

FOURIER COEFFICIENTS OF NON-ANALYTIC AUTOMORPHIC FUNCTIONS OF SEVERAL VARIABLES

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1. Introduction. A well known theorem of H. Hamburger [2] states that the Riemann zeta function can be determined from its functional equation, under certain conditions of regularity. This result has been generalized by E. Hecke [3] to the zeta function of an imaginary quadratic field, over the field of rational numbers. Since then the general problem of determining all meromorphic functions, $\phi(s)$, which are expressible as a Dirichlet series absolutely convergent in some right half plane and satisfying functional equations of the type $\xi(s) = \xi(k - s)$, with

$$\xi(s) = \left(\frac{2\pi}{\lambda}\right)^{-s} (\Gamma(s))^a \left(\Gamma\left(\frac{s}{2}\right)\right)^b \left(\Gamma\left(\frac{s+1}{2}\right)\right)^c \phi(s),$$

has been studied. This problem has been solved for the functional equations of the type satisfied by the Dedekind zeta function for a real quadratic field over the field of rational numbers by H. Maass [5]. For this purpose Maass has introduced analogues of analytic automorphic functions, which he called non-analytic automorphic functions. Such functions are defined as complex valued functions, $f(\tau)$, of the two real variables x and y , with $\tau = x + iy$, satisfying the wave equation

$$\left[y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \mu^2 \right] f(\tau) = 0$$

in the upper half plane $y > 0$ and possessing transformation properties for the transformation group generated by the mappings $\tau \rightarrow \tau + \lambda$, $\tau \rightarrow -1/\tau$ (similar to analytic automorphic forms in the classical sense), and with the further requirement that $f(\tau)$ has growth restrictions as τ approaches the boundary of the upper half plane $\tau = x + iy$, $y > 0$. Later, Maass [6] generalized these functions of two real variables to functions of several variables and he called these functions non-analytic automorphic functions of several variables. The precise definition of these functions is as follows.

We consider the $k + 1$ dimensional hyperbolic space as a subspace of

the euclidean space (x_0, x_1, \dots, x_k) , with $x_k > 0$, is endowed with the fundamental metric form

$$ds^2 = \frac{dx_0^2 + dx_1^2 + \dots + dx_k^2}{x_k^2}.$$

The wave equation corresponding to this metric takes the form

$$(1) \quad \left[x_k^{k+1} \sum_{j=0}^k \frac{\partial}{\partial x_j} \left(x_k^{1-k} \frac{\partial}{\partial x_j} \right) + \left(r^2 + \frac{k^2}{4} \right) \right] f = 0.$$

Here we have replaced the usual wave parameter μ^2 by $r^2 + k^2/4$, where r is, like μ , an arbitrary parameter which we hereafter assume to be real. The motions of this hyperbolic space have been characterised by K. Th. Valen [8] using the Clifford number system. Let C_k denote the Clifford number system of rank 2^{k+1} generated over the field of real numbers by the hypercomplex units i_1, i_2, \dots, i_k , satisfying the relations

$$i_p^2 + 1 = 0, i_p i_q + i_q i_p = 0 \quad (p, q = 1, 2, \dots, k; p \neq q).$$

Let V_k denote the subspace of vectors

$$u = u_0 + u_1 i_1 + u_2 i_2 + \dots + u_k i_k \quad (u_p \text{ real}, p = 0, \dots, k)$$

of C_k . Then, to every point (x_0, x_1, \dots, x_k) of the hyperbolic space, there corresponds a unique vector $x = x_0 + x_1 i_1 + \dots + x_k i_k$, with the $(k + 1)^{\text{th}}$ component positive.

The proper motions of the hyperbolic space can be represented as linear fractional transformations

$$x \rightarrow (\alpha x + \beta) (\gamma x + \delta)^{-1}$$

with suitable coefficients $\alpha, \beta, \gamma, \delta \in C_{k-1}$. We consider the hyperbolic motions represented by the vector transformations $x \rightarrow x + \alpha, x \rightarrow -x^{-1}$, where $\alpha \in V_{k-1}$. We choose a fixed lattice T in V_{k-1} .

By a non-analytic automorphic function of the $k + 1$ variables, x_0, x_1, \dots, x_k , we mean a complex valued function, $f(x) = f(x_0, x_1, \dots, x_k)$, defined in the upper half space $x_k > 0$ (which is a subspace of V_k), satisfying the following conditions:

- (a) $f(x)$ is a twice continuously differentiable function in the subspace $x_k > 0$ and is a solution of the wave equation (1);
- (b) $f(x)$ satisfies the growth conditions $f(x) = O(x_k^{\lambda_1})$ as $x_k \rightarrow \infty$ and $f(x) = O(x_k^{-\lambda_2})$ as $x_k \rightarrow 0$, uniformly in x_0, x_1, \dots, x_{k-1} for some positive constants λ_1 and λ_2 ;
- (c) $f(x + \alpha) = f(x)$ for all α in T , where T is a fixed lattice in V_{k-1} ;
- d) $f(x)$ satisfies the transformation formula $f(-x^{-1}) = f(x)$.

