin{bmatrix} z \\ z' \end{bmatrix},$$

this can be written as $0 = Z^*JY|_a^b$, or

$$0 = [Z^*(a), Z^*(b)]\begin{bmatrix} -J(a) & 0 \\ 0 & J(b) \end{bmatrix}\begin{bmatrix} Y(a) \\ Y(b) \end{bmatrix}.$$

Now let C and D be chosen so that the square matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is nonsingular. Let

$$\begin{bmatrix} \tilde{A}^* & \tilde{C}^* \\ \tilde{B}^* & \tilde{D}^* \end{bmatrix}$$

satisfy

$$\begin{bmatrix} \tilde{A}^* & \tilde{C}^* \\ \tilde{B}^* & \tilde{D}^* \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} -J(a) & 0 \\ 0 & J(b) \end{bmatrix}.$$

Then

$$\tilde{A}^*A + \tilde{C}^*C = -J(a)$$
$$\tilde{A}^*B + \tilde{C}^*D = 0$$
$$\tilde{B}^*A + \tilde{D}^*C = 0$$
$$\tilde{B}^*B + \tilde{D}^*D = J(b),$$

and Green's formula becomes

$$0 = [Z^*(a)Z^*(b)]\begin{bmatrix} \tilde{A}^* & \tilde{C}^* \\ \tilde{B}^* & \tilde{D}^* \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} Y(a) \\ Y(b) \end{bmatrix},$$

or

$$0 = (\tilde{A}Z(a) + \tilde{B}Z(b))^*(AY(a) + BY(b))$$
$$+ (\tilde{C}Z(a) + \tilde{D}Z(b))^*(CY(a) + DY(b)).$$

Since $AY(a) + BY(b) = 0$, while $CY(a) + DY(b)$ may be arbitrary, we find the following theorem.

THEOREM. *If z is in D_L^*, the domain of L^*, then $\tilde{C}Z(a) + \tilde{D}Z(b) = 0$.*

Conversely, the domain of L^*, D_L^*, *consists of those elements* y *in* $L^2(a, b;w)$ *satisfying*:
  1. $\ell z = (1/w) \, ((py')' + qy)$ *exists a.e. and is in* $L^2(a, b; w)$; *and*
  2. $\tilde{C}Z(a) + \tilde{D}Z(b) = 0$ *or* $c_{j1}z(a) + c_{j2}z((a) + d_{j1}z(b) + d_{j2}z'(b) = 0$, $j = 1, \ldots, 4\text{-}m$.

There are other forms. If

$$A\,Y(a) + B\,Y(b) = 0$$
$$C\,Y(a) + D\,Y(b) = \psi,$$

then, by using the coefficient equations derived earlier, we find

$$Y(a) = -J(a)^{-1}\tilde{C}^*\psi$$
$$Y(b) = J(b)^{-1}\tilde{D}^*\psi.$$

These are parametric boundary conditions. (See [7]). Likewise, if

$$\tilde{C}Z(a) + \tilde{D}Z(b) = 0$$
$$\tilde{A}Z(a) + \tilde{B}Z(b) = \phi,$$

then

$$Z(a) = -J(a)^{*-1}A^*\phi$$
$$Z(b) = J(b)^{*-1}B^*\phi,$$

the adjoint parametric boundary conditions. These parametric forms are fully equivalent to the originals.

It is an easy calculation to show that the following holds.

THEOREM. *The differential operator* L *is self-adjoint if and only if*

$$AJ(a)^{-1}A^* = BJ(b)^{-1}B^*.$$

An equivalent equation is

$$\tilde{C}J(a)^{-1}\tilde{C} = \tilde{D}J(b)^{-1}\tilde{D}.$$

One might inquire what happens if the matrices A and B are multiplied by a nonsingular matrix F, thus mixing up the boundary conditions defining D_L. In the calculation of Green's formula this amounts to inserting two extra matrices in the middle:

$$0 = [Z^*(a)Z^*(b)]\begin{bmatrix}\tilde{A}^* & \tilde{C}^* \\ \tilde{B}^* & \tilde{D}^*\end{bmatrix}\begin{bmatrix}F^{-1} & 0 \\ -H^{-1}GF^{-1} & H^{-1}\end{bmatrix}\begin{bmatrix}F & 0 \\ G & H\end{bmatrix}\begin{bmatrix}A & B \\ C & D\end{bmatrix}\begin{bmatrix}Y(a) \\ Y(b)\end{bmatrix},$$

where H likewise is nonsingular. When this is expanded,

$$0 = ((F^{-1*}\tilde{A} - F^{-1*}G^*H^{-1*}C)Z(a)$$
$$+ (F^{-1*}\tilde{B} - F^{-1*}G^*H^{-1*}\tilde{D})Z(b))^* \cdot (FAY(a) + FBY(b))$$
$$+ (H^{-1}\tilde{C}Z(a) + H^{-1}\tilde{D}Z(b))^*((GA + HC)Y(a) + (GB + HD)Y(b)).$$

The constraint on the adjoint domain is now

$$H^{-1}\tilde{C}Z(a) + H^{-1}\tilde{D}Z(b) = 0$$

which is clearly equivalent to the original.

III.4. Singular boundary conditions. The techniques of the last section are not applicable in singular situations, which occur when p, q, w, $1/p$ become infinite or when (a, b) is infinite. The trouble, quite simply, is that limits at a and b may not exist. Something else must be done. Our procedure, therefore, will be to first establish solutions of

$$(py')' + qy = \lambda wy$$

in $L^2(a, b; w)$, and then using these, to define appropriate boundary conditions. It will not be too surprising to see that they are extensions of regular boundary conditions.

We follow the path described in [2]. First note that the differential equation above can be written in system format

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \left(\lambda \begin{bmatrix} w & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -q & 0 \\ 0 & -1/p \end{bmatrix} \right) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

if $y_1 = y$, $y_2 = -py'$. Now let λ be complex. If we premultiply by $[\bar{y}_1 \ \bar{y}_2]$, we have

$$[\bar{y}_1 \bar{y}_2] \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = [\bar{y}_1 \bar{y}_2] \left(\lambda \begin{bmatrix} w & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -q & 0 \\ 0 & -1/p \end{bmatrix} \right) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Its conjugate transpose is

$$- [\bar{y}_1 \bar{y}_2]' \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = [\bar{y}_1 \bar{y}_2] \left(\bar{\lambda} \begin{bmatrix} w & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -q & 0 \\ 0 & -1/p \end{bmatrix} \right) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

If this is subtracted from the first, we find

$$\left([\bar{y}_1 \bar{y}_2] \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right)' = (\lambda - \bar{\lambda}) [\bar{y}_1 \bar{y}_2] \begin{bmatrix} w & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

If c is chosen in (a, b) and $c < b' < b$, we integrate from c to b'. Further we let $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be specific solutions so that $Y = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$ is a fundamental matrix for the system, and we assume that $Y(c) = I$. Then $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ for some constants c_1 and c_2. The left side becomes

$$[\bar{c}_1 \bar{c}_2]\begin{bmatrix} \bar{u}_1 & \bar{u}_2 \\ \bar{v}_1 & \bar{v}_2 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\Big|_c^{b'}$$

$$= [\bar{c}_1 \bar{c}_2]\begin{bmatrix} \bar{u}_1 & \bar{u}_2 \\ \bar{v}_1 & \bar{v}_2 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}(b') - [\bar{c}_1 \bar{c}_2]\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

The right side is $2i \, \mathrm{Im}(\lambda) \int_c^{b'} |y_1|^2 \, w \, dt$. Hence

$$2 \, \mathrm{Im}(\lambda) \int_c^{b'} |y|^2 \, w \, dt$$

$$= [\bar{c}_1 \bar{c}_2]\begin{bmatrix} \bar{u}_1 & \bar{u}_2 \\ \bar{v}_1 & \bar{v}_2 \end{bmatrix}\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}(b') - [\bar{c}_1 \, \bar{c}_2]\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

