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SPECTRUM OF NONPOSITIVE CONTRACTIONS ON C(X).

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ABSTRACT. Known results in the spectral theory of Markov operators are shown to have analogues which are valid for general contractions. For instance we discuss the group structure of the unimodular eigenfunctions, and the representation of an irreducible operator as a rotation of a compact group, followed by a multiplication.

1. Introduction. Throughout, X will be a compact T^2 space and $C(X)$ the continuous scalar valued functions on X, where the scalar field may be either the real or the complex numbers. T will be a contraction on $C(X)$, i.e., a linear operator will $\|T\| \leqq 1$. T is called a Markov operator in case $T \geqq 0$ and $T1 = 1$. In areas such as ergodic theory and spectral theory, the theory of Markov operators is much more developed than that of general contractions. The reason is that positivity is a great convenience when measures come into play. However there exists a device which enables us to bring positivity into the picture even when T is nonpositive. Let $F(T^*) = \{m \text{ in } C(X)^*\colon T^*m = m\}$, let m be an extreme point of the unit ball $F_1(T^*)$, and let φ_m be the Radon-Nikodym derivative $dm/d|m|$. This was introduced in [3] for the special case where $T^2 = T$, and used in [1] to transfer results from the ergodic theory of Markov operators to general contractions. In this paper we make use of the functions φ_m to prove results in spectral theory already well known for Markov operators [4, 6, 7, 8]. For instance we show that the unimodular eigenfunctions form a group under an operation a little more complicated than pointwise multiplication, and that if T is irreducible and the unimodular eigenfunctions "strongly separate" X, then T is essentially a rotation of a compact group, followed by a multiplication.

It should be noted that in contrast to the Markov case it is possible that $F(T^*) = \{0\}$. On the other hand it is easy to manufacture nontrivial examples: let R be a Markov operator, ψ a unimodular continuous func-

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tion, and define $Tf = R(f\psi)\psi^{-1}$. By [**1**; Proposition 2.9(b)], there is a bijective correspondence between $F(T^*)$ and $F(R^*)$, so $F(T^*) \neq \{0\}$.

2. Notation. If x is in X, let t_x be the measure representing the linear functional $f \to Tf(x)(f$ in $C(X))$, so that for f in $C(X)$, $Tf(x) = \int f\, dt_x$. We may extend the domain of T to include the Baire functions by defining $Tg(x) = \int g\, dt_x$ for g Baire. An easy transfinite induction over the Baire classes shows that Tg is again a Baire function. Moreover if m is in $F(T^*)$, i.e., $T^*m = m$, then transfinite induction gives $\int Tg\, dm = \int g\, dm$ for each Baire function g.

If m is a positive measure, supp m is the smallest closed set of full measure, and if m is a signed or complex measure, supp m = supp$|m|$. We let M = closure $\bigcup\{\text{supp } m\text{: } m \text{ in } F(T^*)\}$ = closure $\bigcup\{\text{supp } m\text{: } m \text{ extreme in } F_1(T^*)\}$. (The second equality follows from Krein-Milman [**1**].) Our results will be valid only on the set M rather than on all X. In 3.3 below we show that M is invariant, i.e., x in M implies supp $t_x \subset M$, i.e., $f|M = 0$ implies $Tf|M = 0$ (see [**9**]). As in [**1**], this implies that we may define a contraction T_0 of $C(M)$ by $T_0 f(x) = T\bar{f}(x)$, where x is in M and $\bar{f}$ is any continuous extension of f to all X. In view of this we are justified in assuming throughout that $X = M$ and $T = T_0$.

A Baire function f is called a unimodular eigenfunction if $|f(x)| = 1$ for all x, and $Tf = \lambda f$ with $|\lambda| = 1$. For each m extreme in $F_1(T^*)$, we let $\varphi_m = dm/d|m|$ (cf. [**1**, 2.1.]), so that φ_m can be taken as a Baire function with $|\varphi_m| = 1$ on supp m. Note that $\bar{\varphi}_m = d|m|/dm$, and on supp m, $\bar{\varphi}_m = \varphi_m^{-1}$. We may further assume that $|\varphi_m| \leqq 1$ everywhere.

3. Baire eigenfunctions. Lemma 3.1 was proved in [**1**, Lemma 2.5], but with the restrictive assumption that a nonvanishing continuous fixed point for T exists. Lemma 3.5 generalizes a well known characterization of unimodular eigenfunctions for Markov operators. (See [**8**, p. 558], [**6**, p. 24], and [**4**, p. 1044].) The group operation in Theorem 3.6 was defined for positive operators in [**7**, p. 188].