The conditions a), b), and c) yield [6], for $f(x)$, the Fourier expansion

$$(3) \quad f(x) = u(x_k) + \sum_{\substack{\beta \in S \\ \beta \neq 0}} a(\beta) x_k^{k/2} K_{i_r}(2\pi|\beta|x_k) e^{2\pi i \operatorname{Re}(\beta x)}.$$

The summation on the right side of (3) is over all the vectors β of a lattice S determined uniquely by the given lattice T as follows. Let $\alpha_1, \dots, \alpha_k$ be an arbitrary basis of T . Let β_1, \dots, β_k be vectors satisfying $\operatorname{Re}(\alpha_\mu b_\nu) = \delta_{\mu\nu}$, for $\mu, \nu = 1, 2, \dots, k$, $\delta_{\mu\nu}$ being the Kronecker symbol. The lattice S , then, is generated by the vectors β_1, \dots, β_k . S can be characterized as the set of all vectors $\beta \in V_{k-1}$, satisfying $\operatorname{Re}(\alpha\beta) \equiv 0(1)$, for $\alpha \in T$. The relation between the lattices T and S is symmetric. Further, $|\beta|$ denotes the length of the vector β , $\operatorname{Re} a$ denotes the real part of the element $a \in C_k$, and $K_\nu(z)$ the Bessel function of "purely imaginary argument" usually so denoted [11] and which is a solution of the differential equation

$$z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} - (z^2 + \nu^2) w = 0$$

having the asymptotic behaviour $K_\nu(z) \sim \sqrt{\pi/2z} e^{-z}$ as $z \rightarrow \infty$.

Further,

$$(4) \quad u(x_k) = \begin{cases} a_1 x_k^{k/2+ir} + a_2 x_k^{k/2-ir}, & \text{for } r \neq 0, \\ a_1 x_k^{k/2} + a_2 x_k^{k/2} \log x_k, & \text{for } r = 0 \end{cases}$$

a_1, a_2 being constants.

Our aim here is to consider the averages $\sum_{0 < |\beta|^2 \leq x, \beta \in S} a(\beta) P_n(\beta) (x - |\beta|^2)^\gamma$, and express them as a convergent series of analytic functions for suitable positive real numbers γ . Here, $a(\beta)$ denotes the Fourier coefficient occurring in (3) and $P_n(\beta) = P_n(b_0, b_1, \dots, b_{k-1})$ ($\beta = b_0 + b_1 i_1 + \dots + b_{k-1} i_{k-1}$) is an arbitrary spherical function of order n in k variables in the sense of E. Hecke [4] and x is a positive real number. Problems of the type mentioned here, namely to express $\sum_{0 < a_n \leq x} a_n (x - n)^\gamma$ as a series of analytic functions, a_n being an arithmetical function or the Fourier coefficient of an analytic modular form in the classical sense, have been considered by various authors like Voronöi, Hardy, Landau, Walfisz, Wilton and others. For instance, it is well known that if a_n denotes the number of ways of expressing the integer n as the sum of m squares, then

$$\sum_{0 < n \leq x} a_n (x - n)^\gamma = c_0 x^{\gamma+m} + P_\gamma(x),$$

where $P_\gamma(x) = -x^\gamma + \pi^{-\gamma} \Gamma(\gamma + 1) \sum_{n=1}^\infty a_n (x/n)^{m/4+\gamma/2} J_{m/2+\gamma}(2\pi\sqrt{nx})$, $J_\mu(x)$ being the Bessel functional of the first kind, with the series on the right converging absolutely for $\gamma > (1/2)(m - 1)$, conditionally for $\alpha > (1/2)(m - 3)$ and summable by Riesz typical means (R, n, ℓ) for $0 \leq \gamma \leq$

