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APPROXIMATION IN THE MEAN BY POLYNOMIALS

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ABSTRACT A new condition on crescent shaped regions Ω is given which insures that polynomials are dense in the Bergman spaces $A^p(\Omega)$. The relation of this condition to the classical results is described. Finally, a characterization of the dual of $A^p(\Omega)$ $(1 < p < \infty)$ is given for the region Ω bounded by two internally tangent circles.

1. Throughout this paper Ω will denote an open region of the complex plane $\mathbf{C}$ and σ will denote two-dimensional Lebesque measure. For a fixed Ω, $L^p(\Omega)$ denotes the usual space of Lebesgue-measurable, p-integrable functions on Ω, and $A^p(\Omega)$ will denote the space of functions which are both analytic on Ω and belong to $L^p(\Omega)$. Of course $A^p(\Omega)$ is a closed subspace of $L^p(\Omega)$. Finally, $H^p(\Omega)$ will denote the closure of the polynomials in $L^p(\Omega)$, so $H^p(\Omega)$ is a closed subspace of $A^p(\Omega)$.

2. The results of §3 and §4 come under the broad topic of approximation in the mean by polynomials on subdomains of the complex plane. The domains considered in this paper will be restricted to crescents lying between two internally tangent circles with one multiple boundary point. Much of the older work in this area appeared in the Soviet literature. The following theorem gives the flavor of the classical theory. The sufficiency was established by M. M. Dzrbasjan and the necessity by A. L. Saginjan.

THEOREM. *Let Ω be a crescent with multiple boundary point at the origin such that Ω is situated between the two circles $|z-1| = 1$ and $|z-(1/2)| = 1/2$. Denote by $\ell(r)$ the linear measure of $(|z| = r) \cap \Omega$ and assume that $r(\ell'(r)/\ell(r)) \uparrow +\infty$ as $r \downarrow 0$. Then in order for $H^p(\Omega) = A^p(\Omega)$ for any p it is necessary and sufficient that*

$$\int_0 \log \ell(r)dr = -\infty.$$

A survey paper by S. N. Mergelyan [7] describes the state of the art in

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this subject at this time. More recently James Brennan [2] has studied many of the remaining problems. Brennan has obtained a plethora of results in various directions.

In §5 we give a characterization of the dual space of $A^p(\Omega)$ $1 < p < \infty)$ where Ω is a crescent bounded by two internally tangent circles. In §6 we list some questions which arose during the course of this work.

3. In this section we study some properties of the number $\|z^{-n}\|_p$ for the purpose of establishing a sufficient condition for $H^p(\Omega) = A^p(\Omega)$. We remark, to begin with, that $\|z^{-x}\|$ is a long-convex function of x for $x \geqq 0$. This is an immediate consequence of Hölder's inequality. The following result, which goes back to Carleman [4] will be needed for the proof of Theorem 2 below. A proof is included for completeness.

THEOREM 1. *Let* $\{M_n\}_{n=0}^{\infty}$ *be a log-convex sequence, i.e.,* $M_0 = 1$ *and* $M_n^2 \leqq M_{n-1}M_{n+1}$ *for* $n = 1, 2. \ldots$

(a) If $\sum_{n=1}^{\infty} M_n^{-1/n} = \infty$ *and* f *is an analytic function in the unit disk* D *satisfying* $|f(z)| \leqq M_n\, |1 - z|^n$ *for all* $z \in D$, $n = 0, 1, 2, \ldots$, *then* f *vanishes identically.*

(b) *If* $\sum_{n=1}^{\infty} M_n^{-1/n} < \infty$ *then there exists a nontrivial outer function* f *such that* $|f(z)| \leqq M_n\, |1 - z|^n$ *for all* $z \in D$, $n = 0, 1, 2, \ldots.$

PROOF. Let $\tau(r) = \inf_{n \geqq 0} r^n M_n$. The Denjoy-Carleman Theorem [8, p. 376] with $x = \arctan \theta$ implies that $\sum M_n^{-1/n}$ diverges if and only if $\int \log \tau(\theta)\, d\theta$ diverges. To prove (a) it suffices to note that $|f(e^{i\theta})| \leqq \tau(\theta)$, so $\log |f|$ is not integrable but $f \in H^\infty$. For (b), the integrability of $\log \tau(\theta)$ guarantees the existence of an outer function f such that $|f(e^{i\theta})| = \tau(\theta)$ a.e., and the inequalities follow since $(1 - z)^n$ is outer for each n.