Lemma 3.1. *If m is extreme in $F_1(T^*)$ and x is in* supp m, *then* supp $t_x \subset$ supp m.

Proof. Suppose there exists x in supp m with supp $t_x \backslash$supp $m \neq \emptyset$. By complete regularity there exists f in $C(X)$ with $0 \leqq f \leqq 1$, $f = 1$ on supp m, and $f(w) < 1$ for some w in supp t_x. Ldt $Z = f^{-1}(1) \supset$ supp m. We show first that the set $F = \{y \text{ in supp } m\text{: supp } t_y \subset Z\}$ is closed. Let $y(a)$ be a net in F with $y(a) \to y$. If y is not in F, then supp $t_y \not\subset Z$. Choose h in $C(X)$ with $h = 0$ on Z and $h(v) \neq 0$ for some v in supp t_y. Now, supp t_y has the characterization (see, e.g., [**2**, p. 121]). supp $t_y = \bigcap\{k^{-1}(0)\text{: } k \text{ in } C(X) \text{ and } 0 = \int kg\, dt_y \text{ for all } g \text{ in } C(X)\}$. Hence there exists g in $C(X)$ with Tgh(y)

$= \int gh\, dt_y \neq 0$, while since supp $t_{y(a)} \subset Z$, we have $Tgh(y(a)) = 0$ for all a. Thus $Tgh(y(a)) \nrightarrow Tgh(y)$, contrary to $y(a) \to y$.

Now the set $V = \{y$ in supp m: supp $t_y \not\subset Z\}$ is open and non-void in supp m. Further, if y is in V, then supp $t_y \backslash Z$ is a non-void open subset of supp t_y, and since $|f\bar{\varphi}_m| < 1$ on this set, we have $|Tf\bar{\varphi}_m(y)| = |\int f\bar{\varphi}_m\, dt_y| < 1$. Since V is open in supp m, we then get $|\int Tf\bar{\varphi}_m\, dm| < 1$. On the other hand, $\int f\bar{\varphi}_m\, dm = \int \bar{\varphi}_m\, dm = \int d|m| = 1$, and hence $\int Tf\bar{\varphi}_m\, dm \neq \int f\bar{\varphi}_m\, dm$, contrary to $T^*m = m$ (since $f\bar{\varphi}_m$ is a Baire function).

EXAMPLE 3.2. The result can fail if $\|T\| > 1$. For instance let $X = \{1, 2, 3\}$, so that $C(X)$ is essentially $\mathbf{R}^3$ or $\mathbf{C}^3$, and define T by the matrix

$$\begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ 0 & 0 & 0 \end{bmatrix},$$

i.e., $t_1 = T^*\delta_1 = (1/2)\delta_1 + (1/2)\delta_2 + (1/2)\delta_3$, $t_2 = (1/2)\delta_1 + (1/2)\delta_2 - (1/2)\delta_3$, and $t_3 = 0$. If m is the measure given by the row vector (1/2, 1/2, 0), then m is extreme in $F_1(T^*)$, but 1 is in supp $m = \{1, 2\}$, while supp $t_1 = \{1, 2, 3\} \not\subset \{1, 2\}$.

COROLLARY 3.3. *If* $f|M = 0$, *then* $Tf|M = 0$ (f in $C(X)$), *so that the* T_0 *of the Introduction is well defined. Hence we may assume throughout that* $M = X$.

PROOF. This follows from 3.1, just as in [**1**, Corollary 2.6].

LEMMA 3.4. *If* m *is extreme in* $F_1(T^*)$, *and* $\alpha = \bar{\varphi}_m = d|m|/dm$, *then* $T\alpha = \alpha$ m-ae.

PROOF. We have $|\alpha| = 1$ on supp m, and 3.1 implies $|T\alpha| \leqq 1$ on supp m, so $\int \alpha^{-1} T\alpha\, d|m| = \int T\alpha\, dm = \int \alpha\, dm = \int d|m| = 1$. Since $|m|$ is a probability, $\alpha^{-1}T\alpha = 1$ m-ae.

LEMMA 3.5. *Let* m *and* α *be as in* 3.4. *and let* $W = W_m = \{x\colon T\alpha(x) = \alpha(x)\}$. *Let* g *be a unimodular Baire function which is an eigenfunction on the set* W, *say* $Tg(x) = \lambda g(x)$ *for* x *in* W, *where* $|\lambda| = 1$. *For* x *in* W *we have* $g(s) = \lambda g(x)\alpha(x)^{-1}\alpha(s)$ t_x-a.e.

