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ON THE EXISTENCE OF METRIC POLARIZATIONS

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1. Introduction. In [6] Duflo has shown how to give an orbital parameterization of the generic portion of the unitary dual of an arbitrary Lie group G. In [11] the author has proven the following: Suppose for a co-adjoint orbit $\Omega \subseteq \mathfrak{g}^*$, there exists a metric polarization for Ω with certain natural properties; then the harmonically induced representation determined by that polarization—which is a natural generalization of the realization of square-integrable representations in a space of L_2 harmonic forms—is in the class (in $\hat{G}$) that corresponds (by Duflo's parameterization) to Ω. It follows that the harmonically induced representation is irreducible and that its class is independent of the polarization. A natural and important problem therefore is to find conditions under which such metric polarizations exist. In this paper we shall consider that problem for real algebraic groups.

More precisely let $\mathscr{B}(G)$ be the orbital parameter space defined in [11]. For each $\Omega \in \mathscr{B}(G)$, we write $\pi_\Omega \in \hat{G}$ for the class of irreducible unitary representations defined by Duflo. If $\mathfrak{a}$ is a metric polarization for Ω which is invariant, and satisfies the strong Pukanszky condition (see §2), then we can define the harmonically induced representation $\pi(\Omega, \mathfrak{a})$ (see [11, §2]). The main result of [11, Thm. 2.12] is that, under certain additional conditions on $\mathfrak{a}$, $\pi(\Omega, \mathfrak{a}) \in \pi_\Omega$. Now we are concerned with the existence of such metric polarizations $\mathfrak{a}$. We shall show in this paper that if we drop the requirement of invariance, then we can always find polarizations having the remaining properties. This is the content of Theorem 3.1. The representation $^\circ\pi(\Omega, \mathfrak{a})$ constructed from such a polarization is in general not irreducible; but breaks up as a finite direct sum of irreducible subrepresentations (when G is algebraic)—see [1], [12, §4], and the comments after Remark 2.2. This is very nice, but we still seek invariant polarizations. However in order to guarantee invariance we must impose further assumptions on the structure of G. To see this we consider the special case that G is reductive. It is false that to every orbit $\Omega \in \mathscr{B}(G)$ (i.e., to every regular semisimple orbit) there exists an invariant metric polarization. For that to be true one needs to assume the Harish-Chandra class condition

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$$Ad_{\mathfrak{g}_c} G \subseteq Ad\mathfrak{g}_c. \tag{1.1}$$

How is that reflected in a structural property for general groups? In fact Duflo's orbital parameterization is constructed by a recursion procedure that reduces eventually to reductive groups. Unfortunately that procedure is difficult to reconcile with our method of proof of existence of metric polarizations. Instead we shall use an alternate procedure—one suggested to me by Roger Howe. Our reduction will lead to groups $G = SN$ where N is Heisenberg, and S is reductive and fixes the center of N. Even then the condition (1.1) on S is too weak to deduce existence of invariant metric polarizations. We must assume in addition that the reductive group S has abelian Cartan subgroups. Indeed we shall show that corresponding to any $\Omega \in \mathscr{B}(G)$, there is canonically defined (up to conjugacy) a reductive subgroup S_Ω of G. The second main result of the paper (Theorem 5.3) is that if S_Ω satisfies (1.1) and has abelian Cartan subgroups, then Ω has an invariant metric polarization.

In [**11**] we needed to impose two additional conditions—the Satake condition and endless admissibility (see [**11**, Defs. 3.2 and 3.4])—on metric polarizations to deduce the result $\pi(\Omega, \mathfrak{a}) \in \pi_\Omega$. However, because of the utility of our recursion procedure, we don't need those conditions to demonstrate existence of metric polarizations. It also follows—as we shall explain in more detail in §4 (Remark 4.4)—that for polarizations of the type discussed in this paper, these extra conditions hold automatically and so are not necessary for the main result of [**11**, Thm. 2.12].

Here are some further comments relevant to the existence of invariant metric polarizations. Suppose the orbit Ω satisfies $G_\varphi = G_\varphi^\circ \cdot \text{Cent } G$ for some $\varphi \in \Omega$. Then invariance is automatic. It is interesting to compare this condition with the condition on Ω that forces π_Ω to be square-integrable mod the center—namely $G_\varphi/\text{Cent } G$ compact [**6**, Thm. VII. 3]. We shall consider the relationship between square-integrable representations and the existence of invariant metric polarizations in a future paper. We shall also consider the following. If G is reductive, but not of Harish-Chandra class, then there exist orbits Ω to which there do not correspond invariant metric polarizations. However I conjecture that the set of such orbits is of Plancherel measure zero. I intend to tackle this problem for general algebraic groups in a future publication. (See *Generic representations are induced from square-integrable representations*, Trans. Amer. Math. Soc. (1984).)

