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PROPERTIES OF PFAFFIANS

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This paper is dedicated to E.G. Straus and R.A. Smith

1. Introduction. Several properties of Pfaffians of real skew-symmetric matrices are studied. In §2, the Pfaffian of a skew-symmetric matrix A is expressed in terms of traces of powers of A. Pfaffians of inverses and Kronecker products are studied next. If A and B are $2(2m+1) \times 2(2m+1)$ skew-symmetric matrices whose product is skew-symmetric, then at least one of the two matrices is singular. If A and B are $4\ell \times 4\ell$ non-singular skew-symmetric matrices whose product is skew-symmetric, then it is shown that Pf(A), Pf(B) and Pf(AB) are either all positive all or negative. Finally, a multilinearity property of Pfaffian functions similar to the one in [**4**] for determinant functions is obtained, and a simple expression for the Frechet derivative of a Pfaffian function is obtained.

2. Pfaffian in terms of traces. If A is a real skew-symmetric matrix of order $2n$ (denoted by $A \in M(2n, \mathbf{R})$) then det(A) is the square of a polynomial in matrix elements of A. This polynomial is called the Pfaffian of A, and is denoted by Pf(A). It is well known, (see [**3**, **6**, **8**]) that

$$\mathrm{Pf}(A) = \sum_P \varepsilon_P a_{i_1 i_2} a_{i_3 i_4} \cdots a_{i_{2n-1} i_{2n}},$$

where P is the permutation

$$\begin{vmatrix} 1 & 2 \cdots & 2n \\ i_1 & i_2 & i_{2n} \end{vmatrix},$$

ε_P its sign, and the sum is taken over all permutations with the restrictions $i_1 < i_2, i_3 < i_4, \ldots, i_{2n-1} < i_{2n}, i_1 < i_3 < i_5 \cdots < i_{2n-1}$. If

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{bmatrix},$$

then $\mathrm{Pf}(A) = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}$.

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LEMMA 2.1. *Let $A \in M(2n, \mathbf{R})$. Then there exists a $T \in M(2n, \mathbf{R})$ such that*
(1) $TA = AT$, and
(2) $T^2 = -I$.

PROOF. Let θ be a real $2n \times 2n$ orthogonal matrix such that

$$\theta A \theta^{-1} = \begin{bmatrix} 0 & a_1 & & & & & \\ -a_1 & 0 & & & & & \\ & & 0 & a_2 & & & \\ & & -a_2 & 0 & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & \cdot \end{bmatrix}.$$

Define

$$\hat{T} = \begin{bmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & \cdot \end{bmatrix}.$$

Then $\hat{T}(\theta A \theta^{-1}) = (\theta A \theta^{-1})\hat{T}$ and $\hat{T}^2 = -I$. Define $T = \theta^{-1}\hat{T}\theta$ (so T is skew-symmetric). Then

$$(1) \qquad \begin{aligned} TA &= \theta^{-1}\hat{T}\theta A = \theta^{-1}\hat{T}(\theta A \theta^{-1})\theta \\ &= \theta^{-1}(\theta A \theta^{-1})\hat{T}\theta = A\theta^{-1}\hat{T}\theta = AT \end{aligned}$$

$$(2) \qquad T^2 = (\theta^{-1}\hat{T}\theta)^2 = \theta^{-1}\hat{T}^2\theta = \theta^{-1}(-I)\theta = -I.$$

LEMMA 2.2. *Let $T \in M(2n, \mathbf{R})$ be such that $T^2 = -I$. Then*

$$\mathrm{Pf}(T) = \pm 1.$$

PROOF. From $T^2 = -I$, it follows that $(\det(T))^2 = (\mathrm{Pf}(T))^4 = 1$. Since T is real, $\mathrm{Pf}(T) = \pm 1$.

THEOREM 2.1. *Let $n \in \mathbf{N}$, A and $T \in M(2n, \mathbf{R})$, be such that*
(1) $TA = AT$,
(2) $T^2 = -I$.

Then

$$\mathrm{Pf}(A) = \frac{1}{n!2^n}\,\mathrm{Pf}(T)\begin{vmatrix} -T_1 & 2 & 0 & 0\;0\;0 & \cdots & \\ T_2 & -T_1 & 4 & 0\;0\;0 & \cdots & \\ -T_3 & T_2 & -T_1 & 6\;0\;0 & \cdots & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \cdot \\ (-1)^n T_n & (-1)^{n-1}T_{n-1} & & \cdots & & -T_1 \end{vmatrix},$$

where $T_r = \mathrm{tr}(A^r T^r)$, $\mathrm{Pf}(T) = \pm 1$.

PROOF. Let θ be a real $2n \times 2n$ orthogonal matrix which transforms both A and T to skew-symmetric canonical forms.
Thus

$$\theta T \theta^{-1} = \begin{bmatrix} 0 & \pm 1 & & & & & \\ \mp 1 & 0 & & & & & \\ & & 0 & \pm 1 & & & \\ & & \mp 1 & 0 & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & \cdot \end{bmatrix}.$$

Since

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \varepsilon \\ -\varepsilon & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -\varepsilon \\ \varepsilon & 0 \end{pmatrix}$$

we can assume that

$$\theta T \theta^{-1} = \begin{bmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & \cdot \end{bmatrix}.$$

Let

$$\theta A \theta^{-1} = \begin{bmatrix} 0 & a_1 & & & & & \\ -a_1 & 0 & & & & & \\ & & 0 & a_2 & & & \\ & & -a_2 & 0 & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & \cdot \end{bmatrix}.$$

so that $\mathrm{Pf}(\theta A\theta^{-1}) = \prod_{j=1}^{n} a_j$. If $S_r = \sum_{j=1}^{r} a_j^r$, then, by one of Newton's identities for symmetric functions,

$$\prod_{j=1}^{n} a_j = \begin{bmatrix} S_1 & 1 & 0 & 0 & 0 & \cdots & \\ S_2 & S_1 & 2 & 0 & 0 & \cdots & \\ S_3 & S_2 & S_1 & 3 & 0 & \cdots & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \ddots & \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & & & & & & n-1 \\ S_n & S_{n-1} & & & & & S_1 \end{bmatrix}.$$