$(1/2)(m - 3)$, where $\iota > (1/2)(m - 3) - \gamma$. We adopt the usual convention that in $\sum_{0 < n \leq x} a_n(x - n)^\iota$, if $\gamma = 0$ and x is an integer representable as a sum of m squares, the last term in the sum is to be multiplied by $1/2$. Thus, in this classical case of “number of representations of n as the sum of squares”, the series of analytical functions is a series involving Bessel functions of the first kind. If a_n denotes the number of positive divisors of n then the series involve the Bessel functions $Y_\nu(x)$ and $K_\nu(x)$ and the functions become more and more complicated as the Fourier coefficients a_n become more “complicated”. The author has earlier considered two such cases. In the first instance [9], a_n represents the “measure of representation” of integral representations of n by an indefinite, integral quadratic form in m variables, and in the second instance [10], a_n is the Fourier coefficient of a non-analytic automorphic form of a certain type considered by Maass [7]. In the second case the analytic functions occurring are very complicated and can be identified in terms of known functions only in special cases. In the present paper, we express $\sum_{0 < |\beta|^2 \leq x, \beta \in S} a(\beta) P_n(\beta) (x - |\beta|^2)^\iota$ as a series of analytic functions and we will identify the analytic function as a Meijer function of the type $G_{5,1}^1(y|_{b_1}^{a_1, a_2, a_3, a_4, a_5})$ or as $G_{4,0}^0(y|a_1, a_2, a_3, a_4)$, depending on the parameter r being non-zero or zero respectively.

2. Non-analytic automorphic functions of several variables and their associated Dirichlet series. For every vector $\beta = b_0 + b_1 i_1 + \dots + b_{k-1} i_{k-1}$ in C_k , we define its conjugate vector β' as $b_0 - b_1 i_1 - \dots - b_{k-1} i_{k-1}$ and, for any spherical function $P_n(\beta) = P_n(b_0, b_1, \dots, b_{k-1})$, we define its conjugate function P'_n by $P'_n(\beta) = P_n(\beta')$. We define $P_0(\beta) = 1$. Let $f(x) = f(x_0, x_1, \dots, x_k)$ be a non-analytic automorphic function, as defined by (2), with the Fourier expansion (3). Let

$$(5) \quad F_n(y, P_n) = u_n(y) + \sum_{\substack{\beta \in S \\ \beta \neq 0}} a(\beta) P_n(\beta) y^{k/2+n} K_{ir}(2\pi|\beta|y),$$

where

$$(6) \quad u_n(y) = \begin{cases} u(y), & \text{for } n = 0 \\ 0, & \text{for } n > 0 \end{cases}$$

with $u(y)$ as defined in (4). Then it follows [6] that

$$F_n\left(\frac{1}{y}, P_n\right) = (-1)^n F_n(y, P'_n),$$

for $n = 0, 1, 2, \dots$ and for any spherical function P_n of order n in k variables. Further, Maass has shown [6] that to every such non-analytic automorphic function, one can associate a meromorphic function, $\phi(s, P_n)$, having the following properties:

$$(7) \quad \begin{aligned} & 4 \int_0^\infty (F_n(y, P_n) - u_n(y)) y^{2s-n-k/2-1} dy \\ & = \pi^{-2s} \Gamma\left(s + \frac{ir}{2}\right) \Gamma\left(s - \frac{ir}{2}\right) \phi(s, P_n); \end{aligned}$$

and

$$(8) \quad \left(s - \frac{k+ir}{2}\right) \left(s - \frac{k-ir}{2}\right) \phi(s, P_0) \text{ and } \phi(s, P_n),$$

for $n > 0$, are entire functions of s of finite genus.

The functional equation

$$(9) \quad \xi\left(\frac{k}{2} + n - s, P_n\right) = (-1)^n \xi(s, P_n) \quad (\text{for } n \geq 0)$$

is valid, where

$$(10) \quad \xi(s, P_n) = \pi^{-2s} \Gamma\left(s + \frac{ir}{2}\right) \Gamma\left(s - \frac{ir}{2}\right) \phi(s, P_n).$$

The functions $\phi(s, P_n)$ can be expressed as Dirichlet series

$$(11) \quad \phi(s, P_n) = \sum_{\substack{\beta \in S \\ \beta \neq 0}} \frac{a(\beta) P_n(\beta)}{|\beta|^{2s}},$$

with a finite abscissa of absolute convergence. Conversely, to every such system of functions, $\phi(s, P_n)$, it is possible [6] to associate a non-analytic automorphic function of the type described by (2). The coefficients a_1, a_2 in $u(y)$ are determined by the conditions that

$$\phi(s, P_0) - \frac{2a_1}{M_r \left(s - \frac{k+ir}{2}\right)} - \frac{2a_2}{M_{-r} \left(s - \frac{k-ir}{2}\right)}$$

and

$$\phi(s, P_0) - \left[\frac{2a_1}{M_0} - \frac{2a_2}{M_0} \left(\frac{\Gamma'\left(\frac{k}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} - \log \pi \right) \right] \frac{1}{s - \frac{k}{2}} - \frac{a_2}{M_0 \left(s - \frac{k}{2}\right)^2}$$

are entire functions of s , according as $r \neq 0$ or $r = 0$ respectively, where

$$M_r = \pi^{-k-ir} \Gamma\left(\frac{k}{2}\right) \Gamma\left(\frac{k}{2} + ir\right).$$