Here is a short section-by-section description of the contents of the paper. In §2 we establish the basic notation, terminology and properties of complex polarizations that we shall need. In §3 we state the main result that any $\Omega \in \mathscr{B}(G)$ has an essentially metric polarization—i.e., metric but perhaps not invariant (see Definition 2.1). The proof of Theorem 3.1 takes up §§3 and 4. The alternate recursive procedure is described and executed in §4. Finally in §5 we prove the existence of invariant metric polarizations for orbits Ω whose reductive data (i.e., S_Ω) is Harish-Chandra class and has abelian Cartan subgroups (see Theorem 5.3).

Some of the ideas in §3 originated in conversations I had with Jonathan Rosenberg several years ago. It's my pleasure to thank him.

2. Notation and terminology. G will denote a real algebraic group, that is $G = \mathbf{G}(\mathbf{R})$ where $\mathbf{G}$ is a (complex) algebraic group def/$\mathbf{R}$. G is in particular a real Lie group. The Lie algebra of G is denoted by $\mathfrak{g}$ and its real dual by $\mathfrak{g}^*$. We consider the set $\mathscr{AP}(G)$ of admissible, well-polarizable linear functionals $\varphi \in \mathfrak{g}^*$. The precise definition of the set $\mathscr{AP}(G)$ may be found in [**11**, §2] or [**6**, Ch. II], but the reader should think of its elements as "regular" and "integral" linear functionals. Corresponding to each $\varphi \in \mathscr{AP}(G)$ there is associated a canonical finite set of irreducible unitary representations defined (by Duflo in [**6**]) as follows. We let G_φ denote the stability group of φ in G, and G_φ° its identity component. Since G_φ preserves the symplectic form $B_\varphi(X, Y) = \varphi[X, Y]$, $X, Y \in \mathfrak{g}$, it maps homomorphically into $\mathrm{Sp}(\mathfrak{g}/\mathfrak{g}_\varphi, B_\varphi)$. If $\mathrm{Mp}(\mathfrak{g}/\mathfrak{g}_\varphi, B_\varphi)$ denotes the canonical 2-fold covering group, i.e., the metaplectic group, then the 2-fold cover $\tilde{G}_\varphi = (\tilde{G}_\varphi)_{B_\varphi}$ is defined by $\tilde{G}_\varphi = \{(g, m) \in G_\varphi \times \mathrm{Mp}(\mathfrak{g}/\mathfrak{g}_\varphi, B_\varphi)\colon g$ and m have the same image in $\mathrm{Sp}(\mathfrak{g}(\mathfrak{g}_\varphi, B_\varphi)\}$. Let $\tilde{G}_\varphi^\circ$ denote the inverse image of G_φ° in $\tilde{G}_\varphi$ and ε the non-trivial central element that maps to 1. Then admissibility of φ is tantamount to the fact that

$$\mathfrak{X}(\varphi) = \mathfrak{X}_G(\varphi) = \{\text{unit. reps. } \tau \text{ of } \tilde{G}_\varphi\colon d\tau = i\varphi|_\varphi,\ \tau(\varepsilon) = -Id\}$$

is non-empty (see [**6**, Ch. II]); or equivalently, there exists a unitary character χ_φ of $\tilde{G}_\varphi^\circ$ such that $d\chi_\varphi = i\varphi|_{\mathfrak{g}_\varphi}$, $\chi_\varphi(\varepsilon) = -1$. We set $\mathscr{B}(G) = \{(\varphi, \tau)\colon \varphi \in \mathscr{AP}(G),\ \tau \in \mathfrak{X}_G(\varphi)$ is irreducible$\}$. In [**6**] Duflo has constructed a map $(\varphi, \tau) \to \pi_{\varphi,\tau}$ from $\mathscr{B}(G)$ to the set $\hat{G}$ of equivalence classes of irreducible unitary representations of G. The set $\mathscr{B}(G)$ is naturally a G-space and the above map factors to an injection $\mathscr{B}(G)/G \to \hat{G}$. The image consists of generic classes in the following sense: since G is algebraic, it is type I; the image of Duflo's map is co-null with respect to the Plancherel measure class. When G is reductive the image consists of the tempered representations with regular infinitesimal character.