Since

$$(\theta A\theta^{-1})(\theta T\theta^{-1}) = \begin{bmatrix} -a_1 & & & & \\ & -a_1 & & & \\ & & -a_2 & & \\ & & & -a_2 & \\ & & & & \ddots \end{bmatrix},$$

it is easily seen that

$$T_r = t_r(A^r T^r) = t_r((AT)^r) = \mathrm{tr}\begin{bmatrix} (-a_1)^r & & & & \\ & (-a_1)^r & & & \\ & & (-a_2)^r & & \\ & & & (-a_2)^r & \\ & & & & \ddots \end{bmatrix}$$

$$= 2(-1)^r \sum_{j=1}^{r} a_j^r = 2(-1)^r S_r, \quad r = 1, \ldots, n,$$

or

$$S_r = \frac{(-1)^r}{2} T_r.$$

Therefore

$$\mathrm{Pf}(A) = |\theta|\,\mathrm{Pf}(\theta A\theta^{-1}) = \frac{|\theta|}{n!}\begin{vmatrix} -\frac{1}{2}T_1 & 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2}T_2 & -\frac{1}{2}T_1 & 2 & 0 & 0 & \cdots \\ -\frac{1}{2}T_3 & \frac{1}{2}T_2 & -\frac{1}{2}T_1 & 3 & 0 & \cdots \\ \vdots & & & & & \end{vmatrix},$$

or

$$\mathrm{Pf}(A) = \frac{|\theta|}{n!2^n} \begin{vmatrix} -T_1 & 2 & 0 & 0 & 0 & 0 & \cdots \\ T_2 & -T_1 & 4 & 0 & 0 & 0 & \cdots \\ -T_3 & T_2 & -T_1 & 6 & 0 & 0 & \cdots \\ \vdots & & \ddots & & & & \end{vmatrix}.$$

Finally,

$$\mathrm{Pf}(T) = |\theta|\ \mathrm{Pf}(\theta T\theta^{-1}) = |\theta| \mathrm{Pf} \begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & & \ddots \end{bmatrix} = |\theta|$$

implies the theorem.

3. Pfaffians of inverses. Let $A \in M(2n, \mathbf{R})$ be a non-singular matrix. From

$$[\mathrm{Pf}(A^{-1})]^2 = |A|^{-1} = |A^{-}|^1 = ([\mathrm{Pf}(A)]^2)^{-1} = [\mathrm{Pf}(A)]^{-2}$$

it follows that $\mathrm{Pf}(A^{-1}) = \pm[\mathrm{Pf}(A)]^{-1}$. In this section it is proved that $\mathrm{Pf}(A^{-1}) = (-1)^n[\mathrm{Pf}(A)]^{-1}$.

LEMMA 3.1. *Let $A \in M(2n, \mathbf{R})$ be a non-singular matrix in orthogonal canonical form. Then*

$$\mathrm{Pf}(A^{-1}) = (-1)^n[\mathrm{Pf}(A)]^{-1}.$$

PROOF.

$$A = a_1\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus a_2\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \cdots \oplus a_n\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and $\mathrm{Pf}(A) = a_1 a_2 \cdots a_n$.

$$A^{-1} = \left[\left[-a_1^{-1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] \oplus \left[-a_2^{-1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] \oplus \cdots \oplus \left[-a_n^{-1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right]\right]$$

and $\mathrm{Pf}(A^{-1}) = (-a_1^{-1})(-a_2^{-1}) \cdots (-a_n^{-1}) = (-1)^n[\mathrm{Pf}(A)]^{-1}$.

THEOREM 3.1. *Let $A \in M(2n, \mathbf{R})$ be non-singular. Then*

$$\mathrm{Pf}(A^{-1}) = (-1)^n[\mathrm{Pf}(A)]^{-1}.$$

PROOF. Let θ be a real $2n \times 2n$ orthogonal matrix which transforms A to its canonical form $\hat{A}$. Then $\hat{A} = \theta A\theta^{-1}$, and $\hat{A}^{-1} = \theta A^{-1}\theta^{-1}$. By Lemma 3.1, $\mathrm{Pf}(\hat{A}^{-1}) = (-1)^n[\mathrm{Pf}(\hat{A})]^{-1}$, so

$$\text{Pf}(\theta A^{-1}\theta^{-1}) = (-1)^n[\text{Pf}(\theta A\theta^{-1})]^{-1}.$$

Also $\text{Pf}(\theta A^{-1}\theta^{-1}) = |\theta|\ \text{Pf}(A^{-1})$ and $\text{Pf}(\theta A\theta^{-1}) = |\theta|\ \text{Pf}(A)$, so

$$\text{Pf}(A^{-1}) = (-1)^n[\text{Pf}(A)]^{-1}.$$

EXAMPLE.

$$A = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad A^2 = -I, \qquad A^{-1} = -A, \qquad \text{Pf}(A) = -1$$

$$\text{Pf}(A^{-1}) = (-1)^2[\text{Pf}(A)]^{-1} = -1.$$

4. Pfaffians of Kronecker products. Let $A \in M(2m, \mathbf{R})$, and B be a real $n \times n$ symmetric matrix. Then $A \otimes B \in M(2mn, \mathbf{R})$ and $[\text{Pf}(A \otimes B)]^2 = |A \otimes B| = |A|^n|B|^{2m} = [\text{Pf}(A)]^{2n}|B|^{2m}$. This implies

$$\text{Pf}(A \otimes B) = \pm[\text{Pf}(A)]^n|B|^m.$$

In this section it is proved that

$$\text{Pf}(A \otimes B) = (-1)^{mn(n-1)/2}[\text{Pf}(A)]^n|B|^m$$

and

$$\text{Pf}(B \otimes A) = [\text{Pf}(A)]^n|B|^m.$$

LEMMA 4.1. *Let $A \in M(2m, \mathbf{R})$, and B be an $n \times n$ real symmetric matrix. Then there exists a $2mn \times 2mn$ permutation matrix P such that*

$$B \otimes A = P(A \otimes B)P^{-1}$$

(*see* [**1**, **2**, **5**, **7**]), *and so*

$$\text{Pf}(B \otimes A) = |P|\text{Pf}(A \otimes B).$$

EXAMPLE.