3. Proof of the main result. We consider the Dirichlet series

$$\phi(s, P_n) = \sum_{\substack{\beta \in S \\ \beta \neq 0}} \frac{a(\beta) P_n(\beta)}{|\beta|^{2s}},$$

which converges absolutely for $\text{Re } s > \sigma_0 = (n + \lambda_2)/2 + (3k)/4$. By Perron's formula in the theory of Dirichlet series, it follows that (for $x > 0$ and $\alpha \geq 0$)

$$(13) \quad \sum'_{\substack{0 < |\beta|^2 \leq x \\ \beta \in S}} a(\beta) P_n(\beta) (x - |\beta|^2)^\alpha = x^\alpha \Gamma(\alpha + 1) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^s \Gamma(s) \phi(s, P_n)}{\Gamma(\alpha + 1 + s)} ds$$

($c > 0, c \geq \sigma_1$) where the dash on the left of the summation sign in (13) indicates that if $\alpha = 0$ and $x = |\beta|^2$, where $\beta \in S$, the last term in the summation has to be multiplied by $1/2$ and the line $\text{Re}(s) = \sigma_1$ lies in the half plane of absolute convergence of the Dirichlet series (11). We now choose σ_1 so that $\sigma_1 > 0$ and then we can choose c to be σ_1 . We wish to transform the integral on the right of (13) to an integral taken on the line $\text{Re}(s) = k/2 + n - \sigma_1$, and for this purpose we need an estimate for the integrand on the right of (13) in the strip $k/2 + n - \sigma_1 \leq \text{Re}(s) \leq \sigma_1$. Since the Dirichlet series $\phi(s, P_n)$ converges absolutely for $\sigma \geq \sigma_1$, it follows that

$$\phi(s, P_n) = O(1) \text{ for } \sigma \geq \sigma_1.$$

(We are assuming as usual that $s = \sigma + it$, σ and t real.) Since $\phi(s, P_n)$ satisfies the functional equation (9), it follows that

$$\begin{aligned} \phi(s, P_n) &= (-1)^n \pi^{4s-k-2n} \frac{\Gamma\left(\frac{k}{2} + n - s + \frac{ir}{2}\right) \Gamma\left(\frac{k}{2} + n - s - \frac{ir}{2}\right)}{\Gamma\left(s + \frac{ir}{2}\right) \Gamma\left(s - \frac{ir}{2}\right)} \\ &\times \phi\left(\frac{k}{2} + n - s, P'_n\right). \end{aligned}$$

We set $s = k/2 + n - \sigma_1 + it$ in the above and, using the Sterling approximation,

$$\Gamma(\sigma + it) \sim \sqrt{2\pi} e^{-\pi/2|t|} |t|^{\sigma-1/2} \quad \text{as } |t| \rightarrow \infty,$$

we obtain

$$\phi\left(\frac{k}{2} + n - \sigma_1 + it, P_n\right) = O(|t|^{4\sigma_1-k-2n}).$$

Since $\phi(s, P_n)$, for $n > 0$) and $(s - (k + ir)/2)$ $(s - (k - ir)/2)\phi(s, P_0)$ are entire functions of finite genus, it follows by the principle of Phragmén-Lindelöf, that

$$\phi(\sigma + it, P_n) = O(|t|^{\tau(\sigma)}),$$

as $|t| \rightarrow \infty$, uniformly in the strip $k/2 + n - \sigma_1 \leq \sigma \leq \sigma_1$, where $\tau(\sigma)$ is the linear function passing through the points $(k/2 + n - \sigma_1, 4\sigma_1 - k - 2n)$ and $(\sigma_1, 0)$ and we obtain $\tau(\sigma) = 2(\sigma_1 - \sigma)$. We thus obtain

$$(14) \quad \phi(s, P_n) = O(|t|^{4\sigma_1 - k - 2n}), \quad \text{for } |t| \rightarrow \infty,$$

uniformly in $k/2 + n - \sigma_1 \leq \sigma \leq \sigma_1$.

