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MAPPINGS INTO SETS OF MEASURE ZERO

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ABSTRACT. Let f and g be functions of bounded variation on [0, 1] and let λ denote Lebesgue outer measure. We give a necessary and sufficient condition that $\lambda gS = 0$ implies $\lambda fS = 0$, for all subsets $S \subset [0, 1]$. This condition is $\lambda fX = 0$, where X is a particular set depending on f and g.

In this paper, f and g are real valued functions of bounded variation on [0, 1] and λ denotes Lebesgue outer measure. F and G are their total variation functions, $F(x) = V_0^x(f)$ and $G(x) = V_0^x(g)$ for $0 \leqq x \leqq 1$. We will give a necessary and sufficient condition that $\lambda gS = 0$ implies $\lambda fS = 0$, for any set $S \subset [0, 1]$. This condition is disclosed by the status of just one set determined by f and g. Our work will generalize and unify a number of more or less known corollaries concerning functions satisfying property N, absolutely continuous functions, saltus functions, and finite Borel measures on [0, 1].

Define the set

$$X = \{x \in (0, 1): \text{either } \lim_{h \to \infty} |(f(x+h) - f(x))/(g(x+h) - g(x))| = \infty \text{ or } x \text{ lies in the interior of the set } g^{-1}g(x)\}.$$

(Here we omit those h for which $g(x+h) = g(x)$.) We offer

THEOREM 1. *A necessary and sufficient condition that*

$$(*) \qquad \lambda fX > 0$$

holds is that there exists some set $S \subset [0, 1]$ *such that* $\lambda gS = 0 < \lambda fS$. *Moreover,* $\lambda gX = 0$ *whether* $(*)$ *holds or not.*

In other words, the question whether $\lambda gS = 0$ implies $\lambda fS = 0$, for all sets $S \subset [0, 1]$, is settled by the status of the one set, fX. Before developing a proof of Theorem 1, let us discuss some of its consequences. A

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function is said to satisfy property N (or be an N-function) if it maps sets of measure 0 to sets of measure 0. From Theorem 1, follows

COROLLARY 1. *In Theorem 1 let g be an N-function. Then f is also an N-function if $(*)$ dies not hold. In particular, f is absolutely continuous if f is continuous and $(*)$ does not hold.*

It follows from [**1**, pp. 125,100] that $\lambda P = \lambda FP = \lambda fP = 0$, where P is the set of all points where f is not finitely or infinitely differentiable. We set $g(x) = x$ to obtain

COROLLARY 2. *In Theorem 1, let $X_+ = \{x: f'(x) = \infty\}$ and $X_- = \{x: f'(x) = -\infty\}$. Then f is an N-function if and only if $\lambda f(X_+ \cup X_-) = 0$. When f is continuous, f is absolutely continuous if and only if $\lambda f(X_+ \cup X_-) = 0$.*

Corollaries 1 and 2 can also be obtained from [**1**, p.127]. The fact that $\lambda(X_+ \cup X_-) = 0$ can be regarded as a special case of the last statement in Theorem 1.

COROLLARY 3. *In Theorem 1, let $\lambda g[0, 1] = 0$. Then $\lambda f[0, 1] = 0$ if and only if $(*)$ does not hold.*

Note that $\lambda fP = 0$ if $f' = 0$ on the set P[1, p.271]. We set $g(x) = x$ to obtain

COROLLARY 4. *In Theorem 1, let $X_+ = \{x: f'(x) = \infty\}$ and $X_- = \{x: f'(x) = -\infty\}$. Let $f' = 0$ a.e. Then $\lambda f[0, 1] = 0$ if and only if $\lambda f(X_+ \cup X_-) = 0$.*

When $\lambda f[0, 1] = 0$, f is called a saltus function or a generalized step function. We will have more to say about saltus functions later.