If $\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$ is examined, it is found to be symmetric with eigenvalues $\lambda = \pm$ 1. Since $Y(t)$ is never singular,

$$\begin{bmatrix} \bar{u}_1 & \bar{u}_2 \\ \bar{v}_1 & \bar{v}_2 \end{bmatrix}\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$$

is never singular, and also has one positive, one negative eigenvalue. Let $\mathrm{Im}(\lambda) > 0$. Choose $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ so that $|c_1|^2 + |c_2|^2 = 1$ and

$$[\bar{c}_1 \bar{c}_2]\begin{bmatrix} \bar{u}_1 & \bar{u}_2 \\ \bar{v}_1 & \bar{v}_2 \end{bmatrix}\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} < 0$$

when $t = b'$. Then

$$\int_c^{b'} |y_1|^2 \, w \, dt \leqq \frac{-1}{2\mathrm{Im}(\lambda)} (\bar{c}_1, \bar{c}_2)\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \leqq \frac{1}{2\mathrm{Im}(\lambda)}$$

since $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ is unitary. This is equivalent to

$$[\bar{c}_1, \bar{c}_2]\left(\int_c^{b'} \begin{bmatrix} \bar{u}_1 & \bar{u}_2 \\ \bar{v}_1 & \bar{v}_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} dt\right)\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \leqq \frac{1}{2\mathrm{Im}(\lambda)},$$

which implies that the matrix in the brackets has an eigenvalue less than $1/2\mathrm{Im}(\lambda)$ and $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ may be chosen to be the eigenvector associated with this small eigenvalue. Since all these eigenvectors lie on the unit circle, as $b' \to b$ a subcollection can be chosen which converges to $\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$. Through this subcollection, we find that

$$[\bar{k}_1, \bar{k}_2]\left(\int_c^{b} \begin{bmatrix} \bar{u}_1 & \bar{u}_2 \\ \bar{v}_1 & \bar{v}_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} dt\right)\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \leqq \frac{1}{2\mathrm{Im}(\lambda)}$$

or if

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix},$$

then

$$\int_c^{b} |y_1|^2 \, w \, dt \leqq \frac{1}{2\mathrm{Im}(\lambda)}.$$

We have proven

THEOREM. *If* $\text{Im}(\lambda) \neq 0$, *the differential equation* $(py')' + qy = \lambda wy$ *has a solution* y_1 *which is in* $L^2(c, b; w)$, $a < c < b$.

Only minor modifications are needed when $\text{Im}(\lambda) < 0$.

The proof fails if λ is real, but if there exist two solutions in $L^2(c, b; w)$ for any λ, then variation of parameters can be used [14] to show that every solution for any λ is in $L^2(c, b; w)$.

We have two situations which can arise.

1. For all λ, $\text{Im}(\lambda) \neq 0$, there exists only one solution which is in L^2 $(c, b; w)$. This is commonly called the limit-point or limit-1 case due to the geometry involved in Herman Weyl's original proof [35].

2. For all λ, every solution is in $L^2(c, b; w)$. This is called the limit-circle or limit-2 case.

If the interval (a, c) is examined, the same situation occurs. This yields in general four different cases.

By using the solutions which are in $L^2(c, b; w)$ we can now define boundary conditions at b (see [11]). Although we do not assume that $Y(c) = I$, let us require instead that, at c,

$$\det Y = u_1 v_1' - u_1' v_1 = 1.$$

Then it is automatically 1 for all t in (a, b). If y is in D_M (§ III.1), let $y_1 = y$, $y_2 = -py'$; let $L_M y = f$, i.e.,

$$(py')' + qy = wf.$$

Then let $\begin{bmatrix} -b_v \\ b_u \end{bmatrix}$ be the algebraic solutions of

$$\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \begin{bmatrix} -b_v \\ b_u \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

so

$$b_v = p(y_1 v_1' - y_1' v_1)$$
$$b_u = p(y_1 u_1' - y_1' u_1).$$

If their derivatives are calculated,

$$b_v' = -(fv_1 - \lambda y_1 v_1)w$$
$$b_u' = -(fu_1 - \lambda y_1 u_1)w.$$

If v_1 is in $L^2(c, b; w)$, then Schwarz's inequality shows that b_v' is in $L^1(c, b)$, and so

$$B_v(y) = \lim_{x \to b} b_v$$

exists. Likewise, if u_1 is in $L^2(c, b; w)$, then $B_u(y) = \lim_{x \to b} b_u$ exists. It is

these expressions which replace such regular boundary terms as $y(b)$, $y'(b)$, etc. In the regular case, by choosing u_1, v_1, u_2, v_2 appropriately 0 or 1 at b, the singular limits above can be made to generate the regular expressions.

We formalize this discussion with the following.

THEOREM. *Let u be a solution of $\ell y = \lambda y$ in $L^2(c, b; w)$. Then, for all y in D_M,*

$$B_u(y) = \lim_{x \to b} b_u(y) = \lim_{x \to b} p(yu' - y'u)$$

exists, and, in the sense of Dunford and Schwartz [9], *is a boundary value at b.*

It is tempting to use Green's formula to attempt to establish the relation between a given set of singular boundary conditions and its adjoint collection. This cannot be done, as we shall show, and so it is necessary to defer the definition of self-adjoint problems to the next section.

Green's formula does, however, illustrate the difficulties involved. Let us set

$$L_M y = ((py')' + qy)/w = f$$
$$L_M z = ((pz')' + qz/w = g$$

in $L^2(a, b; w)$. Further set $y_1 = y$, $y_2 = -py'$, $z_1 = z$, $z_2 = -pz'$. Then

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' + \begin{bmatrix} q & 0 \\ 0 & 1/p \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$
$$= \begin{bmatrix} w & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} f \\ 0 \end{bmatrix} - [\bar{z}_1, \bar{z}_2]'\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + [\bar{z}_1, \bar{z}_2]\begin{bmatrix} q & 0 \\ 0 & 1/p \end{bmatrix}$$
$$= [\bar{g}, 0]\begin{bmatrix} w & 0 \\ 0 & 0 \end{bmatrix},$$

Premultiply the first by $[\bar{z}_1 \bar{z}_2]$, postmultiply the second by $[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}]$, add, and integrate from a to b:

$$[\bar{z}_1, \bar{z}_2]\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\bigg|_a^b = \int_a^b z_1 wf\, dx - \int_a^b \bar{g} w y_1\, dx$$
$$= (L_M y, z) - (y, L_M z).$$

The left side can be transformed as follows. Let $\bar{c}_v = b_v(\bar{z})$, $\bar{c}_u = b_u(\bar{z})$, the boundary Wronskians associated with $\bar{z}$. Then

$$\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}\begin{bmatrix} -\bar{c}_v \\ \bar{c}_u \end{bmatrix} = \begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{bmatrix}.$$

Since

$$\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \begin{bmatrix} b_v \\ -b_u \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

the limit as $x \to b$ can be written as

$$[\bar{c}_v, \ -\bar{c}_u] \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \begin{bmatrix} -b_v \\ b_u \end{bmatrix} (b) = [\bar{c}_v b_u - \bar{c}_u b_v] \, (b).$$

Likewise, at $x = a$, a similar limit can be found. If this is expressed as $(\bar{c}_v b_u - \bar{c}_u b_v) \, (a)$, then Green's formula can be expressed by

$$(L_M y, z) = (\bar{c}_v b_u - \bar{c}_u b_v) \, (b) - (\bar{c}_v b_u - \bar{c}_u b_v) \, (a).$$

Unfortunately, even though the terms on the right exist, the limits of individual terms do not necessarily exist. If only u is in $L^2(c, b; w)$ then $\bar{c}_u \to \bar{B}_u(z)$, $b_u \to B_u(y)$, but the other terms may be infinite. Further, the individual products $\bar{c}_v b_u$ and $\bar{c}_u b_v$ may become infinite, with cancellation between them occurring to allow $\bar{c}_v b_u - \bar{c}_u b_v$ to have a limit. Only when the problem is limit circle (at both ends) can the individual terms be separated. In order to overcome this difficulty, a different approach is needed.

III.5.a. Singular boundary value problems. In order to determine singular self-adjoint boundary value problems a new approach is required, that of finding a Green's function. Since the Green's function is a bounded operator, its properties are well known, and, through it, its unbounded counterpart, the differential operator, can be more readily attacked.

Let us choose u and $\tilde{u}$ to be solutions of $(py')' + qy = \lambda_0 wy$, which are analytic in λ_0, with u in $L^2(c, b; w)$ and $\tilde{u}$ in $L^2(a, c; w)$, $a < c < b$. In addition we assume that $b_u(\tilde{u}) = p(\tilde{u}u' - \tilde{u}'u) = 1$. Hence

$$b_{\tilde{u}}(u) = p(u\tilde{u}' - u'\tilde{u}) = -1.$$

DEFINITION. We denote, by D_L, those elements y in $L^2(a, b; w)$ satisfying:
1. $\ell y = (1/w) \, ((py')' + qy)$ exists a.e. and is in $L^2(a, b; w)$;
2. For all λ_0 for which u is in $L^2(c, b; w)$,

$$B_u(y) = \lim_{x \to b} b_u(y) = \lim_{x \to b} p(yu' - y'u) = 0;$$

and

3. For all λ_0 for which $\tilde{u}$ is in $L^2(a, c; w)$,

$$B_{\tilde{u}}(y) = \lim_{x \to a} b_{\tilde{u}}(y) = \lim_{x \to a} p(y\tilde{u}' - y'\tilde{u}) = 0.$$

DEFINITION. We define the operator L by setting

$$Ly = \ell y,$$

for all y in D_L.