In the remainder of the section Ω will denote a crescent lying between the circles $|z - 1| = 1$ and $|z - (1/2)| = 1/2$. The method we use is inspired by the classical use of the Cauchy transform. For $z, w \in \mathbf{C}$, $z \neq w$, let $C_w(z) = (z - w)^{-1}$. Clearly for each $w \notin \bar{\Omega}$, $C_w \in A^p(\Omega)$, and if ϕ is a bounded linear functional on $L^p(\Omega)$, then $\hat{\phi}(w) = \phi(C_w)$ is an analytic function of w on the complement of $\bar{\Omega}$. Of course $\hat{\phi}(w)$ is the Cauchy transform of some function in $L^q(\Omega)$ $((1/p) + (1/q)) = 1)$. If ϕ annihilates $H^p(\Omega)$, then $\phi(z^n) = 0$ for $n = 0, 1, 2, \ldots,$ and an easy calculation shows that $\hat{\phi} = 0$ in the unbounded component of $C - \bar{\Omega}$.

THEOREM 2. *Suppose that* $z^{-n} \in H^p(\Omega)$ *for all positive integers* n *and that*

$$\sum_{n=1}^{\infty} \|z^{-n}\|_p^{-1/n} = \infty.$$

Then $H^p(\Omega) = A^p(\Omega)$.

PROOF. It suffices to prove that each bounded linear functional ϕ on

$L^p(\Omega)$ which annihilates $H^p(\Omega)$ also annihilates $A^p(\Omega)$. An easy application of Runge's theorem shows that the functions $\{(z - 1/4)^n\}_{n=-\infty}^{\infty}$ have dense span in $A^p(\Omega)$. So it will suffice to show that $\phi((z - (1/4)^n) = 0$ for $n = 1, 2, \ldots$, if $\phi((z - (1/4))^n) = 0$ for $n = 0, 1, 2, \ldots$. To this end, let Δ denote the disk $|z - (1/4)| < 1/4$. There is a positive constant c such that $|z - w| \geqq c|z|^2$ for all $z \in \Omega$ and $w \in \Delta$. The functional ϕ corresponds to a function g under the usual pairing. Since

$$C_w(z) = \frac{1}{z} + \frac{w}{z^2} + \frac{w^{n-1}}{z^{n-1}} + \frac{w^n}{z^n(z - w)},$$

and the functions z^{-n} belong to $H^p(\Omega)$ for $n = 1, 2, \ldots$, it follows that

$$\begin{aligned} \hat{\phi}(w) &= \phi\left(\frac{1}{z} + \frac{w}{z^2} + \frac{w^{n-1}}{z^{n-1}} + \frac{w^n}{z^n(z - w)}\right) \\ &= \phi\left(\frac{w^n}{z^n(z - w)}\right) \\ &= w^n \int \frac{g(z)}{z^n(z - w)}\, d\sigma(z), \qquad \text{for } n = 1, 2, \ldots . \end{aligned}$$

Hence

$$\begin{aligned} |\hat{\phi}(w)| &\leqq \frac{|w|^n}{c} \int |g(z)|\, |z|^{-n-2}\, d\sigma(z) \\ &\leqq \frac{|w|^n}{c} \|\phi\|\, \|z^{-n-2}\|_p, \qquad w \in \Delta. \end{aligned}$$

It follows from Theorem 1 that $\hat{\phi}(w) = 0$ for all $w \in \Delta$, and consequently for all w in the bounded component of $C - \bar{\Omega}$. Clearly, for $w \notin \bar{\Omega}$,

$$\hat{\phi}^{(n)}(w) = n!\, \phi((z - w)^{-n-1}), \qquad n = 0, 1, 2, \ldots .$$

In particular,

$$\phi\left(\left(z - \frac{1}{4}\right)^{-n-1}\right) = \frac{\hat{\phi}^{(n)}\left(\frac{1}{4}\right)}{n!} = 0, \qquad n = 0, 1, 2, \ldots,$$

as desired.