Now, let μ_1 and μ_2 be finite nonatomic Borel measures on [0, 1]. Let

$$Y = \{x\colon \text{either } \lim_{\lambda I \to 0} \mu_1 I/\mu_2 I = \infty \text{ where } I \text{ is an interval containing } x, \text{ or } \mu_2 \text{ vanishes on some interval containing } x\}.$$

COROLLARY 5. μ_1 is absolutely continuous with respect to μ_2 if and only if $\mu_1 Y = 0$.

PROOF. Let $f(x) = \mu_1[0, x]$ and $g(x) = \mu_2[0, x]$, for $0 \leqq x \leqq 1$. Then f and g are continuous nondecreasing functions on [0, 1], and in Theorem 1, $X = Y$. By [**1**, p. 100], we have $\mu_1 Y = \lambda f Y = \lambda f X$. Now, $\mu_2 S = \lambda g S = 0$ implies $\mu_1 S = \lambda f S = 0$ for all Borel sets S if and only if μ_1 is absolutely continuous with respect to μ_2. The rest follows from Theorem 1.

Corollary 5 can also be obtained from the Radon-Nikodym Theorem.

We say that a nondecreasing continuous function f on [0, 1] is singular if $f' = 0$ a.e. on [0, 1]. We will see that this is equivalent to the existence of a set $E \subset [0, 1]$, satisfying $\lambda([0, 1] \backslash E) = \lambda f(E) = 0$. (Consult the comments before Lemma 3.) From Theorem 1 follows

COROLLARY 6. *Let f and g be continuous nondecreasing functions in Theorem 1 and let g be singular. Then f is also singular if* (*) *does not hold.*

COROLLARY 7. *Let g_1 be a singular function and g_2 be an N-function of bounded variation. Let f be of bounded variation. Let* (*) *not hold for f and g_1, and not hold for f and g_2. Then f is a saltus function.*

PROOF. Corollaries 1 and 6.

Here is our only lemma that does not require bounded variation.

LEMMA 1. *Let w be a real valued function on the interval $[a, b]$ such that the left limit $w(x-)$ exists for $a < x \leqq b$ and the right $w(x+)$ limit exists for $a \leqq x < b$. Then*

$$\sup w[a, b] - \inf w[a, b] \leqq \lambda w[a, b] + \textstyle\sum_{a<x\leq b} |w(x -) - w(x)| + \sum_{a\leq x<b} |w(x +) - w(x)|.$$

Moreover, if w is monotone on $[a, b]$, then equality holds.

PROOF. Of course w has at most countably many points of discontinuity, so each sum has at most countably many summands. Let (I_n) denote the sequence of all nondegenerate intervals of the form $(w(x+), w(x))$, or $(w(x), w(x+))$, or $(w(x-), w(x))$, or $(w(x), w(x-))$. Now, let $y \notin w[a, b]$, and $\inf w[a, b] < y < \sup w[a, b]$. Without loss of generality, we let $w(b) > y$ for definiteness. Let x_0 be the sup of the set $\{x\colon a \leqq x < b$ and $w(x) < y\}$. It follows that w is discontinuous at x_0 and $y \in \bar{I}_n$, for some n. Thus

$$(\inf w[a, b], \sup w[a, b]) \subset w[a, b] \cup \textstyle\bigcup_n \bar{I}_n$$

and the inequality follows, Finally, if w is nondecreasing on $[a, b]$, then the intervals I_n are mutually disjoint and disjoint from $w[a, b]$, so equality holds.

Our next lemma states much more than we actually need, but it may be of some intrinsic interest. Note that if $E \subset [0, 1]$, then $\lambda gE \leqq \lambda GE$. This follows from the fact that, for any interval I, $\lambda g(G^{-1}I) \leqq \lambda I$. In particular, $\lambda gE = 0$ if $\lambda GE = 0$. Lemma 2 will tell us, among other things, that the converse is also true, i.e., $\lambda GE = 0$ if $\lambda gE = 0$.

LEMMA 2. *For integers i and m, $0 < i \leqq 2^m$, let $J_{im} = [(i - 1)2^{-m}, i2^{-m}]$. Let E be any subset of* $[0, 1]$. *Then*

$$\lambda GE = \lim_{m\to\infty} \sum_{i=1}^{2^m} \lambda g(J_{im} \cap E).$$

In particular, $\lambda GE = 0$ *if* $\lambda gE = 0$.