This definition is in fact equivalent to that in III.e when the boundary conditions in III.3 are separated.

We emphasize that conditions 2 and 3 in the definition of D_L are to hold for all λ_0 for which they are defined. In particular this includes all complex λ_0. Finally we note that the limits involved do exist, no matter what limit case holds at a and b.

We now solve the problem

$$(L - \lambda_0)y = f.$$

That is, we assume that λ_0 is such that u is in $L^2(c, b; w)$, $\tilde{u}$ is in $L^2(a, c; w)$. Then we solve

$$(py')' + qy - \lambda_0 wy = wf,$$
$$B_u(y) = 0, \quad B_{\tilde{u}}(y) = 0.$$

Variation of parameters establishes that

$$y = u \int_c^x \tilde{u}wfd\xi - \tilde{u} \int_c^x uwfd\xi + Au + B\tilde{u},$$

for some c in (a, b). Then

$$b_u(y) = - \int_c^x uwfd\xi + B.$$

As $x \to b$, the $\lim_{x \to b} b_u(y)$ clearly exists. Setting this limit, $B_u(y) = 0$, we find

$$B = \int_c^b uwfd\xi.$$

Likewise

$$b_{\tilde{u}}(y) = - \int_c^x \tilde{u}wfd\xi - A.$$

As $x \to a$, the $\lim_{x \to a} b_{\tilde{u}}(y)$ clearly exists. Setting this limit, $B_{\tilde{u}}(y) = 0$, we find

$$A = - \int_c^a \tilde{u}wfd\xi.$$

Consequently,

$$y = u \int_a^x \tilde{u}wfd\xi + \tilde{u} \int_x^b uwfd\xi.$$

This formula is well known to yield an expression for y in $L^2(a, b; w)$ (see [15].).

If we define the Green's function $G(\lambda_0, x, \xi)$ by setting

$$G(\lambda_0, x, \xi) = u(x, \lambda_0)\tilde{u}(\xi, \lambda_0), \quad a \leqq \xi < x \leqq b$$
$$= u(\xi, \lambda_0)\tilde{u}(x, \lambda_0), \quad a \leqq x < \xi \leqq b,$$

then

$$y = \int_a^b G(\lambda_0, x, \xi)w(\xi)f(\xi)d\xi.$$

We have proved

THEOREM. *Let λ_0 be a complex number such that u is in $L^2(c, b; w)$, $\tilde{u}$ is in $L^2(a, c; w)$, and $p(\tilde{u}u' - \tilde{u}'u) = 1$. Then λ_0 is in the resolvent of L. If $(L - \lambda_0)y = f$, then*

$$y = \int_a^b G(\lambda_0, x, \xi)f(\xi)d\xi.$$

If $Rf = (L - \lambda_0)^{-1}f$, then it is easy to see that $R^*g = (L^* - \bar{\lambda}_0)^{-1}g$ is generated by $G(\bar{\lambda}_0, \xi, x)$, for

$$\int_a^b \overline{(R^*g)}f \, wdx = \langle f, R^*g \rangle$$

$$= \langle Rf, g \rangle$$

$$= \int_a^b \overline{g(x)} \left(\int_a^b G(\lambda_0, x, \xi)f(\xi)w(\xi)\,d\xi \right)w(x)\,dx$$

$$= \int_a^b \left(\int_a^b G(\bar{\lambda}_0, \xi, x)g(\xi)w(\xi)d\xi \right)^* f(x)w(x)dx.$$

Since f can be arbitrary in $L^2(a, b; w)$,

$$R^*g(x) = \int_a^b G(\bar{\lambda}_0, \xi, x)g(\xi)w(\xi)d\xi.$$

This proves

THEOREM. $((L - \lambda_0)^{-1})^* = (L - \bar{\lambda}_0)^{-1}$,

Taking inverses and cancelling $\bar{\lambda}_0$, we have

THEOREM. *L is self-adjoint. $L = L^*$.*

The situation now needs a bit of polishing. We shall show in the remaining part of this section that if the limit point case holds at either a or b, then the boundary condition required in the definition of D_L is automatically satisfied, and, hence, does not need to be stated. Second, if the limit circle case holds at either a or b, and if the boundary condition at a or b is satisfied for a particular choice of λ_0, then it is satisfied for all λ_0. Third, if the limit circle case holds at both a and b, then the boundary conditions can be made considerably more flexible. In particular, boundary conditions at a and b can be mixed together, just as in the regular prob-

lem. Indeed, the regular problem is a special example of a limit circle problem.

We first consider the limit point case.

THEOREM. *If the limit point case holds at a, then the boundary condition $B_{\tilde{u}}(y) = 0$ holds automatically for all λ_0, $\mathrm{Im}(\lambda_0) \neq 0$.*

If the limit point case holds at b, then the boundary condition $B_u(y) = 0$ holds for all λ_0, $\mathrm{Im}(\lambda_0) \neq 0$.

PROOF. If the limit point case holds at a, then $B_{\tilde{u}}(y) = 0$ is equivalent to requiring that y be in $L^2(a, c; w)$. Since this is assumed, $B_{\tilde{u}}(y) = 0$ automatically.

At b, the same argument can be applied.

A word is in order about how the L^2 solutions u and $\tilde{u}$ can be continued from one value of λ to another. Let u_0, $\tilde{u}_0$ be these solutions for $\lambda = \lambda_0$. For $\lambda \neq \lambda_0$, any solution of $(py')' = qy = \lambda wy$ satisfies $(py')' + (q - \lambda_0 w)y = (\lambda - \lambda_0)wy$ and, if it is in $L^2(c, b; w)$, it can be written as

$$y = (\lambda - \lambda_0)\left[u_0 \int_c^x \tilde{u}_0 wy\,d\xi + u_0 \int_x^b u_0 wy\,d\xi \right] + \alpha u_0 + \beta u_0.$$

(In the limit point case at b, $\beta = 0$.) We wish to consider the case $\alpha = 1$, $\beta = 0$. Then

$$y = (\lambda - \lambda_0)\left(u_0 \int_c^x \tilde{u}_0 wy\,d\xi + \tilde{u}_0 \int_x^b u_0 wy\,d\xi \right) + u_0$$

$$= (\lambda - \lambda_0)Gy + u_0,$$

where G is bounded on $L^2(c, b; w)$. If $|\lambda - \lambda_0|\,\|G\| < 1$, the sequence $y_0 = u_0, \ldots, y_n = Gy_{n-1} + u_0$ converges to a function u in $L^2(c, b; w)$. Furthermore, Riesz and Sz. Nagy [30] show that u can be extended analytically to all λ, $\mathrm{Im}(\lambda) \neq 0$. It is this solution we identify as "the" continuation of u_0.

THEOREM. *The function u defined above satisfies*:
 (i) $(pu')' + qy = \lambda u$, $\mathrm{Im}(\lambda) \neq 0$
 (ii) $u\,|_{\lambda=\lambda_0} = u_0$; *and*
 (iii) $\lim_{x \to b} p(u'u_0 - u_0'u) = 0$.

Only the last statement needs verification. Since u satisfies

$$u = (\lambda - \lambda_0)\left(u_0 \int_c^x \tilde{u}_0 wu\,d\xi + \tilde{u}_0 \int_x^b u_0 wu\,d\xi \right) + u_0,$$

a simple computation shows

$$p(u'u_0 - u_0'u) = - \int_0^x u_0 wy\,d\xi.$$

Hence, as $x \to b$, the limit is 0.

Continue $\tilde{u}$ in a similar manner. It satisfies

$$\tilde{u} = (\lambda - \lambda_0)\left(u_0 \int_a^x \tilde{u}_0 w \tilde{u} d\xi + \tilde{u}_0 \int_x^c u_0 w \tilde{u} d\xi\right) + \tilde{u}_0.$$

In the limit point cases these are the only L^2 solutions. In limit circle cases they are not. Nonetheless, even in the limit circle cases, the constraints placed on D_L by boundary conditions $B_{\tilde{u}}(y) = 0$ and $B_u(y) = 0$ can be considerably weakened.

THEOREM. Let y be in $L^2(c, b; w)$; let $\ell y = (1/w)\,((py')' + qy)$ be in $L^2(c, b; w)$ and satisfy $B_u(y) = 0$ for $\lambda = \lambda_0$. Then $B_u(y) = 0$, for all λ for which u is in $L^2(c, b; w)$.