$$m = 1, \qquad n = 2, \qquad P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} |P| = -1.$$

THEOREM 4.1. *Let $A \in M(2m, \mathbf{R})$ be nonsingular, and B be a real $n \times n$ nonsingular symmetric matrix. Then*

$$\text{Pf}(B \otimes A) = [\text{Pf}(A)]^n|B|^m.$$

PROOF. Let $J_0 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, S and R be real $2m \times 2m$ and $n \times n$ rotation matrices such that

$$SAS^{-1} = \begin{bmatrix} a_1 J_0 & & & & \\ & a_2 J_0 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & a_m J_0 \end{bmatrix}$$

and

$$RBR^{-1} = \begin{bmatrix} b_1 & & & & \\ & b_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & b_n \end{bmatrix}.$$

Let $\hat{A} = SAS^{-1}$, $\hat{B} = RBR^{-1}$. Then

$$T(B \otimes A)T^{-1} = (RBR^{-1}) \otimes (SAS^{-1}) = \hat{B} \otimes \hat{A},$$

where $T = R \otimes S$. Therefore

$$\text{Pf}(B \otimes A) = \text{Pf}(T^{-1}(\hat{B} \otimes \hat{A})T) = |T|\text{Pf}(\hat{B} \otimes \hat{A}) = \text{Pf}(\hat{B} \otimes \hat{A}).$$

Now

$$\hat{B} \otimes \hat{A} = \begin{bmatrix} a_1 b_1 J_0 & & & & & & & & & \\ & a_2 b_1 J_0 & & & & & & & & \\ & & \ddots & & & & & & & \\ & & & a_m b_1 J_0 & & & & & & \\ & & & & a_1 b_2 J_0 & & & & & \\ & & & & & a_2 b_2 J_0 & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & a_m b_2 J_0 & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & a_m b_n J_0 \end{bmatrix},$$

$$\begin{aligned} \text{Pf}(\hat{B} \otimes \hat{A}) &= a_1^n a_2^n \cdots a_m^n b_1^m b_2^m \cdots b_n^m \\ &= [\text{Pf}(\hat{A})]^n |\hat{B}|^m, \end{aligned}$$

and

$$\text{Pf}(\hat{A}) = |S|\text{Pf}(A) = \text{Pf}(A)$$

so

$$\text{Pf}(B \otimes A) = [\text{Pf}(A)]^n |B|^m.$$

THEOREM 4.2. *Let A and B be the same as in Theorem* 4.1. *Then*

$$\mathrm{Pf}(A \otimes B) = (-1)^{mn(n-1)/2}[\mathrm{Pf}(A)]^n|B|^m.$$

PROOF. First let $A \in M(2m, \mathbf{R})$, and B be a real $n \times n$ matrix. Then $B \otimes A = P(A \otimes B)P^{-1}$, where $|P| = (-1)^{n(n-1)/2}$. This follows from the fact that the number of transpositions needed to transform $A \otimes B$ to $B \otimes A$ under row and column interchanges is $1 + 2 + \cdots + (n-1) = (n(n-1))/2$. Now let

i) $J_0 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$,

ii) $$J = \begin{bmatrix} J_0 & & & & \\ & J_0 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & J_0 \end{bmatrix} \Bigg\} mJ_0\text{'s},$$

iii) $S \in M(2m, \mathbf{R})$ be non-singular and such that $A = SJS^t$,

iv) $I \in M(n, \mathbf{R})$ be the identity matrix,

v) $T = S \otimes I$ (so $T \in M(2mn, \mathbf{R})$).

Then

$$A \otimes B = (SJS^t) \otimes B = T(J \otimes B)T^t,$$

so

$$\begin{aligned} \mathrm{Pf}(A \otimes B) &= |T|\mathrm{Pf}(J \otimes B) = |S|^n\mathrm{Pf}(J \otimes B) \\ &= [Pf(A)]^n\mathrm{Pf}(J \otimes B). \end{aligned}$$

Now

$$J \otimes B = \begin{bmatrix} J_0 \otimes B & & & & \\ & J_0 \otimes B & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & J_0 \otimes B \end{bmatrix}$$

so

$$\begin{aligned} \mathrm{Pf}(J \otimes B) &= [\mathrm{Pf}(J_0 \otimes B)]^m = \{\mathrm{Pf}[P^{-1}(B \otimes J_0)P]\}^m \\ &= \{|P|\mathrm{Pf}(B \otimes J_0)\}^m = (-1)^{mn(n-1)/2}[\mathrm{Pf}(B \otimes J_0)]^m. \end{aligned}$$

By the last theorem, $\mathrm{Pf}(B \otimes J_0) = [\mathrm{Pf}(J_0)]^n|B|^m = |B|^m$, so

$$\mathrm{Pf}(A \otimes B) = [\mathrm{Pf}(A)]^n\mathrm{Pf}(J \otimes B) = (-1)^{mn(n-1)/2}[\mathrm{Pf}(A)]^n|B|^m.$$

EXAMPLE. For $A = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $n = 2$, $n = 1$, we have

$$A \otimes B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \qquad B \otimes A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

$$-1 = \mathrm{Pf}(A \otimes B) = (-1)^1[\mathrm{Pf}(A)]^2|B| = -1,$$

and

$$1 = \mathrm{Pf}(B \otimes A) = [\mathrm{Pf}(A)]^2\,|B| = 1.$$

5. Pfaffian of AB, where A, B ∈ M(2(2m + 1), R). Let A and B be skew symmetric real matrices. A necessary and sufficient condition for AB to be skew symmetric is that $AB = -BA$ because

$$(AB)^t = -AB \Leftrightarrow B^tA^t = -AB \Leftrightarrow BA = -AB.$$

LEMMA 5.1. *Let*

$$J = \begin{bmatrix} 0 & 1 & & & & & & \\ -1 & 0 & & & & & & \\ & & 0 & 1 & & & & \\ & & -1 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & & 0 & 1 \\ & & & & & & -1 & 0 \end{bmatrix}$$

and B be $2(2m+1) \times 2(2m+1)$ real skew symmetric matrices such that $JB = -BJ$. Then $|B| = 0$.