A second quantity which is intimately related to the norms $\|z^{-n}\|_p$ is defined as follows. For $0 < h < 1$, let $D_h = \{z \in \Omega \mid h < |z| \leqq 2h\}$ and define

$$M(x) = \sup_{0<h<1} h^{-x}\sigma(D_h),$$

for $x \geqq 0$. We note the following propositions.

PROPOSITION 1. *$M(x)$ is an increasing log-convex function of x.*

PROPOSITION 2. *If $M(x)$ is finite for some $x > pn$ then $z^{-n} \in A^p(\Omega)$ and*

$$\|z^{-n}\|_p^p \leqq \frac{M(x)}{1 - \left(\frac{1}{2}\right)^{x-pn}}.$$

PROOF. Note that $\Omega = \bigcup_{k=0}^{\infty} D_{2^{-k}}$ (disjoint union). Thus

$$\int_\Omega |z|^{-np}\, d\sigma(z) = \sum_{k=0}^{\infty} \int_{D_{2^{-k}}} |z|^{-np}\, d\sigma(z)$$

$$\leqq M(x) \sum_{k=0}^{\infty} (2^{np-x})^k$$

$$= \frac{M(x)}{1 - \left(\frac{1}{2}\right)^{x-pn}}.$$

PROPOSITION 3. *If $z^{-n} \in A^p(\Omega)$, then $z^{-n} \in H^p(\Omega)$.*

PROOF. The functions $(z + (1/k))^{-n}$ $(k = 1, 2, \ldots)$ are in $H^p(\Omega)$ and converge pointwise to z^{-n}. The Lebegsue dominated convergence theorem then implies that $z^{-n} \in H^p(\Omega)$.

PROPOSITION 4. *$M(x)$ is finite for all $x \geqq 0$ if any only if $z^{-n} \in H^p(\Omega)$ for all $n = 1, 2, \ldots$.*

Finally we obtain a second sufficient condition, expressed only in terms of $M(x)$.

THEOREM 3. If $\sum_{n=1}^{\infty} M(pn + 1)^{-1/pn} = \infty$, *then $H^p(\Omega) = A^p(\Omega)$.*

PROOF. For each n, let $x = pn + 1$ in Proposition 2. Then

$$\|z^{-n}\|_p^p \leqq 2\, M\, (pn + 1).$$

So

$$M(np + 1)^{-1/pn} \leqq 2^{1/p}\, \|z^{-n}\|_p^{-1/n} \leqq \|z^{-n}\|_p^{-1/n},$$

and Theorem 3 follows from Theorem 2.

4. We now construct examples of domains Ω which will serve to illustrate the connection between the classical results and the results of §3. This will be done by specifying the "width" $\ell(t)$ of the domain Ω at a distance t from $z = 0$. We take the circle $|z - 1| = 1$ for the outer boundary of Ω, and determine the inner boundary from $\ell(t)$ by demanding that Ω be symmetric with respect to the real axis. In fact we will always include in Ω the set $\{z \mid |z - 1| < 1, |z| \geqq 1\}$, and so the function $\ell(t)$ is only relevant for $0 \leqq t \leqq 1$. Since it is the behavior of $\ell(t)$ for small t shich is important, this creates no problems.

For the examples, if β is a positive number, let Ω_β be the region obtained by setting $\ell(t) = e^{-t^{-\beta}}$. First notice that

$$t\frac{\ell'(t)}{\ell(t)} = \beta t^{-\beta},$$

so that $t(\ell'(t))/(\ell(t)) \uparrow +\infty$ as $t \downarrow 0$. Next, $\log \ell(t) = t^{-\beta}$, so that

$$\int_0^1 \log \ell(t)dt > -\infty$$

if and only if $\beta < 1$. Thus by the classical results, $A^p(\Omega_\beta) = H^p(\Omega_\beta)$ if and only if $\beta \geqq 1$.