Let y be in $L^2(a, c; w)$; let $\ell y = (1/w)\,((py')' + qy)$ be in $L^2(a, c; w)$ and satisfy $B_{\tilde{u}}(y) = 0$ for $\lambda = \lambda_0$. Then $B_{\tilde{u}}(y) = 0$ for all λ for which $\tilde{u}$ is in $L^2(a, c; w)$.

PROOF. Consider (c, b). Only the limit circle case needs to be considered. Now, y satisfies

$$(py')' + (q - \lambda_0 w)y = w(f - \lambda_0 y).$$

Consequently

$$y = u_0 \int_c^x \tilde{u}w(f - \lambda_0 y)d\xi + \tilde{u}_0 \int_x^b u_0 w(f - \lambda_0 y)d\xi + \alpha u_0.$$

There is no additional term $\beta\tilde{u}_0$ since $B_u(y) = 0$. Using the integral equation for u as well, a tedious computation shows

$$b_u(y) = -(\lambda - \lambda_0)\int_c^x \tilde{u}_0 w(f - \lambda_0 y)d\xi \int_x^b u_0 w u d\xi$$

$$+ (\lambda - \lambda_0)\int_x^b u_0 w(f - \lambda_0 y)d\xi \int_c^x \tilde{u}_0 w y d\xi$$

$$+ \int_x^b u_0 w(f - \lambda_0)d\xi - (\lambda - \lambda_0)\int_x^b u_0 w u d\xi.$$

As $x \to b$, all integrals remain finite, and the integrals from x to b vanish.

The situation at a is similar.

We summarize:

THEOREM. Let u and $\tilde{u}$ be solutions of $(py')' + qy = \lambda_0 wy$, u in $L^2(c, b; w)$, $\tilde{u}$ in $L^2(a, c; w)$. Let them be extended to all λ, $\mathrm{Im}(\lambda) \neq 0$. If the limit circle case holds at b, or a, respectively, let u and $\tilde{u}$ continue to denote extensions even for real λ. Let $p(\tilde{u}u' - \tilde{u}'u) = 1$. Then D_L consists of those elements y in $L^2(a, b; w)$ satisfying:

1. $\ell y = (1/w) ((py')' + qy)$ exists a.e. and is in $L^2(a, b; w)$;
2. If the limit circle case holds at b, then for any fixed λ_0,

$$B_u(y) = \lim_{x \to b} b_u(y) = \lim_{x \to b} p(yu' - y'u) = 0;$$

and

3. If the limit circle case holds at a, then for any fixed λ_0,

$$B_{\tilde{u}}(y) = \lim_{x \to a} b_{\tilde{u}}(y) = \lim_{x \to a} p(y\tilde{u}' - y'\tilde{u}) = 0.$$

If L is defined by setting $Ly = \ell y$ for all y in D_L, then L is self-adjoint.

In the limit circle cases $\lambda_0 = 0$ is often a convenient choice.

III.5.b. *Extensions*, I. If the limit circle case holds at a or b, or both, there are other self-adjoint operators in addition to those already found. Let us first consider what can happen if a is in the limit circle case and b is in the limit point case:

LEMMA. *Let b be in the limit point case, and let y, z be in D_M. Then*

$$\lim_{x \to b} p(y\bar{z}' - y'\bar{z}) = 0.$$

This implies no boundary condition is required at b. To see this, choose any boundary condition at a, $B_{\tilde{u}} = 0$. For y, z in D_M, modify them so they are 0 in a neighborhood of a. Then $B_{\tilde{u}}(y) = 0$, $B_{\tilde{u}}(z) = 0$, so y and z are in D_L. And L is self-adjoint. Therefore Green's formula shows

$$0 = \int_a^b (\bar{z}(L_M y) - (\overline{L_M z})y)w \, dx = \lim_{x \to b} p(\bar{z}y' - y\bar{z}').$$

Let us now examine Green's formula further. For y, z in D_M,

$$\int_a^b (\bar{z}(L_M y) - (\overline{L_M z})y)w \, dx = \lim_{x \to a} - (\bar{c}_{\tilde{v}} b_{\tilde{u}} - \bar{c}_{\tilde{u}} b_{\tilde{v}}),$$

where the terms on the right were defined in § III.4, the b's referring to y, the c's to z. The right side can be further expressed as

$$- [\overline{B_{\tilde{u}}(z)}, \overline{B_{\tilde{v}}(z)}] \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} B_{\tilde{u}}(y) \\ B_{\tilde{v}}(y) \end{bmatrix}.$$

Now let α, β, γ, δ be complex numbers such that $[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}]$ is non singular, and let its inverse be

$$\begin{bmatrix} -\varepsilon^* & -\eta^* \\ \zeta^* & \theta^* \end{bmatrix}.$$

Then

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -\varepsilon^* & -\eta^* \\ \zeta^* & \theta^* \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

If this is inserted above, the right side of Green's formula becomes

$$- (\zeta B_{\tilde{u}}(z) + \varepsilon B_{\tilde{v}}(z))^*(\alpha B_{\tilde{u}}(y) + \beta B_{\tilde{v}}(y))$$
$$- (\theta B_{\tilde{u}}(z) + \eta B_{\tilde{v}}(z))^*(\gamma B_{\tilde{u}}(y) + \delta B_{\tilde{v}}(y)).$$

If $\tilde{u}$, $\tilde{v}$ are real solutions to the real differential equation $(py')' + qy = \lambda_0 wy$, i.e., if λ_0 is real (It is often convenient to let $\lambda_0 = 0$), then $B_{\tilde{u}}(y)$ and $B_{\tilde{u}}(z)$, $B_{\tilde{v}}(y)$ and $B_{\tilde{v}}(z)$, respectively, represent the same boundary condition as $x \to a$. We pause to state

THEOREM. *If a is in the limit circle case, b is in the limit point case, and if $\tilde{u}$, $\tilde{v}$ are real linearly independent solutions to the real differential equation*

$$(py')' + qy = \lambda_0 wy, \quad (\lambda_0 \text{ real})$$

with $p(\tilde{u}\tilde{v}' - \tilde{u}'\tilde{v}) = 1$, then Green's formula is

$$\int_a^b (\bar{z}(L_M y) - (\overline{L_M z})y)w \, dx =$$

$$- (\zeta B_{\tilde{u}}(z) + \varepsilon B_{\tilde{v}}(z))^*(\alpha B_{\tilde{u}}(y) + \beta B_{\tilde{v}}(y))$$
$$- (\theta B_{\tilde{u}}(z) + \eta B_{\tilde{v}}(z))^*(\gamma B_{\tilde{u}}(y) + \delta B_{\tilde{v}}(y)),$$

where $B_{\tilde{u}}(y)$, $B_{\tilde{v}}(y)$, $B_{\tilde{u}}(z)$, $B_{\tilde{v}}(z)$ are the four boundary values for y and z, respectively, which exist at $x = a$.

DEFINITION. We denote by $D_{\mathscr{L}}$ those elements y in $L^2(a, b; w)$ satisfying:
1. $\ell y = (1/w) ((py')' + qy)$ exists a.e. and is in $L^2(a, b; w)$; *and*
2. For fixed λ_0, with $\tilde{u}$ and $\tilde{v}$ real solutions of $(pu')' + qu = \lambda_0 wu$,

$$\alpha B_{\tilde{u}}(y) + \beta B_{\tilde{v}}(y) = 0.$$

DEFINITION. We define the operator $\mathscr{L}$ by setting

$$\mathscr{L} y = \ell y$$

for all y in $D_{\mathscr{L}}$.

THEOREM. *The domain of $\mathscr{L}^*$, $D_{\mathscr{L}^*}$, consists of those elements z satisfying, for all z in $D_{\mathscr{L}^*}$, $\mathscr{L}^* z = \ell z$:*
1. $\ell z = (1/w) (pz')' + qz)$ *exists a.e. and is in $L^2(a, b; w)$; and*
2. $\theta B_{\tilde{u}}(z) + \eta B_{\tilde{v}}(z) = 0.$

The proof is elementary. Since $\mathscr{L}^* \subset L_M$, the form of $\mathscr{L}^*$ is already known. In Green's formula, since $\alpha B_{\tilde{u}}(y) + \beta B_{\tilde{v}}(y) = 0$, and $\gamma B_{\tilde{u}}(y) + \delta B_{\tilde{v}}(y)$ is arbitrary, it follows that $\theta B_{\tilde{u}}(z) + \eta B_{\tilde{v}}(z)$ must also vanish. This

shows that $\mathscr{L}^*$ is a restriction of the operator with domain $D_{\mathscr{L}^*}$ described above. The converse is trivial.

