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REMARKS ON SHARKOVSKY'S THEOREM

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Dedicated to the Memory of Ernst Straus

1. Introduction. When a continuous function has a cycle of given period, it has cycles of other periods specified through their appearance in a certain ordering of the positive integers, i.e., the Sharkovsky ordering. This note discusses the result and related ones, concentrating on simply-stated conditions which guarantee the existence of certain integer periods. We deliberately avoid referring to the dynamical structure of maps in this brief survey. For problems involving uniqueness, stability, or bifurcations of parameterized families of maps, the reader may consult the recent monograph of Collet and Eckmann [7].

The first section states the Sharkovsky theorem and related theorems for maps of the interval. The second section discusses circle maps. In the third section we inquire about analogous cycles found in discrete-valued maps, and we are led to stating what is believed to be an original combinatorial problem associated with the cycle structure of a certain set of permutations.

2. Sharkovsky's Theorem. Let I be a closed interval and let $f\colon I \to I$ be continuous. Denote self-composition of f inductively by $f^0(x) = x$ and $f^n(x) = f(f^{n-1}(x)), n \geqq 1$. The set $\{x, f(x), f^2(x), \dots\}$ is called the orbit of x. If the orbit is finite, the least positive integer k with $f^k(x) = x$ is the period of x; in this case the orbit is called a k-cycle.

Consider the following ordering of the positive integers

$$1, 2, 4, \dots, 2^n, 2^{n+1}, \dots, 2^{n+1} \cdot 5, 2^{n+1} \cdot 3, \dots 2^n \cdot 5, 2^n \cdot 3, \dots, 2^2 \cdot 5, \dots, 2^2 \cdot 3, \dots, 2 \cdot 5, 2 \cdot 3, \dots, 7, 5, 3.$$

The order writes down, in the manner indicated, the successive powers of 2, then all numbers of the form $2^n \times$ odd integer, and finally the odd integers. A. N. Sharkovsky [18] proved the following beautiful theorem. If f has a point of period m, then f also has a point of period n precisely

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when n is to the left of m in the above ordering. It would be interesting to know if the Sharkovsky ordering also arises in any problems from number theory.

Li and Yorke **[11]**, unaware of Sharkovsky's work at the time, proved (among other things) that a point of period 3 implies points of all periods 1, 2, ... Another corollary is that the existence of a point of period of the form $2^n \times$ odd integer implies infinitely many other periods, whereas period 2^n implies only finitely many periods. Note too, that if f has a point of period $2^n \times$ odd integer, then some power of f has period 3. For detailed proofs of the Sharkovsky theorem see Stefan **[20]** and (for maps with a single critical point) Guckenheimer **[8]**.

The proofs are not easily banalized, but one may say they generally depend on the natural order relation of the real numbers, through which one keeps track of the successive images of certain intervals. For example, the original hypothesis of Li and Yorke assumes there is a point $a \in I$ with $f^3(a) \leqq a < f(a) < f^2(a)$ (so that period 3 is a special case). For k a positive integer, they find a subinterval $Q \subset [f^3(a), a]$ such that $f^k(Q) \supset Q$, from which it follows that f has a point of period k. The argument depends essentially on the following proposition. Let A and B be closed intervals with $A \cap B$ empty or a singleton point. If f: $A \cup B \to R$ is continuous and $f(A) \supset (A \cup B)$ and $f(B) \supset A$, then f has points of all periods. For an n-dimensional version of this proposition see Rogers and Marotto **[17]**.

Li, Misiurewicz, Pianigiani, and Yorke **[12]** have an elegantly stated generalization of the Li-Yorke theorem as follows. For $x_0 \in I$, say the finite orbit $\{x_0, f(x_0), \ldots, f^n(x_0)\}$ has no division if there is no $a \in I$ such that $f^j(x_0) < a$ for all j even, and $f^j(x_0) > a$ for all j odd. They prove that if f has no division and $f^n(x_0) \leqq x_0 < f(x_0)$ or $f^n(x_0) \geqq x_0 > f(x_0)$, then f has a point of odd ($\neq 1$) period.

In regard to applications of Sharkovsky's theorem (difference equation models) it is interesting that the result is sturdy to perturbations in f. Block [5] shows that if f has a point of period n, then there is a neighborhood N of f in $C(I)$ such that for all $g \in N$ and all positive integers k, with k left of n in the Sharkovsky order, g has a point of period k. Kloeden **[10]** and Butler and Pianigiani **[6]** have related results.