Duflo constructs the representation classes $\pi_{\varphi,\tau}$ recursively by induction on dim G. It is natural, since we are dealing with generic representations, to ask if these representations can be realized by some sort of induction procedure via complex polarizations. This issue was addressed in [**11**]. Given a polarization $\mathfrak{a}$ for φ with certain properties, one can form the harmonically induced representation $\pi(\varphi, \tau, \mathfrak{a})$, and the main result of [**11**, Thm. 2.12] is that this representation is in the Duflo class, i.e.,

$\pi(\varphi, \tau, \mathfrak{a}) \in \pi_{\varphi,\tau}$. In particular one gets irreducibility of $\pi(\varphi, \tau, \mathfrak{a})$ and independence of polarization. Thus it is also natural to ask when polarizations with the properties of [11] actually exist. That is the main thrust of this paper. We begin by reviewing the salient properties of polarizations.

DEFINITION 2.1. (i) By a polarization for $\varphi \in \mathfrak{g}^*$ we mean a Legrangian subspace $\mathfrak{a} \subseteqq \mathfrak{g}_c$ (with respect to B_φ) such that $\mathfrak{a}$ and $\mathfrak{a} + \bar{\mathfrak{a}}$ are algebras. As is customary we set $\mathfrak{d} = \mathfrak{a} \cap \mathfrak{g}$, $\mathfrak{e} = (\mathfrak{a} + \bar{\mathfrak{a}}) \cap \mathfrak{g}$. Then $\mathfrak{d} = \mathfrak{e}^\perp$ (with resp. to B_φ). We write exp $\mathfrak{d}$, exp $\mathfrak{e}$ for the connected Lie subgroups of G having $\mathfrak{d}$, $\mathfrak{e}$ as Lie algebras. exp $\mathfrak{d}$ is always closed [3, p. 282].

(ii) $\mathfrak{a}$ is called invariant if $G_\varphi \cdot \mathfrak{a} \subseteq \mathfrak{a}$ (of course $G_\varphi^\circ \cdot \mathfrak{a} \subseteq \mathfrak{a}$ is automatic). If $\mathfrak{a}$ is invariant, then $D = G_\varphi \exp \mathfrak{d}$, $E = G_\varphi \exp \mathfrak{e}$ are well-defined and D is closed [3, p. 282].

(iii) $\mathfrak{a}$ is called real if $\mathfrak{a} = \bar{\mathfrak{a}}$; positive if $i\varphi[X, \bar{Y}]$ is positive semi-definite; and totally complex if $\mathfrak{a} + \bar{\mathfrak{a}} = \mathfrak{g}_c$.

(iv) $\mathfrak{a}$ is called admissible for an ideal $\mathfrak{u}$ of $\mathfrak{g}$ if $\mathfrak{a} \cap \mathfrak{u}$ is a polarization for $\varphi|_\mathfrak{u}$. We say $\mathfrak{a}$ is admissible if it is admissible for the unipotent radical $\mathfrak{n}$ of $\mathfrak{g}$. (When dealing with Lie, perhaps non-algebraic, groups one commonly uses the nilradical in place of the unipotent radical.)

(v) We say $\mathfrak{a}$ satisfies the Pukanszky condition if for any $\lambda \in \mathfrak{g}^*$, $\lambda(\mathfrak{e}) = 0$, $\mathfrak{a}$ is Lagrangian for $\varphi + \lambda$. This is equivalent to: exp $\mathfrak{d} \cdot \varphi$ is closed; or exp $\mathfrak{d} \cdot \varphi = \varphi + \mathfrak{e}^\perp$; or $\varphi + \mathfrak{e}^\perp \subseteq G \cdot \varphi$ [4, Prop. IV.3.1.5]. If in addition E is closed, we say $\mathfrak{a}$ satisfies the strong Pukanszky condition.

(vi) We say $\mathfrak{a}$ is metric if it is invariant and the group of linear transformations $Ad_{\mathfrak{e}/\mathfrak{d}}D$ is compact. We say that $\mathfrak{a}$ is essentially metric if $Ad_{\mathfrak{e}/\mathfrak{d}}$ exp $\mathfrak{d}$ is compact. This allows for the possibility that $\mathfrak{a}$ is not invariant.