COROLLARY. $\mathscr{L}$ is self-adjoint if and only if $\beta\alpha^* = \alpha\beta^*$.

PROOF. If $[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}]$ and $[\begin{smallmatrix} \bar{\zeta}^* & -\eta^* \\ \zeta^* & \theta^* \end{smallmatrix}]$ are mulitplied together, then $-\alpha\eta^* + \beta\theta^* = 0$. Further, if $\mathscr{L}$ is self-adjoint, there must be a nonzero k such that $\theta = k\alpha$, $\eta = k\beta$. Inserting these shows that $\beta\alpha^* = \alpha\beta^*$.

Conversely, if $\beta\alpha^* = \alpha\beta^*$, we find when comparing it to $\beta\theta^* = \alpha\eta^*$, that there is a nonzero number k such that $\alpha = k\theta$, $\beta = k\eta$. Hence $\theta\, B_{\tilde{u}}(z) + \eta B_{\tilde{v}}(z) = 0$ is equivalent to $\alpha B_{\tilde{u}}(z) + \beta B_{\tilde{v}}(z) = 0$. Thus $D_{\mathscr{L}^*} = D_{\mathscr{L}}$. Since $\mathscr{L}$ and $\mathscr{L}^*$ have the same form, $\mathscr{L}$ is self-adjoint.

Of course one might point out that $\alpha B_{\tilde{u}}(y) + \beta B_{\tilde{v}}(y) = B_{\alpha\tilde{u}+\beta\tilde{v}}(y)$, so all that has really been accomplished is to choose a new $\tilde{u}$. This is true, but by varying α and β, the full range of self-adjoint operators is transparent.

Further, since α can be chosen real, the corollary states that $\mathscr{L}$ is self-adjoint if and only if β can be made real as well.

Finally, we note that if a is limit point, b is limit circle, the situation is virtually the same. Replacement of x by $-x$ is all that is required.

III.5.c. *Extensions*, II. If both endpoints a and b are limit circle, then boundary conditions at both ends are required. In fact the situation is more like the regular case than anything else. Since the limits in Green's formula exhibited at the end of § III.4 all individually exist, we have

THEOREM. *If both a and b are in the limit circle case, if $\tilde{u}$, $\tilde{v}$ are real linearly independent solutions to the real differential equation*

$$(py')' + qy = \lambda_0 wy, \quad (\lambda_0 \text{ real})$$

with $p(\tilde{u}\tilde{v}' - \tilde{u}'\tilde{v}) = 1$, if u, v are real linearly independent solutions to the real differential equation

$$(py')' + qy = \lambda_1 wy, \quad (\lambda_1 \text{ real})$$

with $p(uv' - u'v) = 1$, and if, as in §III.3, A and B are $m \times 2$ matrices, C and D are $(4 - m) \times 2$ matrices such that $[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}]$ is non singular, with inverse $[\begin{smallmatrix} \tilde{A}^* & \tilde{C}^* \\ \tilde{B}^* & \tilde{D}^* \end{smallmatrix}]$, then Green's formula is

$$\int_a^b (\bar{z}(L_M y) - (L_M z))w\,dx = (\tilde{A}\tilde{\mathscr{B}}(z) + \tilde{B}\mathscr{B}(z))^*(A\tilde{\mathscr{B}}(y) + B\mathscr{B}(y))$$
$$+ (\tilde{C}\tilde{\mathscr{B}}(z) + \tilde{D}\mathscr{B}(z))^*(C\tilde{\mathscr{B}}(y) + D\mathscr{B}(y)),$$

where

$$\tilde{\mathscr{B}}(z) = \begin{bmatrix} B_{\tilde{u}}(z) \\ B_{\tilde{v}}(z) \end{bmatrix}, \quad \mathscr{B}(z) = \begin{bmatrix} B_u(z) \\ B_v(z) \end{bmatrix}, \quad \tilde{\mathscr{B}}(y) = \begin{bmatrix} B_{\tilde{u}}(y) \\ B_{\tilde{v}}(y) \end{bmatrix}, \quad \tilde{\mathscr{B}}(y) = \begin{bmatrix} B_u(y) \\ B_v(y) \end{bmatrix}.$$

PROOF. From § III.4, the right side of Green's formula is

$$[\tilde{\mathscr{B}}(z)^*, \mathscr{B}(z)^*]\begin{bmatrix} -J & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} \tilde{\mathscr{B}}(y) \\ \mathscr{B}(y) \end{bmatrix},$$

where $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. If this is expanded as in § III.3, the formula immediately follows.

DEFINITION. We denote by D_m those elements y in $L^2(a, b; w)$ satisfying:
1. $\ell y = (1/w)((py')' + qy)$ exists a.e. and is in $L^2(a, b; w)$; and
2. $A\tilde{\mathscr{B}}(y) + B\mathscr{B}(y) = 0$.

DEFINITION. We define the operator m by setting

$$m y = \ell y$$

for all y in D_m.

THEOREM. *The domain of m^*, D_{m^*}, consists of those elements z satisfying*:
1. $\ell z = (1/w)((pz')' + qz)$ *exists a.e. and is in $L^2(a, b; w)$; and*
2. $\bar{C}\tilde{\mathscr{B}}(z) + \bar{D}\mathscr{B}(z) = 0$.
For all z in D_{m^*}, $m^*z = \ell z$.

The proof is similar to those given before.

COROLLARY. *The operator m is self-adjoint if and only if A and B are 2×2 matrices and*

$$AJ^{-1}A^* = BJ^{-1}B^*.$$

There are parametric forms for the boundary conditions here too. They are almost the same as in the regular case. The terms $Y(a)$, $Y(b)$, $Z(a)$, $Z(b)$ are replaced by $\tilde{\mathscr{B}}(y)$, $\mathscr{B}(y)$, $\tilde{\mathscr{B}}(z)$, $\mathscr{B}(z)$.

We note in closing this section that regular ends may be thought of as limit circle cases. If a is a regular point, the boundary functional $y(a)$ may be generated by choosing $\tilde{u}$ so that it satisfies the initial conditions $\tilde{u}(a) = 0$, $\tilde{u}'(a) = 1/p(a)$. Then

$$B_{\tilde{u}}(y) = \lim_{x \to a} p(y\tilde{u}' - y'\tilde{u}) = y(a).$$

If $\bar{v}$ is chosen so that $\bar{v}(a) = -1$, $\bar{v}'(a) = 0$, then

$$B_{\bar{v}}(y) = \lim_{x \to u} p(y\bar{v}' - y'\bar{v}) = p(a)y'(a).$$

Since $p(a) \neq 0$, in the regular case $B_{\bar{v}}(y)/p(a) = y'(a)$.

Likewise if b is a regular point, the boundary functional $y(b)$ may be generated by choosing u so that it satisfies $u(b) = 0$, $u'(b) = 1/p(b)$. Then

$$B_u(y) = \lim_{x \to b} p(yu' - y'u) = y(b).$$

If v is chosen so that $v(b) = -1$, $v'(b) = 0$, then

$$B_v(y) = \lim_{x \to b} p(yv' - y'v) = p(b)v'(b).$$

Since $p(b) \neq 0$ in the regular case $B_v(y)/p(b) = y'(b)$.

III.5.d. *Other boundary constraints.* It is possible to use functions other than solutions to

$$(py')' + qy = \lambda wy$$

to generate boundary conditions at singular ends, but nothing new is found. Let us consider the singular end at b, with r a fixed element in D_M. Let y be any other element in D_M. Then

$$R(y) = \lim_{x \to b} p(y\bar{r}' - y'\bar{r})$$

exists by Green's formula.

THEOREM. *If the limit point case holds at b, then $R(y) = 0$.*

PROOF. Let boundary constraints be imposed at a so the operator L is self-adjoint, and let y and r be modified so they are in D_L. Then Green's formula shows

$$0 = \int_a^b (\bar{r}(Ly) - \overline{(Lr)}y)wdx = R(y).$$

THEOREM. *If the limit circle case holds at b, then there exist constants α and β such that*

$$R(y) = \alpha B_u(y) + \beta B_v(y).$$

PROOF. Let $\angle r = f$. Then

$$r = u \int_a^x \tilde{u}wfd\xi + \tilde{u} \int_x^b uwfd\xi + c_1\tilde{u} + c_2u$$

(see § III.5.a), where $c_2 = 0$ if a is limit point. Then

$$p(yr' - y'r) = p(yu' - y'u) \int_a^x \tilde{u}wfd\xi$$

$$+ p(y\tilde{u}' - y'\tilde{u}) \int_x^b uwfd\xi + c_1p(y\tilde{u}' - y'\tilde{u}) + c_2p(yu' - y'u).$$

As $x \to b$, all the term have finite limits and

$$R(y) = B_u(y) \int_a^b \tilde{u}wfd\xi + c_1 \lim_{x \to b} p(y\tilde{u}' - y'\tilde{u}) + c_2B_u(y).$$

Since $\tilde{u} = c_3 u + c_4 v$,

$$R(y) = B_u(y)\left(\int_a^b \tilde{u}wf\,d\xi + c_1 c_3 + c_2\right) + B_v(y)(c_1 c_4).$$

Letting α and β represent the terms in the brackets completes the proof. At a the situation is similar.