PROOF. For any set S, let $M(g, S)$ denote $\lim_{m\to\infty} \sum_{i=1}^{2m} \lambda g(J_{im} \cap S)$. We first prove the lemma when E is a closed interval $[r, s] = I$. By Lemma 1,

$$\sum_{i=1}^{2m} (\sup g(J_{im} \cap I) - \inf g(J_{im} \cap I))$$
$$\leqq \sum_{i=1}^{2m} \lambda g(J_{im} \cap I) + \sum_{r<x\leqq s} \left|g(x-) - g(x)\right| + \sum_{r\leqq x<s} \left|g(x+) - g(x)\right|.$$

It follows that

$$(1) \qquad V_r^s(g) \leqq M(g, I) + \sum_{r<x\leqq s} |g(x-) - g(x)| + \sum_{r\leqq x<s} |g(x+) - g(x)|,$$

where V denotes total variation. Likewise

$$(2) \qquad V_r^s(G) = \lambda GI + \sum_{r<x\leqq s} |G(x-) - G(x)| + \sum_{r\leqq x<s} |G(x+) - G(x)|.$$

But $V_r^s(G) = V_r^s(g)$, $G(x) - G(x-) = |g(x-) - g(x)|$, and $G(x+) - G(x) = |g(x+) - g(x)|$. It follows from (1) and (2) that $\lambda GI \leqq M(g, I)$. But the inequality $\lambda G(J_{im} \cap I) \geqq \lambda g(J_{im} \cap I)$ is clear, so, in fact, $\lambda GI \geqq M(g, I)$. Hence $M(g, I) = \lambda GI$.

The conclusion must hold when E is an open interval, or the union of mutually disjoint open intervals. (Here an obvious convergence argument is used.) So the conclusion must hold when E is any open subset of $[0, 1]$.

Now let E be an arbitrary subset of $[0, 1]$. Let W be an open set containing $g(J_{im} \cap E)$ such that $\lambda W \leqq \lambda g(J_{im} \cap E) + 2^{-2m}$. Since g is continuous at all but at most countably many points, there is an open set $U \subset J_{im}$ such that $(J_{im} \cap E)\backslash U$ is countable and $gU \subset W$. Thus there is an open set $U_m \subset [0, 1]$ such that $E\backslash U_m$ is countable and

$$(3) \qquad \sum_{i=1}^{2m} \lambda g(J_{im} \cap U_m) \leqq \sum_{i=1}^{2m} \lambda g(J_{im} \cap E) + 2^{-m}.$$

Likewise, there is an open set V_m such that $E\backslash V_m$ is countable and

$$(4) \qquad \lambda G V_m \leqq \lambda GE + 2^{-m}.$$

Put $P = \bigcap_{m=1}^{\infty}(U_m \cap V_m)$. Then $E\backslash P$ is countable and

$$(5) \qquad \lim_{m\to\infty} \lambda G(U_m \cap V_m) = \lambda GE.$$

From (3) and $P \subset U_m \cap V_m$, it follows that

$$(6) \qquad \lim_{m\to\infty} \sum_{i=1}^{2m} \lambda g(J_{im} \cap (U_m \cap V_m)) = M(g, E) = M(g, P).$$

But $\lambda G(U_m \cap V_m) = M(g, U_m \cap V_m))$. In view of (5) and (6) it suffices to prove that $\lim_{m\to\infty} M(g, U_m \cap V_m) = M(g, P)$.