PROOF. First we show that the measure $r(A) = \int_A \alpha(s)\alpha(x)^{-1}dt_x(s)$ is a probability if x is in W. But $r(X) = \int \alpha(s)\alpha(x)^{-1}dt_x(s) = \alpha(x)\alpha(x)^{-1} = 1$; and if f is in $C(X)$ with $\|f\| \leqq 1$, then since by 3.1 $|\alpha| = 1$ on supp t_x, and $\|t_x\| \leqq 1$, we have $|\int f\, dr| \leqq 1$. Now, if x is in W, we have

$$\begin{aligned} 1 = \lambda^{-1}g(x)^{-1}Tg(x) &= \int \lambda^{-1}g(x)^{-1}g(s)dt_x(s) \\ &= \int \lambda^{-1}g(x)^{-1}g(s)\alpha(s)^{-1}\alpha(x)\, dr(s). \end{aligned}$$

Since r is a probability and the integrand has modulus 1 on supp r, we conclude that the integrand is equal to 1 r-a.e., or t_x-a.e.

THEOREM 3.6. (a) *Let H_1 be the set of all unimodular Baire functions which are eigenfunctions on W. Then H_1 is an Abelian group under the operation $f \circ g = fg\bar{\alpha}$.*

(b) *If χ: $H_1 \to C$ is defined by $Th = \chi(h)h$, then χ is a group character.*

(c) *If x is in W and χ_x: $H_1 \to C$ is defined by $\chi_x(h) = h(x)\overline{\alpha(x)}$, then χ_x is a group character.*

PROOF. (a) To prove closure, Let $Tf = \lambda f$ and $Tg = \mu g$ on W. For x in W 3.5 yields

$$
\begin{aligned}
T(fg\bar{\alpha})(x) &= \lambda f(x)\overline{\alpha(x)}\mu g(x)\overline{\alpha(x)} \int \alpha(s)\alpha(s)\overline{\alpha(s)}\, dt_x(s) \\
&= \lambda\, \mu f(x)g(x)\alpha(x)^{-2} \int \alpha(s)\, dt_x(s) \qquad (1)\\
&= \lambda\, \mu f(x)g(x)\alpha(x)^{-2}\alpha(x) = \lambda\, \mu f(x)g(x)\overline{\alpha(x)}.
\end{aligned}
$$

Thus $T(f \circ g) = \lambda\mu f \circ g$ on W. The group identity is α. If g is in H_1, its inverse is $g^{-1}\alpha^2$, for clearly $g \circ (g^{-1}\alpha^2) = \alpha$, and $g^{-1}\alpha^2$ is in H_1, because if x is in W and $Tg = \lambda g$ on W, Lemma 3.5 gives

$$
\begin{aligned}
T(g^{-1}\, \alpha^2)(x) &= \int g(s)^{-1}\, \alpha(s)^2\, dt_x(s) \\
&= \lambda^{-1}\, g(x)^{-1}\, \alpha(x) \int \alpha(s)^{-1}\alpha(s)^2\, dt_x(s) \qquad (2)\\
&= \lambda^{-1}\, g(x)^{-1}\, \alpha(x)^2.
\end{aligned}
$$

(b) If $Tf = \lambda f$ and $Tg = \mu g$, then (1) gives $T(f \circ g) = \lambda\mu f \circ g$.

(c) $\chi_x(g \circ h) = \chi_x(gh\, \alpha^{-1}) = g(x)h(x)\alpha(x)^{-1}\alpha(x)^{-1} = \chi_x(g)\chi_x(h)$.

REMARKS ON ERGODIC THEORY. We shall discuss here how our less restrictive version of [1, Lemma 2.5], namely 3.1 above, leads to less restrictive versions of all other results, in [1]. The need for Lemma A below in proving the unrestricted version of [1, Lemma 2.2] was pointed out by a referee.

LEMMA A. *Let m be extreme in $F_1(T^*)$, f in $C(X)$, and suppose $Tf = f$ on* supp m. *Then $T_n f = f$ on* supp m, *where $T_n = (1/n)\,(I + \cdots + T^{n-1})$, $n \geqq 1$.*

PROOF. If x is in supp m, then by Lemma 3.1 above, supp $t_x \subset$ supp m, and so $T^2f(x) = \int Tf\, dt_x = \int f\, dt_x = Tf(x) = f(x)$. By induction $T^n f(x) = f(x)$, and hence $T_n f(x) = f(x)$.

LEMMA B. (Cf. [1, Lemma 2.2].) *If* $Tf = f$ *on* supp m (*rather than on all* X) *then* $Tf = \int f\,dm\,\bar{\varphi}_m$ m-a.e.

PROOF. Exactly as in [1, Lemma 2.2].