III.6. Examples. By far the best examples of singular boundary problems, indeed one of the main reasons for considering singular boundary value problems, are the problems involving the classic orthogonal polynomials. These, together with the Bessel function problems, are the main reason for the study of singular boundary value problems. We give eight examples:

(a) The Jacobi polynomials, with their special cases the Legendre and Tchebycheff polynomials;

(b) The ordinary and generalized Laguerre polynomials;

(c) The Hermite polynomials; and

(d) Bessel functions.

The Bessel polynomials are not included because a suitable classical weight function and interval has not been found. While it is known that one exists on $[0, \infty)$ as a measure of bounded variation, it cannot be a positive measure. The resulting problem, therefore, is indefinite.

III.7.a. The Jacobi boundary value problem. The Jacobi differential operator is

$$Ly = (1 - x)^{-\alpha}(1 + x)^{-\beta}((1 - x)^{1+\alpha}(1 + x)^{1+\beta}y')',$$

set in $L^2(-1, 1; (1 - x)^{\alpha}(1 + x)^{\beta})$. 1 is in the limit circle case when $-1 < \alpha < 1$. It is limit point when $1 \leq \alpha$. -1 is in the limit circle case when $-1 < \beta < 1$. It is limit circle case when $1 \leq \beta$. Two solutions to $Ly = 0$ are $u = \tilde{u} = 1$ and $v = \tilde{v} = Q_0^{(\alpha, \beta)}(x)$. Boundary conditions, therefore are given by

$$B_u(y) = -\lim_{x \to 1} (1 - x)^{1+\alpha}(1 + x)^{1+\beta}y'(x)$$

$$B_v(y) = \lim_{x \to 1} (1 - x)^{1+\alpha}(1 + x)^{1+\beta}(Q_0^{(\alpha, \beta)\,'}(x)y(x) - Q_0^{(\alpha, \beta)}(x)y'(x))$$

$$B_{\tilde{u}}(y) = -\lim_{x \to -1} (1 - x)^{1+\alpha}(1 + x)^{1+\beta}y'(x)$$

$$B_{\tilde{v}}(y) = \lim_{x \to -1} (1 - x)^{1+\alpha}(1 + x)^{1+\beta}(Q_0^{(\alpha, \beta)\,'}(x)y(x) - Q_0^{(\alpha, \beta)}(x)y'(x))$$

when they are needed.

Boundary value problems involving the Jacobi differential operator which are self-adjoint fall into the category discussed in §III.5.c. The boundary value problem with the Jacobi polynomials as eigenfunctions is

$$Ly = (1-x)^{-\alpha}(1+x)^{-\beta}((1-x)^{1+\alpha}(1+x)^{1+\beta}y')' = -n(n+\alpha+\beta+1)y,$$

$$B_u(y) = -\lim_{x\to 1}(1-x)^{1+\alpha}(1+x)^{1+\beta}y'(x) = 0, \quad -1 < \alpha < 1,$$

$$B_{\tilde{u}}(y) = -\lim_{x\to -1}(1-x)^{1+\alpha}(1+x)^{1+\beta}y'(x) = 0, \quad -1 < \beta < 1.$$

III.7.b. *The Legendre boundary value problem.* In the Jacobi problem, if $\alpha = \beta = 0$, we find immediately the Legendre boundary value problem. The Legendre operator is

$$Ly = ((1 - x^2)y')'$$

in $L^2(-1, 1;1)$. Both -1 and 1 are in the limit circle. Two independent solutions to $((1 - x^2)y')' = 0$ are $u = \tilde{u} = 1$ and $v = \tilde{v} = 1/2 \ln((1 + x)/(1 - x))$. Both are in $L^2(-1, 1;1)$.

Boundary conditions are

$$B_u(y) = -\lim_{x\to 1}(1 - x^2)y'(x)$$

$$B_v(y) = \lim_{x\to 1}(1 - x^2)\left((1 - x^2)^{-1}y(x) - \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)y'(x)\right)$$

$$B_{\tilde{u}}(y) = -\lim_{x\to -1}(1 - x^2)y'(x)$$

$$B_{\tilde{v}}(y) = \lim_{x\to -1}(1 - x^2)\left((1 - x^2)^{-1}y(x) - \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)y'(x)\right).$$

The boundary value problem which has the Legendre polynomials as eigenfunctions is

$$Ly = ((1 - x^2)y')' = -n(n + 1)y,$$

$$B_u(y) = -\lim_{x\to 1}(1 - x^2)y'(x) = 0,$$

$$B_{\tilde{u}}(y) = -\lim_{x\to -1}(1 - x^2)y'(x) = 0.$$

There are other self-adjoint problems involving L with mixed boundary conditions, such as those described in § III.5.c.

III.7.c. *The Tchebycheff problem of the first kind.* In the Jacobi problem, if $\alpha = \beta = -1/2$, we find the first Tchebycheff boundary value problem. Set in $L^2(-1, 1; (1 - x^2)^{-1/2})$ the Tchebycheff differential operator is

$$Ly = (1 - x^2)^{1/2}((1 - x^2)^{1/2} y')'.$$

Both -1 and 1 are limit circle. Two independent solutions to $((1 - x^2)^{-1/2}y')' = 0$ are $u = \tilde{u} = 1$ and $v = \tilde{v} = \sin^{-1}x$. Both are in $L^2(-1, 1, (1 - x^2)^{-1/2})$.

Boundary conditions are

$$B_u(y) = -\lim_{x \to 1} (1 - x^2)^{1/2} y'(x)$$

$$B_v(y) = \lim_{x \to 1} (1 - x^2)^{1/2}((1 - x^2)^{-1/2} y(x) - (\sin^{-1}x)y'(x))$$

$$B_{\tilde{u}}(y) = -\lim_{x \to -1} (1 - x^2)^{1/2}y'(x)$$

$$B_{\tilde{v}}(y) = \lim_{x \to -1} (1 - x^2)^{1/2}((1 - x^2)^{-1/2} y(x) - (\sin^{-1}x)y'(x)).$$

The boundary value problem which has the Tchebycheff polynomials of the first kind as eigenfunctions is

$$Ly = (1 - x^2)^{1/2}((1 - x^2)^{1/2}y')' = -n^2 y,$$

$$B_u(y) = -\lim_{x \to 1} (1 - x^2)^{1/2}y'(x) = 0,$$

$$B_{\tilde{u}}(y) = -\lim_{x \to -1} (1 - x^2)^{1/2}y'(x) = 0.$$

Again there are other self-adjoint problems involving the first Tchebycheff operator.

III.7.d. *The Tchebycheff problem of the second kind.* In the Jacobi problem, if $\alpha = \beta = 1/2$, we find the second Tchebycheff boundary value problem. The Tchebycheff differential operator is

$$Ly = (1 - x^2)^{-1/2}((1 - x^2)^{3/2}y')'.$$

Both -1 and 1 are limit circle. Two independent solutions to $((1 - x^2)^{3/2} y')' = 0$ are $u = \tilde{u} = 1$ and $v = \tilde{v} = (x/(1 - x^2)^{1/2})$. Both are in $L^2(-1, 1; (1 - x^2)^{1/2})$.
Boundary conditions are

$$B_u(y) = \lim_{x \to 1} (1 - x^2)^{3/2}y'(x)$$

$$B_v(y) = \lim_{x \to 1} (1 - x^2)^{3/2}((1 - x^2)^{-3/2} y(x) - (x/(1 - x^2)^{1/2})y'(x))$$

$$B_{\tilde{u}}(y) = -\lim_{x \to -1} (1 - x^2)^{3/2}y'(x)$$

$$B_{\tilde{v}}(y) = \lim_{x \to -1} (1 - x^2)^{3/2}((1 - x^2)^{-3/2} y(x) - (x/(1 - x^2)^{1/2})y'(x)).$$

The boundary value problem which has the Tchebycheff polynomials of the second kind as eigenfunctions is

$$Ly = (1 - x^2)^{-1/2}((1 - x^2)^{3/2}y')' = -n(n + 2)y$$

$$B_u(y) = -\lim_{x \to 1} (1 - x^2)^{3/2}y'(x) = 0$$

$$B_{\tilde{u}}(y) = -\lim_{x \to -1} (1 - x^2)^{3/2}y'(x) = 0.$$

Again there are other self-adjoint problems involving the second Tchebycheff operator.