We consider the integral of $(x^s \Gamma(s) \phi(s, P_n)) / (\Gamma(\alpha + 1 + s))$ over the rectangle with vertices at $\sigma_1 \pm iT$ and $k/2 + n - \sigma \pm iT$ described in the positive sense. For $|T|$ large, there are, at most, poles at $k \pm ir/2$ and the poles of $\Gamma(s)$ in the rectangle. We denote the sum of the residues in the rectangle by $Q_\alpha(x)$. We prove that the integral on the horizontal lines $\sigma \pm iT$, $k/2 + n - \sigma_1 \leq \sigma \leq \sigma_1$ tends to zero, as $|T| \rightarrow \infty$, if α is large. Using Sterling's approximation for $\Gamma(s)$ and the estimate (14), it follows that

$$(15) \quad \int_{k/2+n-\sigma_1 \pm iT}^{\sigma_1 \pm iT} \frac{x^s \Gamma(s) \phi(s, P_n)}{\Gamma(\alpha + 1 + s)} ds = O(x^{\sigma_1} |T|^{4\sigma_1 - k - 2n - \alpha - 1}) \\ = O(1) \text{ as } |T| \rightarrow \infty, \text{ if } \alpha > 4\sigma_1 - k - 2n - 1.$$

We assume that α satisfies this condition. We then obtain

$$\sum_{\substack{0 < |\beta|^2 \leq x \\ \beta \in S}} a(\beta) P_n(\beta) (x - |\beta|^2)^\alpha \\ = \Gamma(\alpha + 1) x^\alpha \left\{ Q_\alpha(x) + \frac{1}{2\pi i} \int_{\sigma=k/2+n-\sigma_1} \frac{x^s \Gamma(s) \phi(s, P_n)}{\Gamma(\alpha + 1 + s)} ds \right\} \\ = \Gamma(\alpha + 1) x^\alpha \left\{ Q_\alpha(x) + (-1)^n \frac{1}{2\pi i} \int_{\sigma=\sigma_1} \frac{x^{k/2+n-s} \Gamma\left(\frac{k}{2} + n - s\right)}{\Gamma\left(\alpha + 1 + \frac{k}{2} + n - s\right)} \right. \\ \left. \times \left[\frac{\pi^{-4s+k+2n} \Gamma\left(s + \frac{ir}{2}\right) \Gamma\left(s - \frac{ir}{2}\right)}{\Gamma\left(\frac{k}{2} + n - s + \frac{ir}{2}\right) \Gamma\left(\frac{k}{2} + n - s - \frac{ir}{2}\right)} \phi(s, P_n') \right] ds \right\},$$

on using the functional equation (9) for $\phi(s, P_n)$. Since $\phi(s, P_n)$ can be represented as a Dirichlet series for $\sigma \geq \sigma_1$, we write $\phi(s, P_n) = \sum_{\beta \neq 0} (a(\beta) P_n'(\beta)) / (|\beta|^{2s})$ in the above integral and obtain, for the right side of (13),

$$(16) \quad \Gamma(\alpha + 1) x^\alpha \left\{ Q_\alpha(x) + (-1)^n x^{k/2+n} \pi^{k+2n} \frac{1}{2\pi i} \int_{\sigma=\sigma_1} \frac{\Gamma\left(\frac{k}{2} + n - s\right)}{\Gamma\left(\alpha + 1 + \frac{k}{2} + n - s\right)} \right. \\ \left. \times \frac{\Gamma\left(s + \frac{ir}{2}\right) \Gamma\left(s - \frac{ir}{2}\right)}{\Gamma\left(\frac{k}{2} + n - s + \frac{ir}{2}\right) \Gamma\left(\frac{k}{2} + n - s - \frac{ir}{2}\right)} \pi^{-4s} x^{-s} \sum_{\substack{\beta \in S \\ \beta \neq 0}} \frac{a(\beta) P_n'(\beta)}{|\beta|^{2s}} ds \right\}.$$

The order of summation and integration in (16) can be interchanged if the series

$$(17) \quad \sum_{\substack{\beta \in S \\ \beta \neq 0}} a(\beta) P_n'(\beta) \times \\ \times \frac{1}{2\pi i} \int_{\sigma=\sigma_1} \frac{\Gamma\left(\frac{k}{2} + n - s\right) \Gamma\left(s + \frac{ir}{2}\right) \Gamma\left(s - \frac{ir}{2}\right) (\pi^4 x |\beta|^2)^{-s}}{\Gamma\left(\alpha + 1 + \frac{k}{2} + n - s\right) \Gamma\left(\frac{k}{2} + n - s + \frac{ir}{2}\right) \Gamma\left(\frac{k}{2} + n - s - \frac{ir}{2}\right)} ds$$

converges absolutely.

