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ON A THEOREM OF BERNSTEIN

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1. Let $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n and $P'(z)$ denote its derivative. Concerning the estimate of $|P'(z)|$ the following result is well known:

THEOREM A. *If $P(z)$ is a polynomial of degree n and* $\max_{|z|=1} |P(z)| = 1$ *then for $|z| \leqq 1$*

$$|P'(z)| \leqq n. \tag{1}$$

There is equality in (1) *if and only if $P(z) \equiv \alpha z^n$, $|\alpha| = 1$.*

Theorem A is known as Bernstein's Theorem. It can be deduced from a result (also known as Bernstein's Theorem) on the derivative of a trigonometric polynomial which can be proved following an interpolation formula obtained by M. Riesz [3]; from where it is also verified that equality in (1) holds only if $P(z) \equiv \alpha z^n$, $|\alpha| = 1$. In **[1]**, S. Bernstein proved the following generalization of Theorem A by the use of Gauss-Lucas Theorem; see also N. G. De Bruijn **[2]**:

THEOREM B. *Let $P(z)$ and $Q(z)$ be polynomials satisfying the conditions that $Q(z)$ has all its zeros in $|z| \leqq 1$ and the degree of $P(z)$ does not exceed that of $Q(z)$. If*

$$|P(z)| \leqq |Q(z)| \quad \text{on} \quad |z| = 1 \tag{2}$$

then

$$|P'(z)| \leqq |Q'(z)| \quad \text{on} \quad |z| = 1. \tag{3}$$

2. In this paper, we study the case when there is equality in (3). In fact, we prove:

THEOREM 1. *Let the hypothesis of* Theorem B *be satisfied. If there is equality in* (3) *at any point μ on $|z| = 1$ where $Q(\mu) \neq 0$ then $P(z) \equiv \alpha Q(z)$, $|\alpha| = 1$.*

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REMARK 1. If $P(z)$ has all its zeros in $|z| \geqq 1$ and $P(1) = 0$, then $Q(z) = z^n \overline{P(1/\bar{z})}$ has all its zeros in $|z| \leqq 1$ and $Q(1) = 0$. Further, $|P(z)| = |Q(z)|$ on $|z| = 1$ and $|P'(1)| = |Q'(1)|$ whereas $P(z) \not\equiv \alpha Q(z)$ for any $|\alpha| = 1$. Hence, the condition that $Q(\mu) \neq 0$ cannot be dropped.

PROOF OF THEOREM 1. Let $P(z)$ and $Q(z)$ be polynomials of degree $\leqq n$ and of degree n respectively satisfying the hypothesis of Theorem 1; $n \geqq 1$. Let μ be a point on $|z| = 1$ where $|P'(\mu)| = |Q'(\mu)|$ and choose a complex number α with absolute value one such that $P'(\mu) - \alpha Q'(\mu) = 0$. Since $Q(z)$ has no zeros in $|z| > 1$ and (2) holds, it follows by the Maximum Modulus Theorem that $P(z) - \alpha Q(z)$ has all its zeros in $|z| \leqq 1$. Further, from the Gauss-Lucas Theorem, $P'(z) - \alpha Q'(z)$ has all its zeros in the convex hull of the zeros of $P(z) - \alpha Q(z)$ which is entirely contained in the unit disc $|z| \leqq 1$. Since μ, $|\mu| = 1$, is a zero of $P'(z) - \alpha Q'(z)$ and the unit disc is strictly convex, μ must also be a zero of $P(z) - \alpha Q(z)$. This implies that $P(z) - \alpha Q(z)$ has a double zero at μ.

Let $z = e^{i\theta}$ and consider the real trigonometric polynomials $T(\theta) = \operatorname{Re} P(e^{i\theta})$, $T^*(\theta) = \operatorname{Im} P(e^{i\theta})$, $S(\theta) = \operatorname{Re}\{\alpha Q(e^{i\theta})\}$ and $S^*(\theta) = \operatorname{Im}\{\alpha Q(e^{i\theta})\}$. We obviously have

$$P(e^{i\theta}) - \alpha Q(e^{i\theta}) = f(\theta) + i f^*(\theta) \tag{4}$$

where $f(\theta) = T(\theta) - S(\theta)$ and $f^*(\theta) = T^*(\theta) - S^*(\theta)$, and both the trigonometric polynomials are of degree at most n and have a double zero at $\varphi = \arg \mu$.

Without any loss of generality, we can assume that $Q(z)$ has all its zeros in $|z| < 1$. In fact, if $Q(z)$ has a zero of order m at $z = \lambda$, $|\lambda| = 1$, then λ is also a zero of order at least m of $P(z)$ and these two polynomials can be written as $P(z) = (z - \lambda)^m \tilde{P}(z)$ and $Q(z) = (z - \lambda)^m \tilde{Q}(z)$. Further, if there holds $|P'(\mu)| = |Q'(\mu)|$, $\mu \neq \lambda$, $|\mu| = 1$, then for some $|\alpha| = 1$, $P(z) - \alpha Q(z) = (z - \lambda)^m(\tilde{P}(z) - \alpha\tilde{Q}(z))$ has a double zero at $\mu \neq \lambda$. Hence $|\tilde{P}'(\mu)| = |\tilde{Q}'(\mu)|$. Thus, to arrive at the conclusion, we can work with $\tilde{P}(z)$ and $\tilde{Q}(z)$ where $\tilde{Q}(z)$ has all its zeros in $|z| < 1$.