PROOF. Observe that $J^t = J^{-1}$, so

$$B = -JBJ^{-1} = -JBJ^t$$

$$\begin{aligned} \mathrm{Pf}(B) &= \mathrm{Pf}(-JBJ^t) = -\mathrm{Pf}(JBJ^t) \text{ (since } n = 2m+1 \text{ is odd)} \\ &= -|J|\mathrm{Pf}(B) = -\mathrm{Pf}(B). \end{aligned}$$

Therefore $\mathrm{Pf}(B) = 0$, and $|B| = 0$.

LEMMA 5.2. *Let*

$$\hat{A} = \begin{bmatrix} 0 & a_1 & & & & & & \\ -a_1 & 0 & & & & & & \\ & & 0 & a_2 & & & & \\ & & -a_2 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & & 0 & a_n \\ & & & & & & -a_n & 0 \end{bmatrix}$$

be a $2n \times 2n$ *matrix, where* $\{a_i\}_1^n \subset \mathbf{R}$, $J_0 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, *and let*

$$B = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ B_{21} & B_{22} & \cdots & B_{2n} \\ \vdots & & & \\ B_{n1} & B_{n2} & \cdots & B_{nn} \end{bmatrix}$$

be a $2n \times 2n$ *real matrix, where each* B_{ij} *is a* 2×2 *submatrix such that* $\hat{A}B = -B\hat{A}$. *Then*
(1) $a_i J_0 B_{ij} = -a_j B_{ij} J_0$,
(2) *if* $i, j \in \{1, 2, \ldots, n\}$ *are such that* $B_{ij} \neq 0$, *then* $a_i^2 = a_j^2$.

PROOF. (1) follows from $\hat{A}B = -B\hat{A}$.
Let $B_{ij} = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$. Then (1) implies

$$\begin{bmatrix} a_i\gamma & a_i\delta \\ -a_i\alpha & -a_i\beta \end{bmatrix} = \begin{bmatrix} a_j\beta & -a_j\alpha \\ a_j\delta & -a_j\gamma \end{bmatrix}$$

or

$$\begin{bmatrix} a_i & -a_j \\ a_j & -a_i \end{bmatrix}\begin{bmatrix} \beta \\ \gamma \end{bmatrix} = 0 \text{ and } \begin{bmatrix} a_i & a_j \\ a_j & a_i \end{bmatrix}\begin{bmatrix} \alpha \\ \delta \end{bmatrix} = 0.$$

Since $B_{ij} \neq 0$, either $\left(\begin{smallmatrix} \beta \\ \gamma \end{smallmatrix}\right) \neq 0$ or $\left(\begin{smallmatrix} \alpha \\ \delta \end{smallmatrix}\right) \neq 0$. Therefore $a_i^2 = a_j^2$.

LEMMA 5.3. *Let* $n, n_1, n_2, \ldots, n_r \in \mathbf{N}$ *be such that* $n_1 + n_2 + \cdots + n_r = n$ and let $a_1, a_2 \ldots, a_r \in \mathbf{R}$ *be distinct. Let*

$$J_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and

$$J_j = \begin{bmatrix} J_0 & & & & \\ & J_0 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & J_0 \end{bmatrix}$$

be the $2n_j \times 2n_j$ *direct sum of* n_j J_0's, for $j = 1, \ldots, r$. *Also let*

$$\hat{A} = \begin{bmatrix} a_1 J_1 & & & & \\ & a_2 J_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & a_r J_r \end{bmatrix},$$

where $\hat{A}$ is $2n \times 2n$, B be a real $2n \times 2n$ matrix such that $\hat{A}B = -B\hat{A}$ and

$$B = \begin{bmatrix} \tilde{B}_{11} & \tilde{B}_{12} & \cdots & \tilde{B}_{1r} \\ \tilde{B}_{21} & \tilde{B}_{22} & \cdots & \tilde{B}_{2r} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \tilde{B}_{r1} & \tilde{B}_{r2} & \cdots & \tilde{B}_{rr} \end{bmatrix},$$

where each $\tilde{B}_{ij}$ is a real $2n_i \times 2n_j$ matrix, for $i, j = 1, \ldots, r$. Then

$$a_i J_i \tilde{B}_{ij} = -a_j \tilde{B}_{ij} J_j \qquad i, j = 1, \ldots, r.$$

If $a_i^2 \neq a_j^2$ $i \neq j$, then $\tilde{B}_{ij} = 0$.

The proof of this lemma follows from Lemma 5.2.

LEMMA 5.4. *With the notation of Lemma 5.3, assume that the numbers $a_1^2, a_2^2, \ldots, a_r^2$ are distinct and non zero, and that n is odd. Then $|B| = 0$.*

PROOF. By Lemma 5.3,

$$B = \begin{bmatrix} \tilde{B}_{11} & & & & \\ & \tilde{B}_{22} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot & \\ & & & & & \tilde{B}_{rr} \end{bmatrix},$$

where all the off diagonal $\tilde{B}_{ij}$'s are zero. By Lemma 5.3, $J_i\tilde{B}_{ii} = -\tilde{B}_{ii}J_i$, for $i = 1, 2, \ldots, r$.