Let

$$(18) \quad G(\alpha, k, n, r, \sigma_1; x) = \\ \frac{1}{2\pi i} \int_{\sigma=\sigma_1} \frac{\Gamma\left(\frac{k}{2} + n - s\right) \Gamma\left(s + \frac{ir}{2}\right) \Gamma\left(s - \frac{ir}{2}\right) x^{-s}}{\Gamma\left(\alpha + 1 + \frac{k}{2} + n - s\right) \Gamma\left(\frac{k}{2} + n - s + \frac{ir}{2}\right) \Gamma\left(\frac{k}{2} + n - s - \frac{ir}{2}\right)} ds,$$

where the line of integration $\sigma = \sigma_1$ is so chosen that none of the poles of the integrand lie on it. Using Sterling's approximation for $\Gamma(s)$, we obtain for the integrand the estimate

$$O\left(\frac{x^{-\alpha_1}}{|t|^{\alpha+1+k+2n-4\sigma_1}}\right) \quad (s = \sigma_1 + it),$$

and hence the integral $G(\alpha, k, n, r, \sigma_1; x)$ converges absolutely if $\alpha > 4\sigma_1 - k - 2n$. The series (17) then becomes

$$O\left(\sum_{\substack{\beta \in S \\ \beta \neq 0}} \frac{|a\beta| |P_n'(\beta)|}{(\pi^4 x |\beta|^2)^{\sigma_1}}\right) a(\beta)$$

and hence converges absolutely, since $\sigma = \sigma_1$ lies in the half plane of absolute convergence for the series $\sum_{\beta \in S, \beta \neq 0} a(\beta) P_n'(\beta) / |\beta|^{2s}$.

The series (17) then becomes

$$(19) \quad \sum_{\substack{\beta \in S \\ \beta \neq 0}} a(\beta) P_n'(\beta) G(\alpha, k, n, r, \sigma_1; \pi^4 x |\beta|^2),$$

absolutely convergent for $\alpha > 4\sigma_1 - k - 2n$.

The function $G(\alpha, k, n, r, \sigma_1; x)$ can be expressed in terms of the Meijer G function defined by

$$\begin{aligned}
 G_{p,q}^{m,n} \left(x \left| \begin{matrix} a_1 \cdots a_p \\ b_1 \cdots b_q \end{matrix} \right. \right) &\equiv G_{p,q}^{m,n} \left(x \left| \begin{matrix} a_r \\ b_s \end{matrix} \right. \right) \\
 (2) \quad &= \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s)} x^s ds,
 \end{aligned}$$

where an empty product is interpreted as 1, $0 \leq m \leq q, 0 \leq n \leq p$ and the parameters are such that no poles of $\Gamma(b_j - s), j = 1, \dots, m$ coincide with any pole of $\Gamma(1 - a_k + s), k = 1, \dots, n$. There are several choices for the path of integration L and we shall choose the path to run from $-i\infty$, or $i\infty$ so that all the poles of $\Gamma(b_j - s), j = 1, \dots, m$ are to the right and all the poles of $\Gamma(1 - a_j + s), k = 1, \dots, n$ lie to the left of L . See [1] for the properties of Meijer's G function. We now will express the function $G(\alpha, k, n, r, \sigma_1; x)$ in terms of the Meijer function. The poles of $\Gamma(k/2 + n - s)$ are at $k/2 + n + m$ and the poles of $\Gamma(s \pm (ir)/2)$ are at $\mp (ir)/2 - m, m = 0, 1, 2, \dots$. We accordingly will choose L as the straight line running from $c - i\infty$ to $c + i\infty$, where $0 < c < k/2 + n$. The path of integration for the integral representing $G(\alpha, k, n, r, \sigma_1; x)$ is $\text{Re}(s) = \sigma_1$, where σ_1 is lying in the region of absolute convergence for the Dirichlet series $\phi(s, P'_n)$. We now assume that $\sigma_1 \geq k/2 + n$. We consider the integral of the integrand of (20) over the rectangle R with vertices at $\sigma_1 \pm iT$ and $c \pm iT$ and show that the integral on the horizontal lines from $\sigma_1 + iT$ to $c + iT$ and $\sigma_1 - iT$ to $c - iT$ tends to zero as $|T| \rightarrow \infty$, if $\alpha > 4\sigma_1 - k - 2n$.

By the Sterling approximation for $\Gamma(s)$, it follows that

$$\begin{aligned}
 &\int_{c \pm iT}^{\sigma_1 \pm iT} \frac{\Gamma\left(\frac{k}{2} + n - s\right) \Gamma\left(s + \frac{ir}{2}\right) \Gamma\left(s - \frac{ir}{2}\right)}{\Gamma\left(\alpha + 1 + \frac{k}{2} + n - s\right) \Gamma\left(\frac{k}{2} + n - s + \frac{ir}{2}\right) \Gamma\left(\frac{k}{2} + n - s - \frac{ir}{2}\right)} x^{-s} ds \\
 &= O\left(\frac{x^{-c}}{|T|^{\alpha+1+k+2n-4\sigma}}\right) = O(1) \quad \text{as } |T| \rightarrow \infty,
 \end{aligned}$$

since $\alpha > 4\sigma - k - 2n$.

