ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 15, Number 1, Winter 1985

# FUNCTIONS WITH PREASSIGNED LOCAL MAXIMUM POINTS

F. S. CATER

In a recent note [1], Posey and Vaughan gave an elementary example of a continuous real valued function that has a proper local maximum at each point of a preassigned countable dense set. Let $A$ and $B$ be disjoint countable sets, each dense in the open interval (0, 1). We will use methods just as elementary as those used in [1] to construct a continuous nowhere differentiable function $F$ on (0, 1) such that $F$ has a proper local maximum at each point of $A$ and a proper local minimum at each point of $B$, and has no other local maximum or minimum points.

By a triadic rational number, we mean a rational number of the form $k3^{-n}$ where $n$ is a positive integer and $k$ is an integer. We say that $k3^{-n}$ is even if $k$ is even, and odd if $k$ is odd. We begin with a lemma that is not very original.

LEMMA 1. *Let $A$ and $B$ be disjoint countable dense subsets of* (0, 1). *Then there is a bijective order preserving mapping $g$ of the set of all triadic rational numbers in* (0, 1) *onto $A \cup B$ such that the odd numbers map to points in $A$ and the even numbers map to points in $B$.*

PROOF. Let the sequence $(a_n)$ be an enumeration of $A$ and $(b_n)$ an enumeration of $B$ with $a_1 < b_1$. Let $g(1/3) = a_1$, $g(2/3) = b_1$. Suppose that $g(k3^{1-n})$ has been defined for some $n > 1$ and for all $k = 1, 2, \ldots, 3^{n-1} - 1$, so that $g$ is injective and order preserving on its domain. Let $g(3^{-n})$, $g(3 \cdot 3^{-n})$, $g(5 \cdot 3^{-n})$, $\ldots$, $g((3^n - 2)3^{-n})$ be the points in $A = \{a_i\}$ with the smallest subscripts that make $g$ still injective and order preserving. Let $g(2 \cdot 3^{-n})$, $g(4 \cdot 3^{-n})$, $g(6 \cdot 3^{-n})$, $\ldots$, $g((3^n - 1)3^{-n})$ be the points in $B = \{b_i\}$ with the smallest subscripts that make $g$ still injective and order preserving. This completes the induction on $n$, and $g$ is the required order preserving bijective mapping onto $A \cup B$.

For each $n > 0$, we define a piecewise linear function $f_n$ on [0, 1] as follows. Let

Recieved by the editors July 7, 1983 and in revised form on October 4, 1983.
*Subject Classification*: 26A27.

$$\begin{aligned} f_n(g(3^{-n})) &= f_n(g(3\cdot 3^{-n})) = f_n(g(5\cdot 3^{-n})) \\ &= \cdots = f_n(g((3^n-2)3^{-n})) = f_n(1) = 1, \\ f_n(0) &= f_n(g(2\cdot 3^{-n})) = f_n(g(4\cdot 3^{-n})) \\ &= f_n(g(6\cdot 3^{-n})) = \cdots = f_n(g(3^n-1)3^{-n})) = 0. \end{aligned}$$

Let $f_n$ be linear on the intervals $[g((k-1)3^{-n}), g(k3^{-n})]$ $(k = 1, 2, \ldots, 3^n)$.

We define an increasing sequence $n(j)$ of positive integers as follows. Put $n(1) = 1$. Now suppose that $n(1), n(2), \ldots, n(j-1)$ have been chosen. Let $F_{j-1} = 2^{-1}f_{n(1)} + 2^{-2}f_{n(2)} + \cdots + 2^{1-j}f_{n(j-1)}$. Let $S$ denote the maximum of the absolute values of the left and right derivatives of the piecewise linear function $F_{j-1}$. Now let $n(j)$ be the smallest index $> n(j-1)$ for which the minimum of the absolute values of the left and right derivatives of $2^{-j}f_{n(j)}$ exceeds $S + j$. This completes the induction on $j$, and $n(j)$ is defined for all $j > 0$.

Put $F = \lim_{j\to\infty} F_j$. Then $F = \sum_{j=1}^{\infty} 2^{-j}f_{n(j)}$ and $F$ is continuous on $(0, 1)$. We claim that $F$ has all the desired properties.

1. Choose any $x \in (0, 1)$ and any number $q > 0$. There is an index $j$ so large that $g((k-1)3^{-n(j)}) \leqq x < g((k+1)3^{-n(j)})$ where $k > 1$, $k$ is odd, and $g((k+1)3^{-n(j)}) - g((k-1)3^{-n(j)}) < q$. Put $a = g((k3^{-n(j)}) \in A$, $b = g((k-1)3^{-n(j)}) \in B$, $c = g((k+1)3^{-n(j)}) \in B$. It follows from the definitions of $n(j)$ and $f_{n(j)}$, that the left and right derivatives of the piecewise linear function $F_j = 2^{-j}f_{n(j)} + F_{j-1}$ exceed $j$ on $(b, a)$ and are exceeded by $-j$ on $(a, c)$. Thus

$$(1) \qquad (F_j(a) - F_j(b))(a-b)^{-1} > j,\ (F_j(c) - F_j(a))(c-a)^{-1} < -j.$$

If follows from (1) that $F_j(a) > F_j(b)$ and $F_j(a) > F_j(c)$. But $f_{n(t)}(a) = 1$ and $f_{n(t)}(b) = f_{n(t)}(c) = 0$ for $t > j$. Since $F = F_j + \sum_{t=j+1}^{\infty} 2^{-t}f_{n(t)}$, it follows from (1) that

$$(F(a) - F(b))(a-b)^{-1} > j,\ (F(c) - F(a))(c-a)^{-1} < -j.$$

Either $b \leqq x \leqq a$ or $a \leqq x \leqq c$. We conclude that there are sequences $(u_j)$, $(v_j) \subseteq A \cup B$ such that $u_j \leqq x \leqq v_j$ for all $j$, $v_j - u_j \to 0$ and

$$|(F(v_j) - F(u_j)(v_j - u_j)^{-1}| \to \infty.$$

If $F$ were differentiable at $x$, we would have

$$|(F(v_j) - F(u_j)(v_j - u_j)^{-1}| \to |F'(x)|$$

which is impossible. So $F$ is nowhere differentiable on $(0, 1)$.

2. Suppose $x \in A$. Choose $j$ and $k$ as in paragraph 1 with $j$ so large that $x = a$. Then $F_j$ has a proper maximum at $x$ on the interval $(b, c)$ and

$f_{n(t)}(x) = 1$ for $t > j$. It follows that $F$ has a proper maximum at $x$ on the interval $(b, c)$.

3. Suppose $x \in (0, 1)\backslash A$. Choose $j$ and $k$ as in paragraph 1. Then $b \leqq x < c$, and just as in paragraph 2, $F$ has a proper maximum at a on the interval $[b, c]$. So $F(x) < F(a)$ and $F$ does not have a maximum at $x$ in $[b, c] \subseteq (x - q, x + q)$. Finally, $F$ does not have a local maximum at $x$.

The proof that $F$ has a proper local minimum at points in $B$ and at no other points is analogous to the paragraphs 1, 2 and 3 with $k$ even.

Note that the sequences $(a_n)$ and $(b_n)$ completely determine the functions $g$ and $F$. Some modification of our arguments would insure that $F$ has no left or right derivative at any point in (0, 1), but we will not include that here.

## Reference

**1.** E.E. Posey & J.E. Baughan, *Functions with a proper local maximum in each interval*, Amer. Math. Monthly **90** #4 (1983) 281–282.

Department of Mathematics, Portland State University, Portland, OR 97207