Since n is odd and $n = n_1 + n_2 + \cdots + n_r$, at least one of the n_i is odd. Let n_m be odd. Since $J_m\tilde{B}_{mm} = -\tilde{B}_{mm}J_m$ it follows from Lemma 4.1 that $|\tilde{B}_{mm}| = 0$. Therefore

$$|B| = \prod_{j=1}^{r} |\tilde{B}_{jj}| = 0.$$

THEOREM 5.1. *Let $A, B \in M(2(2m+1), \mathbf{R})$ be such that AB is skew symmetric. Then at least one of the two matrices A and B is singular.*

PROOF. Assume that A is non-singular. There exists a real orthogonal $2(2m+1) \times 2(2m+1)$ matrix R such that

$$\mathrm{RAR}^{-1} = \begin{bmatrix} 0 & a_1 & & & & & & & & & & & & & & \\ -a_1 & 0 & & & & & & & & & & & & & & \\ & & 0 & a_1 & & & & & & & & & & & & \\ & & -a_1 & 0 & & & & & & & & & & & & \\ & & & & \ddots & & & & & & & & & & & \\ & & & & & 0 & a_1 & & & & & & & & & \\ & & & & & -a_1 & 0 & & & & & & & & & \\ & & & & & & & 0 & a_2 & & & & & & & \\ & & & & & & & -a_2 & 0 & & & & & & & \\ & & & & & & & & & 0 & a_2 & & & & & \\ & & & & & & & & & -a_2 & 0 & & & & & \\ & & & & & & & & & & & \ddots & & & & \\ & & & & & & & & & & & & 0 & a_2 & & \\ & & & & & & & & & & & & -a_2 & 0 & & \\ & & & & & & & & & & & & & & 0 & a_3 \\ & & & & & & & & & & & & & & -a_3 & 0 \\ & & & & & & & & & & & & & & & & \ddots \end{bmatrix},$$

where the real numbers $a_1, a_2, \ldots, a_r$ are positive and distinct.

(1) Let $\hat{A}$, $\hat{B}$ be defined by

$$\hat{A} = RAR^{-1}$$
$$\hat{B} = RBR^{-1}.$$

Thus $\hat{A}, \hat{B} \in M(2(2m + 1),\mathbf{R})$ and $\hat{A}\hat{B} = -\hat{B}\hat{A}$. By Lemma 5.4, $|\hat{B}| = 0$. Therefore $|B| = |R^{-1}\hat{B}R| = |\hat{B}| = 0$.

(2) Assume B is non-singular. Then $BA = -AB$ is skew symmetric and the argument of part (1) shows that $|A| = 0$.

EXAMPLE.

$$A = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

6. Pfaffian of AB, where A, B ∈ M(4ℓ, R). Let $A, B \in M(4\ell, \mathbf{R})$. Then

$$[\mathrm{Pf}(AB)]^2 = |AB| = |A|\,|B| = [\mathrm{Pf}(A)]^2[\mathrm{Pf}(B)]^2$$

implies that $\mathrm{Pf}(AB) = \pm\mathrm{Pf}(A)\mathrm{Pf}(B)$. For AB to be skew symmetric we must have $AB = -BA$. In this section it is shown that $\mathrm{Pf}(A)$, $\mathrm{Pf}(B)$ and $\mathrm{Pf}(AB)$ all have the same sign if A and B are nonsingular.

LEMMA 6.1. *Let $\ell = m/2$, where m is even. Let I be the $m \times m$ identity* *matrix,*

$$A = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \textit{ and } B = \begin{bmatrix} B_{11} & B_{12} \\ -B_{12}^t & B_{22} \end{bmatrix},$$

a $2m \times 2m$ *skew symmetric matrix where each* B_{ij} *is* $m \times m$ *and* $B_{ii}^t = -B_{ii}$ *and let* $AB = -BA$. *Then*

$$\mathrm{Pf}(AB) = \mathrm{Pf}(B) = (-1)^{\ell(2\ell-1)}\,\mathrm{Pf}(A)\mathrm{Pf}(B).$$

PROOF.

$$AB = \begin{bmatrix} -B_{12}^t & B_{22} \\ -B_{11} & -B_{12} \end{bmatrix}, \quad BA = \begin{bmatrix} -B_{12} & B_{11} \\ -B_{22} & -B_{12}^t \end{bmatrix}.$$

$AB = -BA$ implies that $B_{12}^t = -B_{12}$, $B_{22} = -B_{11}$.

$$\text{Thus } B = \begin{bmatrix} B_{11} & B_{12} \\ B_{12} & -B_{11} \end{bmatrix}.$$

Therefore

$$AB = \begin{bmatrix} B_{12} & -B_{11} \\ -B_{11} & -B_{12} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} I & I \\ -I & I \end{bmatrix} B \begin{bmatrix} I & -I \\ I & I \end{bmatrix},$$

so

$$\mathrm{Pf}(AB) = \frac{1}{2^m}\begin{vmatrix} I & I \\ -I & I \end{vmatrix}\mathrm{Pf}(B).$$

Now

$$\begin{bmatrix} I & I \\ -I & I \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \otimes I,$$

so

$$\begin{vmatrix} I & I \\ -I & I \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}^m = 2^m.$$

Therefore $\mathrm{Pf}(AB) = \mathrm{Pf}(B)$ and

$$\mathrm{Pf}(A) = \mathrm{Pf}(J_0 \otimes I) = (-1)^{m(m-1)/2}[\mathrm{Pf}(J_0)]^m|I| = (-1)^{\ell(2\ell-1)},$$

where

$$J_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Therefore

$$\mathrm{Pf}(AB) = (-1)^{\ell(2\ell-1)}\,\mathrm{Pf}(A)\mathrm{Pf}(B).$$

LEMMA 6.2. *Let*

$$J_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad a > 0,$$

$$A = a\begin{bmatrix} J_0 & & & & \\ & J_0 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & J_0 \end{bmatrix} 2\ell\, J_0\text{'s } (so\ A \in M(4\ell, \mathbf{R})),$$