Hence,

$$\begin{aligned}
 &G(\alpha, k, n, r, \sigma_1; x) \\
 &= G_{5,1}^{2,2} \left(x^{-1} \left| \begin{matrix} 1 - \frac{ir}{2}, 1 + \frac{ir}{2}, \alpha + 1 + k + 2n, \frac{k}{2} + n + \frac{ir}{2}, \frac{k}{2} + n - \frac{ir}{2} \\ \frac{k}{2} + n \end{matrix} \right. \right)
 \end{aligned}$$

+ sum of the residues of the integrand of the integral (18) at its poles in the rectangle R . The poles of the integrand are at $k/2 + n + m, m = 0, 1,$

1, 2, ..., p , where p is an integer such that $k/2 + n + p < \sigma_1 \leq k/2 + n + p + 1$. Since the residue of $\Gamma(s)$ at $s = -m$ is $(-1)^m/m!$, it turns out that the sum of the residues at the poles in the rectangle R is

$$\sum_{m=0}^p \frac{(-1)^m}{m!} \frac{\Gamma\left(\frac{k}{2} + n + m + \frac{ir}{2}\right)\Gamma\left(\frac{k}{2} + n + m - \frac{ir}{2}\right)}{\Gamma(\alpha + 1 - m)\Gamma\left(-m + \frac{ir}{2}\right)\Gamma\left(-m - \frac{ir}{2}\right)} x^{-k/2-n-m}.$$

We examine the special case of $r = 0$. In this case,

$$(21) \quad G(\alpha, k, n, 0, \sigma_1; x) = \frac{1}{2\pi i} \int_{\sigma=\sigma_1} \frac{\Gamma^2(s)}{\Gamma\left(\alpha + 1 + \frac{k}{2} - n - s\right)\Gamma\left(\frac{k}{2} + n - s\right)} x^{-s} ds,$$

and we can take L as $\sigma = \sigma_1$. We then obtain

$$G(\alpha, k, n, 0, \sigma_1; x) = G_{4,0}^{0,2}\left(x^{-1} \middle| 1, 1, \alpha + 1 + \frac{k}{2} - n, \frac{k}{2} + n\right).$$

If $r \neq 0$, we obtain

$$(22) \quad \begin{aligned} & G(\alpha, k, n, r, \sigma_1; x) \\ &= \sum_{m=0}^p \frac{(-1)^m}{m!} \frac{\Gamma\left(\frac{k}{2} + n + m + \frac{ir}{2}\right)\Gamma\left(\frac{k}{2} + n + m - \frac{ir}{2}\right)}{\Gamma(\alpha + 1 - m)\Gamma\left(-m + \frac{ir}{2}\right)\Gamma\left(-m - \frac{ir}{2}\right)} x^{-k/2-n-m} \\ & \quad + G_{5,1}^{1,2}\left(x^{-1} \middle| \begin{matrix} a_r \\ b_1 \end{matrix}\right), \end{aligned}$$

where, for brevity,

$$(23) \quad \begin{aligned} & G_{5,1}^{1,2}\left(x \middle| \begin{matrix} a_r \\ b_1 \end{matrix}\right) \\ &= G_{5,1}^{1,2}\left(x \middle| \begin{matrix} 1 - \frac{ir}{2}, 1 + \frac{ir}{2}, \alpha + 1 + k + 2n, \frac{k}{2} + n + \frac{ir}{2}, \frac{k}{2} + n - \frac{ir}{2} \\ \frac{k}{2} + n \end{matrix}\right). \end{aligned}$$

In view of (21) and (22), the series (19) can be written as

$$(24) \quad \sum_{\substack{\beta \in S \\ \beta \neq 0}} a(\beta) P'_n(\beta) G_{4,0}^{0,2}\left(\pi^{-4} x^{-1} |\beta|^{-2} \middle| 1, 1, \alpha + 1 + \frac{k}{2} - n, \frac{k}{2} + n\right), \quad \text{if } r = 0,$$