For each y and set A, let $s(A, y)$ = power of the set $A \cap g^{-1}(y)$. Suppose that $g(J_{im} \cap A)$ is measurable for all J_{im}. It is clear that

$$\lim_{m\to\infty} \sum_{i=1}^{2m} \chi_{g(J_{im}\cap A)}(y) = s(A, y) \text{ a.e.}$$

and $s(A, y)$ is measurable. In particular, when $A = [0, 1]$, we see that $s([0, 1], y)$ is measurable and, by Beppo Levi's theorem,

$$M(g, [0, 1]) = \int s([0, 1], y)dy \leqq V(g) < \infty.$$

But $\sum_{i=1}^{2m} \chi_{g(J_{im}\cap A)}(y) \leqq s(A, y)$ a.e. and

$$\lim_{m\to\infty} \sum_{i=1}^{2m} \chi_{g(J_{im}\cap A)}(y) = s(A, y) \text{ a.e.}$$

By the Lebesgue dominated convergence theorem,

$$\lim_{m\to\infty} \sum_{i=1}^{2m} \lambda_{\bar{g}}(J_{im} \cap A) = \int s(A, y)dy = M(g, A).$$

We may further assume that $U_{m+1} \subset U_m$ and $V_{m+1} \subset V_m$ for all m. Just replace U_m with $U_1 \cap U_2 \cap \cdots \cap U_m$ and V_m with $V_1 \cap V_2 \cap \cdots \cap V_m$.

But g satisfies the property T_1[**1**, p. 277] because $\int s([0, 1], y)dy < \infty$. This implies that

$$\lim_{m\to\infty} s(U_m \cap V_m, y) = s(P, y) \text{ a.e.}$$

Moreover, $\lim_{m\to\infty} \chi_{g(J\cap U_m\cap V_m)} = \chi_{g(J\cap P)}$ a.e., so $g(J \cap P)$ is measurable for any interval J. By the preceding paragraph, $s(P, y)$ is measurable and $M(g, P) = \int s(P, y)dy$. But $s(U_m \cap V_m, y) \leqq s([0, 1], y)$, so, by the Lebesgue dominated convergence theorem,

$$\begin{aligned}\lim_{m\to\infty} M(g, U_m \cap V_m) &= \lim_{m\to\infty} \int s(U_m \cap V_m, y)dy \\ &= \int s(P, y)dy = M(g, P).\end{aligned}$$

Note that E need not be measurable in Lemma 2. In the proof we saw that

$$\begin{aligned} V(F) &= \lambda F[0, 1] + \sum_{0<x\leqq 1} (F(x) - F(x-)) + \sum_{0\leqq x<1} (F(x+) - F(x)) \\ &= \lambda F[0, 1] + \sum_{0<x\leqq 1} |f(x) - f(x-)| + \sum_{0\leqq x<1} |f(x+) - f(x)| = V(f). \end{aligned} \tag{1}$$

We usually call a function f of bounded variation a saltus function if

$$V(f) = \sum_{0<x\leqq 1} |f(x) - f(x-)| + \sum_{0\leqq x<1} |f(x+) - f(x)|. \tag{2}$$

In view of Lemma 2 and equations (1) and (2), we see that f is a saltus function if and only if $\lambda f[0, 1] = 0$ if and only if F is a saltus function. Since f has at most countably many discontinuities, the set $\overline{f[0, 1]} \setminus f[0, 1]$ is at most countable. But $\overline{f[0, 1]}$ is compact, so $f[0, 1]$ has Jordan content 0 if $\lambda f[0, 1] = 0$. Thus, f is a saltus function if and only if $f[0, 1]$ has Jordan content 0.

It follows from [**1**, p. 127], for example, that when F is finitely differentiable on set P, then $\lambda FP = 0$ if and only if $F' = 0$ a.e. on P. In view of Corollary 4, it follows that f is a saltus function if and only if $f' = 0$ a.e. on $[0, 1]$ and $\lambda f(X_+ \cup X_-) = 0$.

The significance of saltus functions is that any function of bounded variation is the sum of a continuous function of bounded variation and a saltus function [**1**, p. 99]. This decomposition is unique within an additive constant.

From [**1**, p. 127] it follows that $F' = 0$ a.e. on $[0, 1]$ if and only if there is a set $E \subset [0, 1]$ satisfying $\lambda([0, 1] \setminus E) = \lambda FE = 0$. Thus, a continuous nondecreasing function f is singular if and only if, for some set $E \subset [0, 1]$ we have $\lambda([0, 1] \setminus E) = \lambda fE = 0$.