LEMMA C. (Cf. [1, Lemma 2.6].) *If* $f|M = 0$, *then* $Tf|M = 0$. (*This is just* 3.3 *above.*)

Using A, B and C in place of their counterparts in [1], we now need only assume in the statements of Theorems 3.2, 3.3 and 3.4 of [1] that there exists α in $C(X)$ such that $T\alpha = \alpha$ on M (rather than on all X), and $\alpha \neq 0$ on M.

4. Continuous eigenfunctions. In this section we assume the existence of a continuous β with $T\beta = \beta$ and $|\beta(x)| = 1$ for all x. This is necessary if we are to consider $H = H_1 \cap C(X)$ as a subgroup, since then β will serve as the identity. If m is an extreme point in $F_1(T^*)$, then by [1, Lemma 2.2], $\beta = k\bar{\varphi}_m$ m-a.e., where $k = \int \beta\,dm$. Since $k \neq 0$, φ_m may be taken as continuous on supp m [1, Remark 2.3]. Note that we now have $W = W_m =$ supp m for each extreme m.

Much of the development here is an adaptation to the general case of results on Markov operators in [6, Paragraph 6].

LEMMA 4.1. *If g is in H, say $Tg = \lambda g$, if m is extreme in $F_1(T^*)$, and if x is in* supp m, *then* $g(s) = \lambda g(x)\,\beta(x)^{-1}\beta(s)$ *on* supp t_x.

PROOF. If $\alpha = \bar{\varphi}_m$, then since $\beta = k\alpha$, 3.5 implies that for t_x-almost all s, $g(s) = \lambda g(x)\alpha(x)^{-1}\alpha(s) = \lambda g(x)\beta(x)^{-1}\beta(s)$. By continuity the equality holds for all s in supp m.

THEOREM 4.2. (a) *H is an Abelian group under the operation* $f \circ g = fg\bar{\beta}$.
(b) *If χ is defined by* $Th = \chi(h)h$, *then χ is a group character.*
(c) *If x is in X, then χ_x defined by* $\chi_x(h) = h(x)\overline{\beta(x)}$ *is a group character.*

PROOF. (a) To prove closure, first let m be an extreme point of $F_1(T^*)$. Lemma 4.1 and the same computation as (1) in Theorem 3.6 yield $T(f\circ g)(x) = \lambda\mu(f\circ g)(x)$ for x in supp m. The union of the supports of such extremes is dense in $M = X$, so by continuity the result holds for all x. Clearly the group identity is β, and the inverse of g is $g^{-1}\beta^2$. The proof that $g^{-1}\beta^2$ is in H is the same computation as (2), except that we use 4.1 and replace α by β.

(b) and (c) are left as exercises.

DEFINITION 4.3. A subset $S \subset C(X)$ strongly separates X if for $x \neq y$, there exists no scalar b such that for all f in S, $f(x) = bf(y)$.

LEMMA 4.4. *Suppose H strongly separates X. Then for each x, there exists* *y (which we shall call πx) such that for all f in $C(X)$, $Tf(x) = k(x)f(\pi x)$,* where $k(x) = \beta(x)\overline{\beta(\pi x)}$.

PROOF. First, the set $H\bar{\beta}$ is closed under pointwise multiplication, since if f and g are in H, then by the proofs of 3.6 and 4.1, $fg\bar{\beta}$ is in H, and so $(f\bar{\beta})(g\bar{\beta}) = (fg\bar{\beta})\bar{\beta}$ is in $H\bar{\beta}$. Also $1 = \beta\bar{\beta}$ is in $H\bar{\beta}$. By the strong separating property of H, $H\bar{\beta}$ separates X, and hence by Stone-Weierstrass the linear span of $H\bar{\beta}$ is a dense subalgebra of $C(X)$.

For fixed x, consider the measure $dr = \beta(x)^{-1}\beta\, dt_x$, which was shown in the proof of 3.5 to be a probability. By 4.2 if f and g are in H, say $Tf = \lambda f$ and $Tg = \mu g$, then

$$\int f\bar{\beta}g\bar{\beta}\, dr = \overline{\beta(x)}T(fg\bar{\beta})(x) = \overline{\beta(x)}^2\lambda\mu f(x)g(x).$$

Also

$$\int f\bar{\beta}\, dr \int g\bar{\beta}\, dr = \overline{\beta(x)}T(f\bar{\beta}\beta)(x)\overline{\beta(x)}T(g\beta\bar{\beta})(x)$$
$$= \overline{\beta(x)}^2\lambda\mu f(x)g(x).$$

Thus the probability measure r is multiplicative on $H\bar{\beta}$, and hence on all $C(X)$, so by [5, 33] we have $r = \delta_{\pi x}$ for some πx in X. Thus for all f in $C(X)$, $f(\pi x) = \int f\, dr = \overline{\beta(x)}T(f\beta)(x)$, or $Tf(x) = T((f\bar{\beta})\beta)(x) = \beta(x)\overline{\beta(\pi x)}f(\pi x)$.