$B \in M(4\ell, R)$ be such that $AB = -BA$. Then

$$\mathrm{Pf}(AB) = \mathrm{Pf}(A)\mathrm{Pf}(B).$$

PROOF. Let I be the $2\ell \times 2\ell$ identity matrix. Then $A = aI \otimes J_0$. Let P be a real $4\ell \times 4\ell$ permutation matrix such that if U is 2×2 and V is $2\ell \times 2\ell$, $V \otimes U = P(U \otimes V)P^{-1}$. Then

$$\frac{1}{a}A = I \otimes J_0 = P(J_0 \otimes I)P^{-1} = P\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}P^{-1}.$$

Let $\hat{A} = J_0 \otimes I = 1/aP^{-1}AP$, $\hat{B} = P^{-1}BP$, so $\hat{A}\hat{B} = -\hat{B}\hat{A}$. By Lemma 6.1 $\mathrm{Pf}(\hat{A}\hat{B}) = \mathrm{Pf}(\hat{B})$ and $\mathrm{Pf}[P^{-1}((1/a)AB)P] = \mathrm{Pf}(P^{-1}BP)$. Therefore

$$\mathrm{Pf}(AB) = a^{2\ell}\mathrm{Pf}(B) = \mathrm{Pf}(A)\mathrm{Pf}(B).$$

LEMMA 6.3. *Let $a > 0$, $J_0 = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$, $\varepsilon_i = \pm 1$, $i = 1, \ldots, 2\ell$,*

$$A = a\begin{bmatrix} \varepsilon_1 J_0 & & & & \\ & \varepsilon_2 J_0 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \varepsilon_{2\ell} J_0 \end{bmatrix}$$

be a $4\ell \times 4\ell$ matrix, and $B \in M(4\ell, \mathbf{R})$ be such that $AB = -BA$. Then

$$\begin{aligned} \mathrm{Pf}(AB) &= \mathrm{Pf}(A)\mathrm{Pf}(B), \qquad \text{if } \mathrm{Pf}(A) > 0. \\ &= -\mathrm{Pf}(A)\mathrm{Pf}(B), \qquad \text{if } \mathrm{Pf}(A) < 0. \end{aligned}$$

PROOF. Define $\sigma_x = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, so σ_x is orthogonal, and $|\sigma_x| = -1$, $\sigma_x J_0 \sigma_x = -J_0$. Let

$$R = \begin{bmatrix} \sigma_x^{\frac{1-\varepsilon_1}{2}} & & & & \\ & \sigma_x^{\frac{1-\varepsilon_2}{2}} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \sigma_x^{\frac{1-\varepsilon_{2\ell}}{2}} \end{bmatrix}$$

be a $4\ell \times 4\ell$ matrix. Then R is orthogonal, $R = R^{-1}$, and

$$|R| = (-1)^{(1-\varepsilon_1/2)+(1-\varepsilon_2/2)+\cdots+(1-\varepsilon_{2\ell}/2)} = (-1)^N,$$

where N is the number of ε_i equal to -1. $\mathrm{Pf}(A) = (-1)^N a^{2\ell} = |R| a^{2\ell}$, and

$$RAR^{-1} = a \begin{bmatrix} J_0 & & & & \\ & J_0 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & J_0 \end{bmatrix}.$$

Now define $\hat{A} = RAR^{-1}$, $\hat{B} = RBR^{-1}$, so $\hat{A}\hat{B} = -\hat{B}\hat{A}$. By Lemma 6.2, $\mathrm{Pf}(\hat{A}\hat{B}) = \mathrm{Pf}(\hat{A})\mathrm{Pf}(\hat{B})$. Therefore

$$\mathrm{Pf}(R(AB)R^{-1}) = \mathrm{Pf}(RAR^{-1})\mathrm{Pf}(RBR^{-1}).$$

So

$$\begin{aligned} \mathrm{Pf}(AB) &= |R|\mathrm{Pf}(A)\mathrm{Pf}(B) \\ &= \mathrm{Pf}(A)\mathrm{Pf}(B), \qquad \text{if } \mathrm{Pf}(A) > 0 \\ &= -\mathrm{Pf}(A)\mathrm{Pf}(B), \qquad \text{if } \mathrm{Pf}(A) < 0. \end{aligned}$$

THEOREM 6.1. *Let $A, B \in M(4\ell, \mathbf{R})$ be non-singular matrices such that all eigenvalues of A have the same absolute value, and $AB = -BA$. Then*

$$\begin{aligned} \mathrm{Pf}(AB) &= \mathrm{Pf}(A)\mathrm{Pf}(B), \qquad \text{if } \mathrm{Pf}(A) > 0 \\ &= -\mathrm{Pf}(A)\mathrm{Pf}(B), \qquad \text{if } \mathrm{Pf}(A) < 0. \end{aligned}$$

PROOF. Let a be the common absolute value of the eigenvalues of A, let $J_0 = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$, and let R be a rotation matrix such that

$$RAR^{-1} = a \begin{bmatrix} \varepsilon_1 J_0 & & & & \\ & \varepsilon_2 J_0 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & \varepsilon_{2\ell} J_0 \end{bmatrix},$$

where $\varepsilon_i = \pm 1$, for $i = 1, 2, \ldots, 2\ell$. Let $\hat{A} = RAR^{-1}$ and $\hat{B} = RBR^{-1}$. Then

$$\hat{A}\hat{B} = -\hat{B}\hat{A}.$$

By Lemma 6.3,

$$\begin{aligned} \mathrm{Pf}(\hat{A}\hat{B}) &= \mathrm{Pf}(\hat{A})\mathrm{Pf}(\hat{B}), \qquad \text{if } \mathrm{Pf}(\hat{A}) > 0 \\ &= -\mathrm{Pf}(\hat{A})\mathrm{Pf}(\hat{B}), \qquad \text{if } \mathrm{Pf}(\hat{A}) < 0. \end{aligned}$$

Since $\mathrm{Pf}(\hat{A}) = \mathrm{Pf}(A)$, $\mathrm{Pf}(\hat{B}) = \mathrm{Pf}(B)$, and $\mathrm{Pf}(\hat{A}\hat{B}) = \mathrm{Pf}(AB)$, the assertion follows.