III.8.a. The generalized Laguerre boundary value problem. The generalized Laguerre differential operator is

$$Ly = x^{-\alpha}e^x(x^{\alpha+1}e^{-x}y')',$$

set in $L^2(0, \infty; x^\alpha e^{-x})$. The point 0 is in the limit circle case if $-1 < \alpha < 1$ and it is limit point if $1 \leq \alpha$, while the point ∞ is always limit point.

The boundary conditions at 0, $-1 < \alpha < 1$ are given by choosing $\tilde{u} = 1$ and

$$\tilde{v} = \int_1^x \frac{e^\xi}{\xi^{\alpha+1}} \, d\xi$$

as independent solutions to $Ly = 0$. Then

$$B_{\tilde{u}}(y) = -\lim_{x \to 0} x^{\alpha+1}e^{-x}y'(x)$$

$$B_{\tilde{v}}(y) = \lim_{x \to 0} x^{\alpha+1}e^{-x}\left(\frac{e^x}{x^{\alpha+1}} y(x) - \int_1^x \frac{e^\xi}{\xi^{\alpha+1}} \, d\xi \, y'(x)\right).$$

Self-adjoint boundary value problems fall into the category discussed in § III.5.b. The boundary condition satisfied by the Laguerre polynomials is $B_{\tilde{u}}(y) = 0$. Hence the boundary value problem with the generalized Laguerre polynomial as eigenfunctions is

$$Ly = x^{-\alpha}e^x(x^{\alpha+1}e^{-x}y')' = -ny,$$

$$B_{\tilde{u}}(y) = -\lim_{x \to 0} x^{\alpha+1}e^{-x}y'(x) = 0, \text{ (required when } -1 < \alpha < 1).$$

III.8.b. *The ordinary Laguerre boundary value problem.* The ordinary Laguerre differential operator is

$$Ly = e^x(xe^{-x}y')',$$

set in $L^2(0, \infty; e^{-x})$. 0 is the limit circle case; ∞ is in the limit point case.

The boundary conditions at 0 are given by choosing $\tilde{u} = 1$ and

$$\tilde{v} = \int_1^x \frac{e^\xi}{\xi} \, d\xi$$

as independent solutions to $Ly = 0$. Then

$$B_{\tilde{u}}(y) = -\lim_{x \to 0} xe^{-x}y'(x)$$

$$B_{\tilde{v}}(y) = \lim_{x \to 0} xe^{-x}\left(\frac{e^x}{x} y(x) - \int_1^x \frac{e^\xi}{\xi} \, d\xi \, y'(x)\right).$$

Self-adjoint boundary value problems fall into the category discussed in

§ III.5.b. The boundary condition satisfied by the Laguerre polynomials is $B_{\tilde{u}}(y) = 0$. Hence the boundary value problem with the ordinary Laguerre polynomials as eigenfunctions is

$$Ly = e^x(xe^{-x}y')' = -ny$$

$$B_{\tilde{u}}(y) = -\lim_{x \to 0} xe^{-x}y'(x) = 0.$$

III.9. The Hermite boundary value problem. The Hermite differential operator is

$$Ly = e^{x^2}(e^{-x^2}y')',$$

set in $L^2(-\infty, \infty; e^{-x^2})$. Both $-\infty$ and ∞ are in the limit point case, and so no boundary conditions are required. The conditions $\lim_{x \to \pm\infty} e^{-x^2}y'(x) = 0$ are automatically satisfied. Consequently the boundary value problem with the Hermite polynomials as eigenfunctions is

$$Ly = e^{x^2}(e^{-x^2}y')' = ny, \quad -\infty < x < \infty.$$

III.10. Bessel functions. Although they are not polynomials, we include a brief discussion of the Bessel boundary value problems because of their importance. There are four different problems encountered. They differ over the intervals involved.

(1) If the interval is $[a, b]$, $0 < a < b < \infty$, the problem is regular. The problem is like that discussed in § III.3.

(2) If the interval is $[0, b]$, $0 < b < \infty$, the problem is singular at 0, regular at b. The problem is like those discussed in § III.5.b or § III.5.c. if the regular end at b is treated as a limit circle case.

(3) If the interval is $[a, \infty)$, $0 < a < \infty$, the problem is regular at a, limit point at ∞. If the regular end is treated as a limit circle case, the problem falls into the category treated in § III.5.c.

(4) If the interval is $[0, \infty)$, the problem is singular at both ends. It falls into either the category of § III.5.a or III.5.b, depending upon what happens at 0.

The Bessel differential operator is

$$Ly = x^{-1}((xy')' - (n^2/x)y)$$

set ln $L^2(a, b; x)$. Two convenient solutions of $Ly = 0$ for finite a and/or b are $u = \tilde{u} = x^n$ and $v = \tilde{v} = x^{-n}$, $n \neq 0$, and $u = \tilde{u} = 1$, $v = \tilde{v} = \log x$ if $n = 0$.

If $a = 0$, it is easy to see that the endpoint 0 is in the limit circle case only if $|n| < 1$, while the limit point case holds when $|n| \geq 1$. Assuming $|n| < 1$ the boundary conditions at 0 are

$$B_{\bar{u}}(y) = \lim_{x \to 0} x(nx^{n-1}y(x) - x^n y'(x))$$

$$B_{\bar{v}}(y) = \lim_{x \to 0} x(-nx^{-n-1}y(x) - x^{-1}y'(x))$$

if $n \neq 0$.

If $n = 0$,

$$B_{\bar{u}}(y) = -\lim_{x \to 0} xy'(x)$$

$$B_{\bar{v}}(y) = \lim_{x \to 0} x(x^{-1}y(x) - \log xy'(x)).$$

The boundary condition traditionally associated with Bessel functions of the first kind is

$$B_{\bar{u}}(y) = \lim_{x \to 0} x(nx^{n-1}y(x) - x^n y'(x))$$

when $n \neq 0$, or

$$B_{\bar{u}}(y) = \lim_{x \to 0} xy'(x) = 0,$$

when $n = 0$.

At a regular point, the conditions involving $y(a)$ and $y'(a)$ or $y(b)$ and $y'(b)$ are preferred.

At ∞ the limit point case always holds, and so no boundary condition is required.

IV.1. Indefinite problems. The astute reader might have noticed some gaps in what has been described so far. The Jacobi boundary value problem was discussed only when α, $\beta > -1$. The generalized Laguerre boundary value problem assumed that $\alpha > -1$. These are artificial constraints which have existed for over 100 years.

Furthermore there was no discussion of the Bessel polynomial operator's boundary value problem.

There are several questions which arise. First, what happens to the polynomials in the Jacobi case when α and/or $\beta < -1$? Second, what is the singular Sturm-Liouville problem like in these circumstances? Third, is there a relation between them? (See [27].)

The same questions may be asked of the generalized Laguerre polynomials. First, what happens to the polynomials when $\alpha < -1$? Second, what is the singular Sturm-Liouville problem like? Third, is there a relation between them? (See [16], [17]).

For the Bessel polynomials we would like to know if there is any boundary value problem for which they are eigenfunctions? Second, what does the Sturm-Liouville problem look like over $[0, b]$, $[a, \infty)$, $[0, \infty)$? Is there any relation between them? (See [27].)

Frankly, little is known. The Jacobi polynomials have only been glanced

at with very incomplete results. The orthogonalizing functional w is known, but little else.

More is known about the generalized Laguerre polynomials. Their boundary value problem lies in an indefinite inner product space. There are still some gaps in the theory. Nothing is known about the Laguerre Sturm-Liouville value problem on $[0, \infty)$, $\alpha < -1$. This has been held for a graduate student's dissertation for several years. There are no results yet. There has obviously been no connection made.

For the Bessel polynomials nothing is known. Only the δ-function expansion of the orthogonalizing weight function and a closely related formula for it have been found. Because of the alternating signs in the moments, any boundary value problem concerning the polynomials must be indefinite.

We shall try to describe what we do know.