REMARKS 4.5. (a) We see the curious result that if f is in H with $Tf = \lambda f$, then for all x, $\lambda f(x)\overline{f(\pi x)} = \beta(x)\overline{\beta(\pi x)}$.

(b) If H strongly separates, then $x \neq y$ implies $\chi_x \neq \chi_y$.

DEFINITIONS 4.6. Let s be a fixed element of X and $H_0 = \{f \text{ in } H: 1 = \chi_s(f) = f(s)\overline{\beta(s)}\}$, a subgroup of H. Let G be the (compact) group of all characters on H_0. Finally, let $c = \chi_{\pi s}$.

THEOREM 4.7. *Assume H strongly separates X. Then*

(a) *the map $x \to \chi_x$ from X to G is a continuous injection,*

(b) *χ_s is the unit of G,*

(c) *$\chi_{\pi s} = \chi|_{H_0}$, where χ is as in* 4.2(b),

(d) *For all x, $\chi_{\pi x} = c\chi_x = \chi_{\pi s}\chi_x$.*

(e) *$\{c^k$: k an integer$\} \subset \{\chi_x$: x in $X\}$.*

PROOF. (a) and (b) are obvious.

(c) If f is in H_0, then $\chi_{\pi s}(f) = f(\pi s)\overline{\beta(\pi s)} = Tf(s)\,\overline{\beta(s)} = \chi(f)f(s)\overline{\beta(s)} = \chi(f)$, the second equality by 4.4, the third by 4.2(b), and the last since f is in H_0.

(d) If f is in H_0, then $\chi_{\pi x}(f) = f(\pi x)\,\overline{\beta(\pi x)} = Tf(x)\overline{\beta(x)} = \chi(f)f(x)\overline{\beta(x)} = \chi_{\pi s}(f)\chi_x(f)$, the last equality following from (c).

(e) In fact, $c_k = \chi^k_{\pi s} = \chi_t$, where $t = \pi^k s$, for all $k \geqq 1$. To show this, note first that by an easy induction, $T^k f(x) = \beta(x)\overline{\beta(\pi^k)}\, f(\pi^k x)$. Then if f is in H_0, $\chi_t(f) = f(\pi^k s)\overline{\beta(\pi^k s)} = T^k f(s)/\beta(s) = T^k f(s) = \chi(f)^k f(s) = \chi_t(f)^k$. That c^k is in $\{\chi_x\colon x \text{ in } X\}$ for all integers follows from [**6**, Lemma 6.4].

THEOREM 4.8. *Assume H strongly separates X, and further that T is irreducible, i.e., there exists in X no proper invariant closed set.* (*In particular,* supp $m = X$ *for every m in $F(T^*)$.*) *Then $K = \{\chi_x\colon x \text{ in } X\}$ is a subgroup of G homeomorphic to X. Define $S\colon C(K) \to C(K)$* by $Sf(\chi_x) = k(x)f(\chi_s\chi_x)$ *and $U\colon C(K) \to C(X)$ by $Uf(x) = f(\chi_x)$. Then $T \circ U = U \circ s$.*

PROOF. That K is a subgroup homeomorphic to X is proved as in [**6**, Theorem 6.9]. If f is in $C(K)$, $T(Uf)(x) = k(x)Uf(\pi x) = k(x)f(\chi_{\pi x})$, and $U(Sf)(x) = Sf(\chi_x) = k(x)f(\chi_s\chi_x)$. But by 4.7(d), $\chi_{\pi x} = \chi_s\chi_x$.

REMARK ON THE SPECTRUM OF T. It follows from the proof of Theorem 3.6(b) that the set of eigenvalues corresponding to Baire eigenfunctions on W is a multiplicative subgroup of the unit circle. Likewise Theorem 4.2(b) implies that the eigenvalues corresponding to continous unimodular eigenfunctions is also a subgroup. In case T is irreducible, we have the stronger result that the set of all eigenvalues of T which have modulus one is a subgroup of the unit circle. Just as in [**4**, p. 1044], one shows that if $Tf = \lambda f$ where $|\lambda| = 1$, and $\|f\| = 1$, then the set $F = \{x\colon |f(x)| = 1\}$ is an invariant set, so that by irreducibility we have $F = X$. Hence λ corresponds to a unimodular eigenfunction.

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