Since $Q(z)$ has all its zeros in $|z| < 1$, it follows from the principle of argument that the image curve $Q(e^{i\theta})$ in the w-plane winds around the origin n times (without ever passing through the origin) as θ varies from 0 to 2π. Hence $S(\theta)$ as well as $S^*(\theta)$ have exactly $2n$ simple zeros in $[0, 2\pi)$. Let the zeros of $S(\theta)$ be $\sigma_1, \sigma_2, \ldots, \sigma_{2n}$ and the zeros of $S^*(\theta)$ be $\tau_1, \tau_2, \ldots, \tau_{2n}$. It is easily seen that $\sigma_1 < \tau_1 < \sigma_2 < \tau_2 < \cdots < \sigma_{2n} < \tau_{2n} < \sigma_1$ and at any two consecutive zeros τ_k and τ_{k+1} of $S^*(\theta)$, $\operatorname{sgn} S(\tau_k) = -\operatorname{sgn} S(\tau_{k+1})$, $k = 1, 2, \ldots, 2n$ and $\tau_{2n+1} = \tau_1$. If it were not so, $S^*(\theta)$ would have to have more than $2n$ zeros in order that the image curve $Q(e^{i\theta})$ wind around the origin n times implying $S^*(\theta) \equiv 0$ and so reducing $Q(z)$ to a constant; a contradiction. Similarly, at any two consecutive zeros σ_k and σ_{k+1} of $S(\theta)$, sgn $S^*(\sigma_k) = -\text{sgn } S^*(\sigma_{k+1})$, $k = 1, 2, \ldots, 2n$; $\sigma_{2n+1} = \sigma_1$. Moreover, from (2) one has

$$|T(\tau_k)| \leqq |T(\tau_k) + i\, T^*(\tau_k)| \leqq |S(\tau_k)| \tag{5}$$

and

$$|T^*(\sigma_k)| \leqq |T(\sigma_k) + i\, T^*(\sigma_k)| \leqq |S^*(\sigma_k)| \tag{6}$$

from which

$$\text{sgn}\{T(\tau_k) - S(\tau_k)\} = -\text{sgn}\{T(\tau_{k+1}) - S(\tau_{k+1})\} \tag{7}$$

provided $T(\tau_j) \neq S(\tau_j)$; $j = k, k + 1$ and

$$\text{sgn}\{T^*(\sigma_k) - S^*(\sigma|)\} = -\text{sgn}\{T^*(\sigma_k) - S^*(\sigma_{k+1})\} \tag{8}$$

provided $T^*(\sigma_j) \neq S^*(\sigma_j)$; $j = k, k + 1$ for $k = 1, 2, \ldots, 2n$.

Now, we show that $f(\theta)$ has at least $2n$ zeros, one in each of the $2n$ intervals $[\tau_k, \tau_{k+1}]$, $k = 1, 2, \ldots, 2n$. The observation relies on geometrical consideration. If $f(\tau_k) \neq 0$, then from (7), $f(\theta)$ has a zero in $[\tau_{k-1}, \tau_k]$ and another zero in $[\tau_k, \tau_{k+1}]$. Next, when τ_k is a simple zero of $f(\theta)$, the graph of $T(\theta)$ meets the graph of $S(\theta)$ either from below or from above at $\theta = \tau_k$. In this case, if $S(\tau_k) > 0$ and the graph of $T(\theta)$ meets the graph of $S(\theta)$ from below (above), then $f(\theta)$ must have a zero in $[\tau_{k-1}, \tau_k) < (\tau_k, \tau_{k+1}] >$ similarly, if $S(\tau_k) < 0$, $f(\theta)$ must have a zero either in $[\tau_{k-1}, \tau_k)$ or $(\tau_k, \tau_{k+1}]$. In consequence, whenever τ_k is a simple zero of $f(\theta)$, we note that $f(\theta)$ has at least two zeros in $[\tau_{k-1}, \tau_{k+1}]$, one in $[\tau_{k-1}, \tau_k]$ and the other in $[\tau_k, \tau_{k+1}]$. If τ_k is a double (or multiple) zero of $f(\theta)$, out of these one may be regarded in $[\tau_{k-1}, \tau_k]$ and the other in $[\tau_k, \tau_{k+1}]$. We repeat the above argument for each k and establish the claim.

Similarly, we can show that $f^*(\theta)$ has at least $2n$ zeros, one in each of the $2n$ intervals $[\sigma_k, \sigma_{k+1}]$, $k = 1, 2, \ldots, 2n$.

Let us suppose that $\varphi \neq \tau_k$ for any k. Since $\varphi = \arg \mu$ is a double zero of $f(\theta)$, we conclude that $f(\theta)$ has at least $2n + 1$ zeros in $[0, 2\pi)$. Hence $f(\theta) \equiv 0$. This conclusion and (2) further implies that $|T^*(\theta)| \leqq |S^*(\theta)|$ for all θ in $[0, 2\pi)$. So the $2n$ simple zeros of $S^*(\theta)$ are also the zeros of $f^*(\theta)$. Since φ is a double zero of $f^*(\theta)$ there are at least $2n + 1$ zeros of $f^*(\theta)$. Thus $f^*(\theta) \equiv 0$.

If $\varphi = \tau_k$ for some k, then $\varphi \neq \sigma_k$ for any k and we begin with $f^*(\theta)$ to arrive at the same conclusion $f^*(\theta) \equiv f(\theta) \equiv 0$.

Consequently, $P(z) \equiv \alpha Q(z)$.

As an immediate consequence of Theorem 1, we observe that equality in (1) holds only if $P(z) \equiv \alpha z^n$, $|\alpha| = 1$; and also have the following variation of Theorem A.

THEOREM 2. *Let $P(z)$ be a polynomial of degree n and* $\max_{|z|=1} |P(z)| = 1$.

if for some α *with* $|\alpha| = 1$, $P(z) - \alpha z^n$ *has a double zero on* $|z| = 1$, *then* $P(z) \equiv \alpha z^n$.

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