The proof of Theorem 1 will emerge from the next two lemmas.

LEMMA 3. $\lambda gX = 0$.

PROOF. Suppose, to the contrary, that $\lambda gX > 0$. Fix any $k > 0$. Let

$$Z = \{x \in X: g \text{ is continuous at } x \text{ and } g \text{ is not constant on any interval containing } x\}.$$

So $g(X \setminus Z)$ is countable and $\lambda gZ > 0$. Let $T = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$ be a partition of $[0, 1]$ such that

$$\text{(i)} \qquad \sum_i |g(t_i) - g(t_{i-1})| \geqq \sum_i V_{t_{i-1}}^{t_i}(g) - \frac{1}{2} \lambda gZ.$$

Also, (i) holds when T is replaced by any refinement of T. Now, each $x \in Z$ lies in an interval $[a, b]$ such that $|f(b) - f(a)| \geqq k|g(b) - g(a)|$ and $(a, b) \cap T = \varnothing$. Moreover, $b - a$ and $\sup g[a, b] - \inf g[a, b]$ can be made as small as we please. Thus, intervals of the form $[\inf g[a, b], \sup g[a, b]]$ constitute a Vitali covering of gZ. By the Vitali covering theorem, there exist countably many pairwise disjoint intervals $[a_i, b_i]$ such that

$$\text{(1)} \qquad \sum_i [\sup g[a_i, b_i] - \inf g[a_i, b_i]] \geqq \lambda gZ > 0,$$

$$\text{(2)} \qquad \sum_i |f(b_i) - f(a_i)| \geqq k \sum_i |g(b_i) - g(a_i)|,$$

and, by (i),

$$\text{(3)} \quad \sum_i |g(b_i) - g(a_i)| \geqq \sum_i (\sup g[a_i, b_i] - \inf g[a_i, b_i]) - \frac{1}{2} \lambda gZ.$$

We combine (1), (2) and (3) to obtain

$$\text{(4)} \qquad \textstyle\sum_i |f(b_i) - f(a_i)| \geqq \frac{1}{2} k\lambda gZ > 0.$$

But k can be made arbitrarily large, so (4) implies that $V(f) = \infty$.

LEMMA 4. *Let* $S \subset [0, 1] \setminus X$ *and* $\lambda gS = 0$. *Then* $\lambda fS = 0$.

PROOF. Suppose, to the contrary, that $\lambda fS > 0$. Now, $S = \bigcup_n S_n$ where $S_n = \{x \in S\colon \liminf_{h\to 0} |(f(x+h) - f(x))/(g(x+h) - g(x))| < n\}$. For some N, $\lambda fS_N > 0$. Then $\lambda fW > 0$, where

$$W = \{x \in S_N\colon f \text{ and } G \text{ are continuous at } x \text{ and } f \text{ is not constant on any interval containing } x\}.$$

Let $T = \{0 = t_0 < t_1 < \cdots < t_m = 1\}$ be a partition of [0, 1] such that

$$\text{(i)} \qquad \textstyle\sum_i |f(t_i) - f(t_{i-1})| \geqq \sum_i V_{t_{i-1}}^{t_i}(f) - \frac{1}{2}\lambda fW.$$

Also, (i) holds when T is replaced by any refinement of T.

Choose any $c > 0$. By Lemma 2, $\lambda GW = 0$ and there exists an open set $U \supset GW$ with $\lambda U < c$. Each $x \in W$ lies in an interval $[a, b]$ such that $G(b) - G(a) \geqq N^{-1}|f(b) - f(a)|$ and $G[a, b] \subset U$ and $(a, b) \cap T = \varnothing$. Moreover, $b - a$ and $\sup f[a, b] - \inf f[a, b]$ can be made as small as we please. The intervals of the form $[\inf f[a,b], \sup f[a,b]]$ constitute a Vitali covering of the set fW. By the Vitali covering theorem, there exist countably many mutually disjoint intervals $[a_i, b_i]$ such that