To discuss the connection with the results of §3 it will be necessary to estimate certain integrals. To this end, for $x > 1$ let

$$K_\beta(x) = \int_0^1 t^{-x}e^{-t^{-\beta}}\,dt = \frac{1}{\beta}\int_1^\infty u^{x-1/\beta-1}e^{-u}\,du$$

under the change of variables $u = t^{-\beta}$. Now

$$\int_1^\infty e^{-w}w^{z-1}\,dw = \Gamma(z)\sum_{n=0}^\infty \frac{(-1)^n}{n!(z+n)}$$

and thus

$$\begin{aligned} K_\beta(x) &= \frac{1}{\beta}\Gamma\left(\frac{x-1}{\beta}\right) - \frac{1}{\beta}\sum_{n=0}^\infty \frac{(-1)^n}{n!\left(\frac{x-1}{\beta}+n\right)} \\ &= \frac{1}{\beta}\Gamma\left(\frac{x-1}{\beta}\right) + 0\left(\frac{1}{x}\right), \qquad \text{as } x \to \infty. \end{aligned}$$

But Stirling's formula yields

$$\Gamma\left(\frac{x-1}{\beta}\right) = \sqrt{2\pi}\left(\frac{x-1}{\beta}\right)^{x-1/\beta-1/2} e^{-(x-1/\beta)}[1+0(1)], \qquad \text{as } x \to \infty.$$

In particular there are positive constants c, C, and K such that

$$cK_\beta(x) \leqq K^x x^{x/\beta} \leqq C\,K_\beta(x),$$

for x sufficiently large. Thus for any $\alpha > 0$

$$\sum_{n=1}^\infty K_\beta(\alpha n)^{-1/\alpha n}$$

converges for $\beta < 1$ and diverges for $\beta \geqq 1$.

Now the significant part of $\|z^{-n}\|_p$ is that obtained by integrating over $\Omega_0 = \Omega \cap \{z \mid |z| < 1\}$, and is easy to see that if $f(z)$ is a continuous function which depends only on $|z|$, then

$$\int_{\Omega_0} f(z)d\sigma(z) = \int_0^1 f(t)\ell(t)dt.$$

Hence to estimate $\|z^{-n}\|_p^p$ it suffices to consider the integral

$$\int_0^1 t^{-np}\ell(t)dt = K_\beta(np).$$

Thus $\|z^{-n}\|_p^{-1/n}$ is essentially $K_\beta(np)^{-1/np}$, and so

$$\sum_{n=1}^{\infty} \|z^{-n}\|^{-1/n}$$

converges for $\beta < 1$ and diverges for $\beta \geqq 1$.

It is thus seen that the divergence criteria are equivalent for this class of examples.

5. Let Ω be a proper simply connected domain in **C** and let $g(w_1, w_2)$ be Green's function for Ω. The Bergman kernel of Ω is given by

$$K(w_1, w_2) = -\frac{2}{\pi}\partial_{w_1}\bar{\partial}_{w_2}g(w_1, w_2),$$

and the Bergman projection for Ω is defined by

$$P_\Omega f(w_1) = \int_\Omega K_\Omega(w_1, w_2)f(w_2)d\sigma(w_2).$$

The Bergman projection can also be defined as the orthogonal projection of $L^2(\Omega)$ onto $A^2(\Omega)$, but it is well known that the two definitions coincide for $f \in L^2(\Omega)$. A basic question concerning the Bergman projection is the following: when is P_Ω a bounded projection from $L^p(\Omega)$ onto $A^p(\Omega)$? An important step toward the solution of this question appears in the paper [1] of Bekollé and Bonami. If w is a nonnegative function on the unit disk D, then they show that the Bergman projection induces a bounded linear operator on $L^p(wd\sigma)$ $(1 < p < \infty)$ if and only if

$$\sup\left\{\frac{1}{|S|}\int_S wd\sigma\right\}\left\{\frac{1}{|S|}\int_S W^{-1/p-1}\,d\sigma\right\}^{p-1}$$

is finite, where the supremum is taken over all Carleson rectangles

$$S = S(\theta, \rho) = \{z = re^{i\phi} \in D \mid 1 - \rho < r < 1, |\theta - \phi| < 2\pi\rho\}.$$

J. Burbea has used this condition in a series of papers (see [3]) to give necessary and sufficient conditions on a domain $\Lambda \subseteq C$ (not necessarily simply connected) for the Bergman kernel to be bounded on $L^p(\Lambda)$ $(1 < p < \infty)$. In this section we will use the Bekollé-Bonami theorem to obtain a result on the boundedness of the Bergman projection on $L^p(\Omega)$ for simply connected Ω. Although this result is implicit in the work of Burbea, we include it because of the elementary nature of the proof in this case. Having produced this characterization, we do a computation which proves that the criteria holds for the crescent bounded by two internally tangent circles. As a direct consequence we obtain a characterization of the dual space of $A^p(\Omega)$ in this case.