(vii) Finally, we say that $\mathfrak{a}$ is a harmonic polarization if it is solvable, admissible, metric and satisfies the strong Pukanszky condition. If $\mathfrak{a}$ is solvable, admissible, essentially metric, satisfies the Pukanszky condition and exp $\mathfrak{e}$ is closed, we say that $\mathfrak{a}$ is essentially harmonic.

REMARK 2.2. This is a change of use of the word harmonic from [11], [12]. There we required two extra properties (endless admissibility and the Satake condition). We shall show later (in Remark 4.4) that we shan't need them for the result $\pi(\varphi, \tau, \mathfrak{a}) \in \pi_{\varphi,\tau}$. This will be done by altering the recursive procedure of Duflo.

Recall that if $\mathfrak{a}$ is a polarization which is metric and satisfies the strong Pukanszky condition—in palticular if it is harmonic—then one can define the harmonically induced representation $\pi(\varphi, \tau, \mathfrak{a})$. The precise definition is in [11, §2], but roughly: τ is twisted by the Duflo shift, extended canonically to D, then square-integrable harmonic forms between D and E are taken, and finally ordinary induction up to G is applied. If $\mathfrak{a}$ is only essentially harmonic, we can still define a representation ${}^\circ\pi(\varphi, \mathfrak{a})$—by twisting, extending from G_φ° to exp $\mathfrak{d}$, taking harmonic forms up to exp $\mathfrak{e}$ and then inducing to G. In general ${}^\circ\pi(\varphi, \mathfrak{a})$ is not irreducible; but one expects to locate irreducibles canonically as subrepresentations (see [**1**], [**12**, §4]). In this paper we show that for every $\varphi \in \mathscr{A}\mathscr{P}(G)$, there exists an essentially harmonic polarization (Theorem 3.1). Then we develop some sufficient conditions for there actually to exist harmonic polarizations (Theorem 5.3).

Here is some further notation. We write N for the unipotent radical of G. Usually we denote a Levi factor by the letter S, $G = SN$. If $\varphi \in \mathfrak{g}^*$, we set $\theta = \varphi|_{\mathfrak{n}}$, G_θ the stability group, $\mathfrak{g}_\theta$ its Lie algebra. We also consider $\mathfrak{n}_\theta$ the stability subalgebra of θ in $\mathfrak{n}$, and $\mathfrak{q}_\theta = \operatorname{Ker} \theta|_{\mathfrak{n}_\theta}$. Finally set $\xi = \varphi|_{\mathfrak{g}_\theta}$. We have the following standard result [**14**, Lemma 2], [**3**, Prop. II. 1.3], [**9**, p. 271].

PROPOSITION 2.3. (1) $N_\theta \cdot \varphi = \varphi + (\mathfrak{g}_\theta + \mathfrak{n})^\perp$, $\mathfrak{n}_\theta \cdot \varphi = (\mathfrak{g}_\theta + \mathfrak{n})^\perp$;
(ii) $(G_\theta)_\xi = G_\varphi N_\theta$, $(\mathfrak{g}_\theta)_\xi = \mathfrak{g}_\varphi + \mathfrak{n}_\theta$.

As a consequence of Proposition 2.3 we know

$$\dim(\mathfrak{g}_\theta)_\xi/\mathfrak{g}_\varphi = \dim \mathfrak{n}_\theta/\mathfrak{n}_\varphi = \dim \mathfrak{g}/(\mathfrak{g}_\theta + \mathfrak{n}). \tag{2.1}$$

Incidentally, Proposition 2.3 does not require that N be the unipotent radical, only that it be a closed connected normal subgroup.

3. Essentially harmonic polarizations. In this section and the next we shall prove the following theorem.

THEOREM 3.1. *Let G be a real algebraic group. Then for any $\varphi \in \mathscr{A}\mathscr{P}(G)$, there exists a polarization $\mathfrak{a}$ for φ which is essentially harmonic.*

The argument has two components. First there is a crucial lemma on the existence of admissible polarizations. Second (in the next section) we perform a refinement and alternation of the recursive procedure of Duflo. Here is the key.