$$\text{(1)} \qquad \textstyle\sum_i (\sup f[a_i, b_i] - \inf f[a_i, b_i]) \geqq \lambda fW > 0,$$

$$\text{(2)} \qquad \textstyle\sum_i [G(b_i) - G(a_i)] \geqq N^{-1} \sum_i |f(b_i) - f(a_i)|,$$

and, by (i),

$$\text{(3)} \quad \textstyle\sum_i |f(b_i) - f(a_i)| \geqq \sum_i (\sup f[a_i, b_i] - \inf f[a_i, b_i]) - \frac{1}{2}\lambda fW.$$

We combine (1), (2) and (3) to obtain

$$\text{(4)} \qquad \textstyle\sum_i [G(b_i) - G(a_i)] \geqq \frac{1}{2} N^{-1} \lambda fW > 0.$$

But $\bigcup_i G[a_i, b_i] \subset U$, and

$$\text{(5)} \qquad c > \lambda u \geqq \textstyle\sum_i [G(b_i) - G(a_i)] \geqq \frac{1}{2} N^{-1} \lambda fW.$$

Since c can be made arbitrarily small, it follows that $\lambda fW = 0$. But $\lambda fW > 0$.

PROOF OF THEOREM 1. Assume that (*) holds. By Lemma 3, $\lambda gX = 0$, so we need only put $S = X$ in the conclusion. Now, assume that (*) does not hold. Then $\lambda fX = 0$, and, for any set S, $\lambda fS \leqq \lambda f(S\backslash X) + \lambda f(S \cap X) = \lambda f(S\backslash X)$. If $\lambda gS = 0$, then $\lambda g(S\backslash X) = 0$ and, by Lemma 4, $\lambda f(S\backslash X) = 0 = \lambda fS$. The last statement of Theorem 1 is just Lemma 3 again.

Let

$$Y = \{x\text{: the (finite or infinite) limit } \lim_{h\to 0} |(f(x+h) - f(x))/(g(x+h) - g(x))| \text{ does not exist and } x \text{ is not in the interior of } g^{-1}g(x)\},$$

$$U = \{x\text{: the (finite or infinite) limit } \lim_{h\to 0}(f(x+h) - f(x))/(g(x+h) - g(x)) \text{ does not exist and } x \text{ is not in the interior of } g^{-1}g(x)\}.$$

In conclusion we show that the sets Y and U are "small" in a sense.

THEOREM 2. *Let Y and U be as described before. Then*
(i) $\lambda fY = \lambda gY = 0$,
(ii) $\lambda fU = \lambda gU = 0$ *if g is nondecreasing on* $[0, 1]$.

PROOF. (i). First assume that f and g are nondecreasing, i.e., $f = F$, $g = G$. Then $Y = \bigcup_{pq} Y_{pq}$, where p and q are positive rational, and

$$Y_{pq} = \{x \in U\text{: } \liminf_{h\to 0}(F(x+h) - F(x))/(G(x+h) - G(x)) < p < q < \limsup_{h\to 0}(F(x+h) - F(x))/(G(x+h) - G(x))\}.$$

For some $p < q$, let W denote the set of points in Y_{pq} where F and G are continuous. Let P be an open set containing GW. By the Vitali covering theorem, there exist countably many mitually disjoint intervals $(a_n, b_n) \subset (0, 1)$ such that, for each n, $(G(a_n), G(b_n)) \subset P$, $F(b_n) - F(a_n) < p(G(b_n) - G(a_n))$, and $\sum_n(F(b_n) - F(a_n)) \geqq \lambda FW$. But $\lambda FW \leqq \sum_n(F(b_n) - F(a_n)) \leqq p \sum_n(G(b_n) - G(a_n)) \leqq p\,\lambda P$. Since P is arbitrary, we obtain $\lambda FW \leqq p\lambda GW$. By an analogous argument, $\lambda FW \geqq q\lambda GW$. Then $\lambda GW = 0$; otherwise $\lambda FW \leqq p\lambda GW < q\lambda GW \leqq \lambda FW$, which is impossible. Hence, $\lambda GW = 0 = \lambda FW$. Since p and q are arbitrary, $\lambda FY = 0 = \lambda GY$.