THEOREM 4. *Let Ω be a simply connected region and let ψ be a conformal mapping of the unit disk D onto Ω. Then, for $1 < p < \infty$, the Bergman projection P_Ω is bounded on $L^p(\Omega)$ if and only if*

$$\sup\left\{\frac{1}{|S|}\int_S |\psi'|^{2-p}\right\}^{1/p}\left\{\frac{1}{|S|}\int_S |\psi'|^{2-q}\right\}^{1/q}$$

is finite, where $q = p/(p-1)$ and the supremum is taken over all Carleson rectangles S.

PROOF: We will show that P_Ω is bounded on $L^p(\Omega)$ if any only if P_D is bounded on $L^p(|\psi'|^{2-p})$. Then the Bekollé-Bonami result applied to $w = |\psi'|^{2-p}$ will finish the proof. It follows from the change of variables formula that $L^p(\Omega)$ is isometrically isomorphic to $L^p(|\psi'|^2)$ under the operator induced by composition with ψ. It follows that P_Ω is bounded on $L^p(\Omega)$ if any only if $\tilde{P}_\Omega$ is bounded on $L^p(|\psi'|^2)$, where $\tilde{P}_\Omega = P_\Omega(f \circ \psi^{-1}) \circ \psi$. It will suffice to show that the boundedness of $\tilde{P}_\Omega$ on $L^p(|\psi'|^2)$ is equivalent to the boundedness of P_D on $L^p(|\psi'|^{2-p})$. It is easily seen that the Bergman kernels of Ω and D are related by

$$K_\Omega(\psi(w_1), \psi(w_2)) = K_D(w_1, w_2)\frac{1}{\psi'(w_1)\overline{\psi'(w_2)}},$$

so that

$$\begin{aligned}\tilde{P}_\Omega g(w_1) &= \int_D g(w_2)K_D(w_1, w_2)\frac{|\psi'(w_2)|^2}{\psi'(w_1)\overline{\psi'(w_2)}}\,d\sigma(w_2)\\ &= \frac{1}{\psi'(w_1)}P_D(g\psi')(w_1).\end{aligned}$$

Therefore, if P_D is bounded on $L^p(|\psi'|^{2-p})$, then

$$\begin{aligned}\int_D |\tilde{P}\Omega|^p|\psi'|^2 &= \int_D |P_D(g\psi')|^p|\psi'|^{2-p}\\ &\leqq \|P_D\|^p \cdot \int |g\psi'|^p|\psi'|^{2-p}\\ &= \|P_D\|^p\int_D |g|^p\,|\psi'|^2.\end{aligned}$$

So $\tilde{P}_\Omega$ is bounded on $L^p(|\psi'|^2)$. Clearly the above reasoning can be reversed. That completes the proof.

THEOREM 5. *Let Ω be a crescent bounded by two internally tangent circles and let ψ be a conformal mapping of the unit disk D onto Ω. Then ψ satisfies the criterion of Theorem 4.*

PROOF. It will suffice to prove the theorem for a specific crescent of the specified shape and a specific mapping function. The general case follows by observing that a change in the crescent or the mapping function only changes the size of the constants appearing in the final inequality. So let Ω be the crescent with multiple boundary point at the origin bounded by the two circles

$$C_1 : \left| w + \frac{i}{\pi} \right| = \frac{1}{\pi},$$

$$C_2 : \left| w + \frac{i}{3\pi} \right| = \frac{1}{3\pi}$$

The mapping function is

$$\psi(z) = \frac{1}{\log \dfrac{1+z}{1-z} + i\pi}.$$

A straightforward calculation shows that there are positive constants d_1 and d_2 such that