and as

$$\begin{aligned}
 & \sum_{m=0}^k \frac{(-1)^m}{m!} \frac{\Gamma\left(\frac{k}{2} + n + m + \frac{ir}{2}\right)\Gamma\left(\frac{k}{2} + n + m - \frac{ir}{2}\right)}{\Gamma(\alpha + 1 - m)\Gamma\left(-m + \frac{ir}{2}\right)\Gamma\left(-m - \frac{ir}{2}\right)} \sum_{\beta \in S} \frac{a(\beta)P'_n(\beta)}{(\pi^4|\beta|^{2k/2+n+m})} \\
 & + \sum_{\substack{\beta \in S \\ \beta \neq 0}} a(\beta)P'_n(\beta)G_{5,1}^2\left(\pi^{-4}x^{-1}|\beta|^{-2}\left|\frac{a_r}{b_1}\right.\right), \quad \text{if } r \neq 0 \\
 (25) \quad & = \sum_{m=0}^k \frac{(-1)^m}{m!} \frac{\Gamma\left(\frac{k}{2} + n + m + \frac{ir}{2}\right)\Gamma\left(\frac{k}{2} + n + m - \frac{ir}{2}\right)\phi\left(\frac{k}{2} + n + m, P'_n\right)}{\Gamma(\alpha + 1 - m)\Gamma\left(-m + \frac{ir}{2}\right)\Gamma\left(-m - \frac{ir}{2}\right)} \frac{(\pi^4x)^{k/2+n+m}}{(\pi^4x)^{k/2+n+m}} \\
 & + \sum_{\substack{\beta \in S \\ \beta \neq 0}} a(\beta)P'_n(\beta)G_{5,1}^2\left(\pi^{-4}x^{-1}|\beta|^{-2}\left|\frac{a_r}{b_1}\right.\right).
 \end{aligned}$$

In the expression (25), $\phi(k/2 + n + m, P'_n)$ is meaningful, as when $r \neq 0$, the only possible singularities of $\phi(s, P'_n)$ are at $k \pm (ir)/2$.

We now obtain the final result of this paper by substituting (24) and (25) for the series (19).

THEOREM. *Let $\sigma = \sigma_1$ lie in the half plane of absolute convergence of the Dirichlet series $\phi(s, P_n)$ and $\phi(s, P'_n)$, $\sigma_1 \geq k/2 + n$, $\alpha \geq 0$, $x > 0$ and $\alpha > 4\sigma_1 - k - 2n$. Then, for $r = 0$,*

$$\begin{aligned}
 (26) \quad & \sum'_{0 < |\beta|^2 \leq x, \beta \in S} a(\beta)P_n(\beta)(x - |\beta|^2)^\alpha = \Gamma(\alpha + 1) x^\alpha Q_\alpha(x) \\
 & + (-1)^n \Gamma(\alpha + 1) x^{\alpha+k/2+n} \pi^{k+2n} \sum'_{\substack{\beta \in S \\ \beta \neq 0}} a(\beta)P'_n(\beta) \\
 & \times G_{8,2}^2\left(\pi^{-4}x^{-1}|\beta|^{-2}\left|1, 1, \alpha + 1 + \frac{k}{2} - n, \frac{k}{2} + n\right.\right),
 \end{aligned}$$

and, for $r \neq 0$,

$$\begin{aligned}
 & \sum'_{0 < |\beta|^2 \leq x, \beta \in S} a(\beta)P_n(\beta)(x - |\beta|^2)^\alpha \times x^{\alpha+k/2+n}\pi^{k+2n} \\
 & = \Gamma(\alpha + 1) x^\alpha Q_\alpha(x) + (-1)^n \Gamma(\alpha + 1) \\
 & \left\{ \sum_{m=0}^k \frac{(-1)^m}{m!} \frac{\Gamma\left(\frac{k}{2} + n + m + \frac{ir}{2}\right)\Gamma\left(\frac{k}{2} + n + m - \frac{ir}{2}\right)}{\Gamma(\alpha + 1 - m)\Gamma\left(-m + \frac{ir}{2}\right)\Gamma\left(-m - \frac{ir}{2}\right)} \phi\left(\frac{k}{2} + n + m, P'_n\right) \right. \\
 & \times \\
 & \left. + \sum_{\substack{\beta \in S \\ \beta \neq 0}} a(\beta)P'_n(\beta)G_{5,1}^2\left(\pi^{-4}x^{-1}|\beta|^{-2}\left|1 - \frac{ir}{2}, 1 + \frac{ir}{2}, \alpha + 1 + k + 2n, \frac{k}{2}\right.\right) \right\},
 \end{aligned}$$

where the dashes on the left sides of the formulas (26) and (27) indicate that the last term on the summation is to be multiplied by 1/2, if $\alpha = 0$ and $x = |\beta|^2$, for some $\beta \in S$, $Q_\alpha(x)$ is the sum of the residues of $(x^s \Gamma(s) \phi(s, P_n)) /$

$\Gamma(\alpha + 1 + s)$ at its singularities in the strip $k/2 + n - \sigma_1 \leq \sigma \leq \sigma_1$, and the series on the right of (26) and (27) converge absolutely.

In conclusion, we remark that, in special cases, the Meijer function may be expressible as a combination of more elementary transcendental functions like the Bessel functions and the Whittaker functions. Such attempts may be useful even though at times frustrating.

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