LEMMA 6.4. *Let* $r \in \{2, 3, 4, \ldots\}$ *and* $\{m_i\}_1^r \subset \mathbf{N}$. *Also, for* $i = 1, 2, \ldots, r$, *let* $A_i, B_i \in M(2m_i, \mathbf{R})$ *be non-singular matrices such that*
(1) $A_iB_i = -B_iA_i$, and
(2) $\mathrm{Pf}(A_iB_i) = \varepsilon_i\mathrm{Pf}(A_i)\mathrm{Pf}(B_i)$,
where $\varepsilon_i = \mathrm{sgn}[\mathrm{Pf}(A_i)]$. *Let*

$$A = \begin{bmatrix} A_1 & & & & \\ & A_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & A_r \end{bmatrix} \text{ and } B = \begin{bmatrix} B_1 & & & & \\ & B_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & B_r \end{bmatrix}.$$

Then

$$\mathrm{Pf}(AB) = \varepsilon\mathrm{Pf}(A)\mathrm{Pf}(B),$$

where $\varepsilon = \mathrm{sgn}[\mathrm{Pf}(A)]$.

PROOF.

$$\mathrm{Pf}(AB) = \mathrm{Pf}\begin{bmatrix} A_1B_1 & & & & \\ & A_2B_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & A_rB_r \end{bmatrix} = \prod_{i=1}^{r} \mathrm{Pf}(A_iB_i)$$

$$= \prod_{i=1}^{r} \varepsilon_i\mathrm{Pf}(A_i)\mathrm{Pf}(B_i) = \prod_{i=1}^{r} \varepsilon_i\mathrm{Pf}(A)\mathrm{Pf}(B).$$

$$\prod_{i=1}^{r} \varepsilon_i = \prod_{i=1}^{r} \mathrm{sgn}(\mathrm{Pf}A_i) = \mathrm{sgn}\left[\prod_{i=1}^{r} \mathrm{Pf}(A_i)\right] = \mathrm{sgn}[\mathrm{Pf}(A)].$$

LEMMA 6.5. *Let* $m, \hat{m} \in \mathbf{N}$, $\alpha, \hat{\alpha} \in (0, \infty)$, $\alpha \neq \hat{\alpha}$, $\varepsilon_i = \pm 1$, $\hat{\varepsilon}_j = \pm 1$, $i = 1, \ldots, m, j = 1, \ldots, \hat{m}$, $J_0 = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$,

$$J = \begin{bmatrix} \varepsilon_1 J_0 & & & & \\ & \varepsilon_2 J_0 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & \varepsilon_m J_0 \end{bmatrix}, \qquad \hat{J} = \begin{bmatrix} \hat{\varepsilon}_1 J_0 & & & & \\ & \hat{\varepsilon}_2 J_0 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & \hat{\varepsilon}_{\hat{m}} J_0 \end{bmatrix},$$

and B *be a real* $2m \times 2\hat{m}$ *matrix such that* $\alpha JB = -\hat{\alpha}B\hat{J}$. *Then* $B = 0$.

PROOF. $\hat{J}^{-1} = \hat{J}^t$, so $JB\hat{J}^t = -\hat{\alpha}/\alpha B$. Thus $(J \otimes \hat{J})\beta = -\hat{\alpha}/\alpha\beta$, where β is a $4m\hat{m} \times 1$ column vector whose elements are the elements of B in appropriate order. Since $-\hat{\alpha}/\alpha \neq \pm 1$ and all eigenvalues of $J \otimes \hat{J}$ being ± 1, it follows that $\beta = 0$ so $B = 0$.

LEMMA 6.6. *Let* $r \in \mathbf{N}$, $a_1, \ldots, a_r \in (0, \infty)$ *be distinct,* $\{m_i\}_1^r \subset \mathbf{N}$, $\varepsilon_{ij} = \pm 1$, for $i = 1, \ldots, r, j = 1, \ldots, m_i$; *let* $J_0 = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$, *and*

$$\hat{A} = a_i \begin{bmatrix} \varepsilon_{i1}J_0 & & & & \\ & \varepsilon_{i2}J_0 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & \varepsilon_{im_i}J_0 \end{bmatrix} \qquad i = 1, \ldots, r.$$

Let $m = \sum_{i=1}^r m_i$,

$$A = \begin{bmatrix} \hat{A}_1 & & & & \\ & \hat{A}_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & \hat{A}_r \end{bmatrix},$$

and

$$B = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1r} \\ B_{21} & B_{22} & \cdots & B_{2r} \\ \cdot & & \cdot & \\ \cdot & & \cdot & \\ \cdot & & \cdot & \\ B_{r1} & B_{r2} & \cdots & B_{rr} \end{bmatrix} \in M(2m, \mathbf{R})$$

be a non-singular matrix, where each B_{ij} *is a* $2m_i \times 2m_j$ *matrix such that* $AB = -BA$. *Then*

(1) $B_{ii}^t = -B_{ii}$, $\quad i = 1, \ldots, r$;

(2) $B_{ij}^t = -B_{ji}$, $\quad i, j = 1, \ldots, r$;

(3) $\hat{A}_i B_{ij} = -B_{ij}\hat{A}_j$, $\quad i, j = 1, \ldots, r$;

(4) $B_{ij} = 0$, $\quad i, j = 1, \ldots, r, i \neq j$;

(5) *each* m_i *is even*;

(6) $\mathrm{Pf}(\hat{A}_i B_{ii}) = \varepsilon_i \mathrm{Pf}(\hat{A}_i)\mathrm{Pf}(B_{ii})$, $\quad i = 1, \ldots, r$, *where* $\varepsilon_i = \mathrm{sgn}[\mathrm{Pf}(\hat{A}_i)]$;

(7) $\mathrm{Pf}(AB) = \varepsilon \mathrm{Pf}(A)\mathrm{Pf}(B)$, *where* $\varepsilon = \mathrm{sgn}\,[\mathrm{Pf}(A)]$.