IV.2. The indefinite Jacobi problem. When either α or $\beta < -1$, the Jacobi weight function

$$w = (1 - x)^\alpha (1 + x)^\beta, \quad -1 \leqq x \leqq 1$$
$$= 0, \qquad\qquad , \quad \text{otherwise}$$

is no longer integrable. As a consequence the moments $\{\mu_n\}_{n=0}^\infty$ used to define the Jacobi polynomials do not satisfy

$$\mu_n = \int_{-\infty}^\infty x^n w \, dx, \quad n = 0, 1, \ldots .$$

They must be found by means of the recurrence relation exhibited in §II.2.a, and there are immediate problems if $\alpha = -1, -2, \ldots$ or $\beta = -1$, $-2, \ldots$, or if $\alpha + \beta = -2, -3, \ldots$. Nonetheless, if these values of α, β, $\alpha + \beta$ are avoided, the polynomials are well defined and are orthogonal with respect to the complicated regularization of the classical weight function, which is exhibited in § II.2.a. The inner product

$$(f, g) = \langle w, f\bar{g} \rangle$$

generates an indefinite inner product space [5] which has not yet been seriously studied to date.

Likewise the singular boundary value problem associated with

$$Ly = (1 - x)^{-\alpha}(1 + x)^{-\beta}((1 - x)^{1+\alpha}(1 + x)^{1+\beta}y')'$$

set in $L^2(-1, 1; (1 - x)^\alpha(1 + x)^\beta)$ has not been studied either when α and/or $\beta < -1$.

IV.3. The indefinite Laguerre problem. If $\alpha < -1$, the orthogonalizing functional for the generalized Laguerre polynomials is

$$(f, g) = \frac{1}{\Gamma(\alpha + 1)} \int_0^\infty x^\alpha \Big(e^{-x} f(x)\overline{g(x)}$$

$$- \sum_{\ell=0}^{j-1} \Big\{ \sum_{k=\ell}^{j-1} \frac{(-1)^k x^k}{\ell!(k-\ell)!} \Big\} (-1)^\ell (f\bar g)^{(\ell)}(0) \Big) dx,$$

where $-j - 1 < \alpha < -j$. This can be expressed as

$$(f, g) = \frac{1}{\Gamma(\alpha + 1)} \int_0^\infty x^\alpha \Big(e^{-x} f(x)\overline{g(x)} - \sum_{\ell=0}^{j-1} (e^{-x} f\bar g)^{(\ell)}(0) \frac{x^\ell}{\ell!} \Big) dx,$$

or, if integration by parts is performed,

$$(f, g) = \frac{(-1)^j}{\Gamma(\alpha + j + 1)} \int_0^\infty x^{\alpha+j} (e^{-x} f\bar g)^{(j)} dx.$$

If $j = 0$, this reduces to the classic formula.

If $F = (f, f', \ldots, f^{(j)})^T$, $G = (g, g', \ldots, g^{(j)})^T$ and $A = (a_{\ell m})$, where

$$a_{\ell m} = j!(-1)^{\ell+m}/(j - \ell - m + 2)!(\ell - 1)!(m - 1)!, \quad \ell + m \leqq j + 2$$
$$= 0, \qquad \qquad \qquad \qquad \qquad \qquad \qquad \ell + m > j + 2,$$

then,

$$(f, g) = \frac{(-1)^j}{\Gamma(\alpha + j + 1)} \int_0^\infty x^{\alpha+j} e^{-x} G^* A F \, dx.$$

Setting f and g in $H^{(j)}([0, \infty); x^{\alpha+j} e^{-x})$, we find that

$$|(f, f)| < C \int_0^\infty x^{\alpha+j} e^{-x} \sum_{\ell=0}^j |f^{(\ell)}(x)|^2 \, dx \leqq C \{f, f\},$$

where $\{\cdot, \cdot\}$ denotes the norm in $H^{(j)}([0, \infty); x^{\alpha+j} e^{-x})$. Thus, for elements in $H^{(j)}$, the expression (f, f) is finite. The expression $\{\cdot, \cdot\}$ is called a Hilbert majorant for $(\cdot, \cdot)$. The matrix A is the Gram operator of $(\cdot, \cdot)$ with respect to $\{\cdot, \cdot\}$ [5; pp. 77, 89]. Hence $(\cdot, \cdot)$ generates an indefinite inner product space H whose elements are precisely those elements in $H^{(j)}([0, \infty); x^{\alpha+j} e^{-x})$.

It can be shown that H can be decomposed into

$$H = H^+ \oplus H^-,$$

where H^+ is the linear space spanned by the Laguerre polynomials satisfying $(f, f) > 0$, and H^- is the linear space spanned by the Laguerre polynomials satisfying $(f, f) < 0$. Further every element f in H can be expanded in terms of the Laguerre polynomials,

$$f(x) = \sum_{n=0}^\infty \frac{(f, L_n^{(\alpha)})}{n!(\alpha + 1)_n} L_n^{(\alpha)}(x),$$

where $(\alpha + 1)_n = (\alpha + 1) \ldots (\alpha + n)$.

Green's formula exists, but is more complicated. Let ℓ be given by

$$\ell y = -xy'' + (1 + \alpha - x)y' = -(x^{\alpha+1}e^{-x}y')'/x^{\alpha}e^{-x}.$$

Then

$$(\ell y, z) - (y, \ell z) = -\lim_{x \to 0} (x^{\alpha+j+1}(e^{-x}W)^{(j)}),$$

where $W = y'\bar{z} - y\bar{z}'$. In order for a differential operator generated by ℓ to be self-adjoint, the right side of Green's formula must be made to vanish by imposing boundary conditions in a symmetric manner. Set

$$B_i(y) = \lim_{x \to 0} x^{\alpha+j+1}(e^{-x}W(y, L_{i-1}^{(\alpha)})^{(j)},$$

and define $\mathscr{D}$ by

DEFINITION. We denote by $\mathscr{D}$ those elements y in H satisfying $B_i(y) = 0$, $i = 1, \ldots, m$, where $m = (j + 2)/2$ if j is even, $m = (j + 1)/2$ if j is odd.

DEFINITION. We denote by $\mathscr{L}$ the operator defined by setting $\mathscr{L}y = \ell y$ for all y in $\mathscr{D}$.

The following theorem can then be proved [16, 17].

THEOREM. *The operator $\mathscr{L}$ is self-adjoint.*

There is more work that needs to be done here, especially on the boundary conditions.

IV.4. The indefinite Bessel polymonial problem. Almost nothing is known for the Bessel polynomial problem. Although R. Boas showed [3] that there is a function of bounded variation on $[0, \infty]$ which acts as a weight function, it has not been found.

Perhaps the best we can say is that the δ-function expansion does serve to make the polynomials orthogonal. The square of the norms generated by it alternate in sign, so the boundary value problem, whatever it is, is indefinite.

There are two additional ways to orthogonalize the polynomials. The function $z^{a-2}e^{-b/z}$ over the unit circle in the complex plane is a complex weight function [23]. Related to it, the measure ψ defined by

$$\psi(\beta) - \psi(\alpha) = \lim_{\varepsilon \to 0} -\frac{1}{\pi} \int_{\alpha}^{\beta} \text{Im}(-b/z)_1F_1(1, a, -b/z)dx, \quad z = x + i\varepsilon,$$

also renders the polynomials orthogonal (see [18]). As noted earlier, when $a = b = 2$,

$$\psi(\beta) - \psi(\alpha) = \lim_{\varepsilon \to 0} -\frac{1}{\pi} \int_{\alpha}^{\beta} \exp\left(\frac{-2x}{x^2 + \varepsilon^2}\right) \sin\left(\frac{2\varepsilon}{x^2 + \varepsilon^2}\right) dx.$$

When $a = b = 1$,

$\psi(\beta) - \psi(\alpha)$

$$= \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_\alpha^\beta \exp\left(\frac{-x}{x^2 + \varepsilon^2}\right) \left(\frac{-x}{x^2 + \varepsilon^2} \sin\left(\frac{\varepsilon}{x^2 + \varepsilon^2}\right) + \frac{\varepsilon}{x^2 + \varepsilon^2} \cos\left(\frac{\varepsilon}{x^2 + \varepsilon^2}\right)\right) dx.$$

V. Remarks and acknowledgements. The theory of second order singular Sturm-Liouville problems and their application to orthogonal polynomials is now essentially complete. There remains only some needed polish on the spectral resolution of the self-adjoint operators in the limit-circle cases. In all the cases discussed, the classic orthogonal polynomials serve as excellent examples.

What has been said for the second order problems can be extended to problems of higher order. Again various orthogonal polynomials sets serve as excellent examples. There are some new and fascinating examples of polynomials satisfying differential equations of fourth or sixth order, which are related to but not quite the same as those satisfying the differential equations of second order. Indeed we plan to follow this survey with Part II, which will discuss those problems.

Throughout the years we have discussed this work with many people. It is clear that they, too, know and have developed much of what is contained herein. It is impossible to separate out who did exactly what, and so we hope that mentioning their names will suffice. We thank them all, and we apologize to those whose names we have wrongfully omitted.

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We encourage others who are interested to join in what remains to be done.

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