Straffin **[21]** used properties of directed graphs to show that an odd period $k \geqq 1$ implies all periods $\geqq k - 1$, generalizing the period 3 theorem of Li and Yorke. The points composing the k-cycle are located on the real line, and these determine $k - 1$ closed subintervals $I_1, I_2, \ldots, I_{k-1}$. The associated digraph has vertices labeled $I_1, I_2, \ldots, I_{k-1}$, and an arrow is drawn from I_i to I_j if $f(I_i) \supset I_j$. Straffin proves that if such a graph has a nonrepetitive circuit of length k, then f has a k-cycle. For example, in the case of period 3, the graph has arrows connecting I_1 to I_2, I_2 to I_1, and a loop at I_2. To conclude period 5 from this graph, go from I_1 to I_2, then go around the loop at I_2 three times, and finally go back to I_1. A graph-theoretic proof of the complete Sharkovsky theorem is due to Ho and Morris [9] and, independently, to Block, Guckenheimer, Misiurewicz and Young [4].

3. Circle maps. Continuous maps of the circle have numerous applications to problems associated with biological clocks. See Winfree [22], for example. Block [3] studied the structure of the periodic points of a continuous circle map f assuming that f has a point of period 1, i.e., a fixed point. A nice result in this case is that if f has a point of odd period $n > 1$, f has points of all period $m > n$; the corollary, that a circle map with periods 1, 2, and 3 has points of all periods, was obtained, independently, by Sieberg [19].

The following weaker theorem is due to Bernhardt [2]. Suppose f has smallest periods p_1 and p_2 with $p_1 < p_2$ and suppose $2p_1 \neq p_2$. Then p_1 and p_2 are relatively prime, and f has points of all periods of the form $\alpha p_1 + \beta p_2$ where α and β are any positive integers. More simply, if f has no fixed points and p_1 and p_2 are relatively prime then the same conclusion holds [1].

A complete discussion of maps of the circle with no fixed points (or generally, circle maps of degree 1) is more complicated; see, e.g., Misiurewicz [14].

4. A combinatorial problem. In this last section we briefly consider a set of permutations which have complicated cycle structure. The problems formulated here are directly motivated by the cycle theory for continuous maps outlined above.

Define $\Delta(n)$ to be the set of f such that (i) f is a permutation of $\{1, 2, \ldots, n\}$, and (ii) f strictly increases up to a unique maximum value and thereafter strictly decreases. Hence $\Delta(n)$ is a discrete analogue of families of single-humped continuous maps (the "quadratic family") whose cycle structure, and other dynamical properties, have been so thoroughly studied [7], [8]. See also May [13] and Rogers [16] for biological applications.

$\Delta(n)$ is contained in the set of all $n!$ permutations of $\{1, 2, \ldots, n\}$. It is easy to verify that the cardinality of $\Delta(n)$ is $2^{n-1} - 2$; essentially one constructs an element in $\Delta(n)$ by sticking in a new maximum value n to the immediate left or right of the old maximum value $n - 1$ belonging to an element in $\Delta(n - 1)$. An element $f \in \Delta(n)$, like any permutation, decomposes into a finite number of cycles. Starting at 1, for example, there is some k, $1 \leqq k \leqq n$, with $f^k(1) = 1$, producing a cycle of length k.

If ℓ_k is the number of cycles of period k which occur in a given $f \in \Delta(n)$, then $1 \cdot \ell_1 + 2\ell_2 + \cdots + n\ell_n = n$. An apparently difficult problem is to compute the cycle classes in $\Delta(n)$. Define $C(\ell_1, \ell_2, \ldots, \ell_n)$ to be the number of elements in $\Delta(n)$ which have ℓ_1 1-cycles, ℓ_2 2-cycles, . . . , ℓ_n n-cycles. For example the two elements in $\Delta(3)$ are (132) = (1) (32) and (231), and so C(1, 1, 0) = 1 and C(0, 0, 1) = 1. For any n the cycle enumerator functions C may be computed, however there is no known general formula. See Riordan **[15]** for the corresponding problem (solved) for the set of all permutations on n objects.

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ADDED IN PROOF

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The number of transitive cycles of Δ(n). The following results were proved after this review was written.

THEOREM (Weiss [23]). *The cardinality π_n of the transitive cycles of $\Delta(n)$ is*

$$\pi_n = \frac{1}{n} \sum_{\substack{d \mid n \\ d \text{ odd}}} \mu(d) 2^{n/d-1}$$

where $\mu(d)$ is the Möbius function.

Thus $C(0,\ldots,0,1)=\pi_n$ when $n \geqq 3$. In addition, a remarkable fact is that smooth unimodal families of maps of the interval run though precisely these combinatorial possiblities:

THEOREM. *The number of orientation-reversing cycles of the quadratic family $x \to ax(1-x)$, $x \in [0, 1]$, $1 \leqq a \leqq 4$, of minimum period n, is π_n; further such cycles are born stable.*

Proofs will appear elsewhere; the second theorem exploits the invariant coordinate method of Milnor and Thurston (see [7]). We thank Leo Jonker for help with the stability result.

A conjecture of Guckenheimer [8] states that the bifurcations of the quadratic family are the generic flips and folds. If true, then π_n counts the number of bifurcations of this family.