More generally, we drop the hypothesis that f and g are nondecreasing. For convenience, let (f, g) denote the quotient $(f(x+h) - f(x))/(g(x+h) - g(x))$. Let V_1 be a set such that $\lambda(F + G)V_1 = 0$ and, for $x \notin V_1$, all the limits

$$\lim_{h\to 0} (F, F + G),\ \lim_{h\to 0} (F + f, F + G),\ \lim_{h\to 0} (G, F + G),\ \lim_{h\to 0} (G + g, F + G)$$

exist, and, hence, the limits $\lim_{h\to 0}(f, F + G)$ and $\lim_{h\to 0}(g, F + G)$ also exist. Now, $(F + G) - F$ and $(F + G) - G$ are nondecreasing, so, in fact, $\lambda F V_1 = \lambda G V_1 = 0$. By Lemma 3, there is a set V_2 such that $\lambda g V_2 = 0$ and for $x \notin V_2$, $\lim_{h\to 0}|(F + G, g)| \neq \infty$. But $|(F + G, g)| \geqq 1$, so, for $x \notin V_1 \cup V_2$,

$$\lim_{h\to 0} |(f, g)| = \lim_{h\to 0} |(F + G, g)| \lim_{h\to 0} |(f, F + G)|.$$

Hence $Y \subset V_1 \cup V_2$ and $\lambda g Y = 0$. By Lemma 4, $\lambda f Y = 0$ also. This proves (i). (It is well to note here that if $g(x + h) \neq g(x)$, then $(F + G)(x + h) \neq (F + G)(x)$.)

(ii). For $x \notin V_1$, the limit $\lim_{h\to 0}(F + G, G)$ exists. By Lemma 3, there is a set V_3 such that $\lambda f V_3 = 0$, and for $x \notin V_3$, $\lim_{h\to 0}|(F + G, f)| \neq \infty$ and $\lim_{h\to 0}(f, F + G) \neq 0$. Again, by Lemma 3, there is a set V_4 such that $\lambda G V_4 = 0$ and for $x \notin V_4$, $\lim_{h\to 0}(F + G, G) \neq \infty$. It follows that, for $x \notin (V_1 \cup V_3) \cap (V_1 \cup V_4)$,

$$\lim_{h\to 0} (f, G) = \lim_{h\to 0} (f, F + G) \lim_{h\to 0} (F + G, G).$$

Then $U \subset (V_1 \cup V_3) \cap (V_1 \cup V_4)$ and $\lambda f U = \lambda g U = 0$.

Part (ii) reduces to [**1**, p. 125, Theorem (9.1)] essentially when $g(x) = x$ for all x. Absolute value is essential in part (i). Consider $f(x) = x$ for all x, $g(x) = 0$ for all irrational x, and $g(n/m) = 2^{-m}$ for rational numbers n/m in lowest terms. The limit does not exist without the absolute value at irrational points.

Finally, we observe that $\lambda g U = 0$ in Theorem 2 whether g is nondecreasing or not. Note first that $\lambda g X = 0$ in Lemma 3 even when $\limsup_{h\to 0}|(f, g)|$ replaces $\lim_{h\to 0}|(f, g)|$ in the definition of X. (This is clear from the proof.) Thus, if $\lambda g U > 0$, then there is a number $k > 0$ such that $\lambda g U_k > 0$, where

$$U_k = \{x \in U: 0 < \limsup_{h\to 0} (f, g) = -\liminf_{h\to 0} (f, g) < k \text{ at } x\}.$$

Now, $\lim_{h\to 0}|(f + kg, g)|$ does not exist at any $x \in U_k$. By Theorem 2(i), $\lambda g U_k = 0$. But $\lambda g U_k > 0$.

Reference

1. S. Saks, *Theory of the Integral*, second revised edition, Dover 1964.

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