PROOF. The proof of (4) follows from (3) and Lemma 6.5. To prove (5) note that $|B| = \prod_{i=1}^r |B_{ii}|$. Since $|B| \neq 0$, $|B_{ii}| \neq 0$. By parts (1), (3) and §5 it follows that each m_i is even. The proof of (6) follows from Theorem 6.1. The proof of (7) follows from (6) and Lemma 6.4.

THEOREM 6.2. *Let* $A, B \in M(4\ell, \mathbf{R})$ *be non-singular matrices such that* AB *is skew symmetric. Then*

$$\mathrm{Pf}(AB) = \varepsilon \mathrm{Pf}(A)\mathrm{Pf}(B),$$

where $\varepsilon = \mathrm{sgn}[\mathrm{Pf}(A)]$.

PROOF. Let R be a real $4\ell \times 4\ell$ rotation matrix such that (with the notation of Lemma 6.6)

$$RAR^{-1} = \begin{bmatrix} \hat{A}_1 & & & & & \\ & \hat{A}_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \hat{A}_r \end{bmatrix},$$

$\hat{A} = RAR^{-1}$, and $\hat{B} = RBR^{-1}$. Then $\hat{A}\hat{B} = -\hat{B}\hat{A}$. By Lemma 6.6, $\mathrm{Pf}(\hat{A}\hat{B}) = \hat{\varepsilon}\mathrm{Pf}(\hat{A})\mathrm{Pf}(\hat{B})$, where $\hat{\varepsilon} = \mathrm{sgn}[\mathrm{Pf}(\hat{A})]$. Since $\mathrm{Pf}(A) = \mathrm{Pf}(\hat{A})$, $\mathrm{Pf}(B) = \mathrm{Pf}(\hat{B})$, and $\mathrm{Pf}(AB) = \mathrm{Pf}(\hat{A}\hat{B})$, the assertion follows.

COROLLARY 6.1. *Let* $A, B \in M(2m, \mathbf{R})$ *be nonsingular matrices such that* AB *is skew-symmetric. Then* $\mathrm{Pf}(A)$, $\mathrm{Pf}(B)$ *and* $\mathrm{Pf}(AB)$ *all have the same sign.*

PROOF. Let $\varepsilon_A = \mathrm{sgn}[\mathrm{Pf}(A)]$, $\varepsilon_B = \mathrm{sgn}[\mathrm{Pf}(B)]$, and $\varepsilon_{AB} = \mathrm{sgn}[\mathrm{Pf}(AB)]$. By Theorem 6.2, $\mathrm{Pf}(AB) = \varepsilon_A \mathrm{Pf}(A)\mathrm{Pf}(B)$. Since $AB = -BA$, $\mathrm{Pf}(AB) = \mathrm{Pf}(-BA) = (-1)^m\mathrm{Pf}(BA)$. By Theorem 5.1, m is even. Therefore $\mathrm{Pf}(AB) = \mathrm{Pf}(BA)$. By Theorem 6.2 $\mathrm{Pf}(BA) = \varepsilon_B\mathrm{Pf}(B)\mathrm{Pf}(A)$. Thus $\varepsilon_A = \varepsilon_B$. Finally,

$$\begin{aligned} \varepsilon_{AB} &= \mathrm{sgn}[\mathrm{Pf}(AB)] \\ &= \varepsilon_A \mathrm{sgn}[\mathrm{Pf}(A)\mathrm{Pf}(B)] = \varepsilon_A \mathrm{sgn}[\mathrm{Pf}(A)]\mathrm{sgn}[\mathrm{Pf}(B)] \\ &= \varepsilon_A^2 \varepsilon_B = \varepsilon_B. \end{aligned}$$

EXAMPLES. For

$$A = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

we have

$$AB = -BA = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad \mathrm{Pf}(A) = \mathrm{Pf}(B) = \mathrm{Pf}(AB) = -1.$$

For

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

we have

$$AB = -BA = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathrm{Pf}(A) = \mathrm{Pf}(B) = \mathrm{Pf}(AB) = 1.$$

7. A multilinearity property of Pfaffian functions and the Frechet derivative of a Pfaffian function. Let $n \in \mathbf{N}$, and $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, let $M(2\mathrm{n}, \mathbf{K})$ be the set of all $2n \times 2n$ skew symmetric matrices over $\mathbf{K}$. Let $\mathrm{Pf}: M(2n,\mathbf{K}) \to \mathbf{K}$ be the Pfaffian function on $M(2n, \mathbf{K})$, and let F_n: $[M(2n,\mathbf{K})]^n \to \mathbf{K}$ be defined by

$$\begin{aligned} F_n(A_1, A_2, \ldots, A_n) &= \mathrm{Pf}(A_1 + A_2 + \cdots + A_n) \\ &\quad - \sum_{i=1}^{n} \mathrm{Pf}(A_1 + \cdots + A_{i-1} + A_{i+1} + \cdots + A_n) \\ &\quad + \sum_{\substack{i,j=1 \\ i<j}}^{n} \mathrm{Pf}(A_1 + \cdots + A_{i-1} + A_{i+1} + \cdots + A_{j-1} + A_{j+1} + \cdots + A_n) \\ &\quad + \cdots \\ &\quad \vdots \\ &\quad + (-1)^{n-2} \sum_{\substack{r,s=1 \\ r<s}}^{n} \mathrm{Pf}(A_r + A_s) + (-1)^{n-1} \sum_{s=1}^{n} \mathrm{Pf}(A_s), \end{aligned}$$

so

$$F_1(A) = \mathrm{Pf}(A).$$

THEOREM 7.1. *$F_n(A_1, A_2, \ldots, A_n)$ is an n-multilinear function.*

THEOREM 7.2. *For all A, $B \in M(2n, \mathbf{K})$, with any norm on $M(2n, \mathbf{K})$,*

$$[\mathrm{Pf}'(A)]B = \frac{1}{(n-1)!} F_n(\underbrace{A, A, \ldots, A}_{n-1 A\text{'s}}, B).$$

The proofs of these theorems are similar to these for similar functions defined for determinants instead of Pfaffians. For details, see **[4]**.

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