$$\frac{d_2}{|1-z^2| \log \left| \dfrac{1+z}{1-z} \right|^2} \leqq |\psi'(z)| \leqq \frac{d_1}{|1-z^2| \log \left| \dfrac{1+z}{1-z} \right|^2}$$

for $z \in D$. By symmetry it suffices to prove that there is a constant $C < 0$ such that

$$(*) \left\{ \frac{1}{|S|} \int_S \frac{d\sigma(z)}{(1-z| \log^2 |1-z)|^{2-p}} \right\}^{1/p} \left\{ \frac{1}{|S|} \int_S \frac{d\sigma(z)}{(|1-z| \log^2 |1-z|)^{2-q}} \right\}^{1-q} \leqq C$$

for all Carleson rectangles S contained in $D \cap \operatorname{Re} z > 0$ with $1 \in \partial S$. Evidently it will suffice to establish (*) with the rectangles S replaced by the sets $A_\varepsilon = \{|z-1| < \varepsilon\} \cap D$ for all ε with $0 < \varepsilon < 1$. To this end define

$$H(\varepsilon, \alpha) = \int_0^\varepsilon \frac{1}{\left(r \log^2 \dfrac{1}{r}\right)^\alpha} r\,dr \qquad -\infty < \alpha < 1.$$

An easy argument shows that

$$\left\{\frac{1}{|A_\varepsilon|}\int_{A_\varepsilon}\frac{d\sigma(z)}{(|1-z|\log^2|1-z|)^{2-p}}\right\}^{1/p}\left\{\frac{1}{|A_\varepsilon|}\int_{A_\varepsilon}\frac{d\sigma(z)}{(|1-z|\log^2|1-z|^{2-q}}\right\}^{1/q}$$
$$\leqq C\,\varepsilon^{-2}\,H^{1/p}(\varepsilon, 2-p)\,H^{1/q}(\varepsilon, 2, -q)$$

where C is an absolute constant. One can prove that

$$H(\varepsilon, \alpha) \leqq C_\alpha\,\varepsilon^{2-\alpha}\log^{-2\alpha}\frac{1}{\varepsilon}$$

and the theorem follows from this inequality.

REMARK. For $1 < p < \infty$, if the Bergman projection P_Ω is bounded on $L^p(\Omega)$ then the dual space of $A^p(\Omega)$ is isomorphic to $A^q(\Omega)$ under the pairing

$$(f, g) = \int_\Omega f(z)\,\overline{g(z)}\,d\sigma(z).$$

6. Finally we discuss several questions arising out of this work. First it would be nice to have necessary conditions corresponding to the sufficient conditions of §3. In particular it should be noted that Theorem 3 has a measure-theoretic flavor. In fact the work in §3 was inspired be a recent result of D. Luecking [5]. His result is (in effect) as follows.

THEOREM. *Let Ω be a region in the unit disk D such that $\delta = \inf(\sigma(\Omega) \cap S/\sigma(S)) > 0$ where the infimum is taken over all Carleson rectangles S. Then there is a constant C depending only on δ and p $(1 \leqq p < \infty)$ such that*

$$\int_D |f|^p\,d\sigma \leqq C\int_\Omega |f|^p d\sigma$$

for all $f \in A^p(D)$.

An elaboration of his ideas may make it possible to establish necessary conditions on crescents Ω for which the "width" function $\ell(t)$ does not satisfy the regularity condition $(\ell'(t)/\ell(t)) \uparrow +\infty$ as $t \downarrow 0$.

A second problem is the following: for which domains Ω and for which values of p (depending perhaps on Ω) is the space $A^p(\Omega)$ complemented in $L^p(\Omega)$? Of course this is immediate if the Bergman projection is bounded on $L^p(\Omega)$. By using Bergman projections corresponding to various weighted area measures on the unit disk, Shields and Williams [9] have shown that $A^1(D)$ is complemented in $L^1(D)$. Closely related to this is the question of finding characterizations of dual spaces similar to the result of Theorem 6. In particular in those cases where $H^p \neq A^p$ $(1 \leqq p < \infty)$ it would be interesting to find useful characterizations of the dual spaces $(A^p)^*$ and $(H^p)^*$. Some recent work of Coifman and Rochberg [11] on atomic decompositions for Bergman spaces suggested a possible line of approach to some of these problems.

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