LEMMA 3.2. *Assume $\theta \neq 0$ and that $\mathfrak{g}_\theta/\mathfrak{n}_\theta$ is reductive, i.e., $\mathfrak{n}_\theta$ is the unipotent radical of $\mathfrak{g}_\theta$. Let $\xi = \varphi|_{\mathfrak{g}_\theta}$ and $\varphi_1 =$ the linear functional obtained from ξ by passage to the quotient $\mathfrak{g}_\theta/\mathfrak{q}_\theta$. Suppose that $\mathfrak{a}_1$ is an essentially harmonic polarization for φ_1. Let $\mathfrak{a}$ be its pullback to $(\mathfrak{g}_\theta)_c$. Then there exists $\mathfrak{b}$, a metric polarization for θ, so that $\mathfrak{c} = \mathfrak{a} + \mathfrak{b}$ is an essentially harmonic polarization for φ.*

Before we begin the proof we supply an alignment lemma that will make our work technically more simple.

LEMMA 3.3. *Let $\mathfrak{s}$ be a Levi factor of $\mathfrak{n}$ in $\mathfrak{g}$. Suppose that $\mathfrak{n}_\theta$ is the unipotent radical of $\mathfrak{g}_\theta$. Then there exists $n \in N$ such that if we set $\varphi' = n \cdot \varphi$, $\theta' = \varphi'|_{\mathfrak{n}} = n \cdot \theta'$, then*

$$\mathfrak{g}_{\theta'} = \mathfrak{s}_{\theta'} + \mathfrak{n}_{\theta'}, G_{\theta'} = S_{\theta'}N_{\theta'}.$$

PROOF. The proof is modelled after [10, Lemma 4.2]. Choose a Levi factor $\mathfrak{h}$ for $\mathfrak{n}_\theta$ in $\mathfrak{g}_\theta$, $\mathfrak{g}_\theta = \mathfrak{h} + \mathfrak{n}_\theta$. Clearly $\mathfrak{h} = \mathfrak{h}_\theta$. Then there must exist $n \in N$ such that $\mathrm{Ad}n(\mathfrak{h}) \subseteq \mathfrak{s}$. Since

$$\begin{aligned}\mathfrak{g}_{n\cdot\theta} &= \mathrm{Ad}n(\mathfrak{g}_\theta) = \mathrm{Ad}n(\mathfrak{h} + \mathfrak{n}_\theta) = \mathrm{Ad}n(\mathfrak{h}_\theta) + \mathrm{Ad}n(\mathfrak{n}_\theta)\\ &= (\mathrm{Ad}n(\mathfrak{h}))_{n\cdot\theta} + \mathfrak{n}_{n\cdot\theta}\\ &\subseteq \mathfrak{s}_{n\cdot\theta} + \mathfrak{n}_{n\cdot\theta}.\end{aligned}$$

The reverse inequality is obvious. We leave the details of the corresponding proof at the group level to the reader.

The properties of Lemma 3.2 and the main Theorem 3.1 are all preserved under inner automorphism. Thus it is no loss of generality in the following to assume that the Levi factor for $\mathfrak{n}_\theta$ in $\mathfrak{g}_\theta$ is $\mathfrak{s}_\theta$, $\mathfrak{g}_\theta = \mathfrak{s}_\theta + \mathfrak{n}_\theta$. But then $\mathfrak{g}_\theta/\mathfrak{q}_\theta = \mathfrak{s}_\theta + (\mathfrak{n}_\theta/\mathfrak{q}_\theta)$ is a direct sum of a reductive ideal and a unipotent abelian one-dimensional ideal. It is obvious that $\mathfrak{a}_1$ is essentially harmonic for $\varphi_1 \Rightarrow \mathfrak{a}$ is essentially harmonic for ξ, $\mathfrak{a} = (\mathfrak{a} \cap (\mathfrak{s}_\theta)_c) + (\mathfrak{n}_\theta)_c$, and $\mathfrak{a} \cap (\mathfrak{s}_\theta)_c$ is essentially harmonic for $\xi|_{\mathfrak{s}_\theta}$. Thus it is enough to prove the following slightly simpler form of Lemma 3.2.

LEMMA 3.2′. *Assume $\mathfrak{g}_\theta = \mathfrak{s}_\theta + \mathfrak{n}_\theta$ is a Levi decomposition. Let $\xi = \varphi|_{\mathfrak{s}_\theta}$ and suppose $\mathfrak{a}$ is an essentially harmonic polarization for ξ. Then there exists $\mathfrak{b}$, a metric polarization for θ, so that $\mathfrak{c} = \mathfrak{a} + \mathfrak{b}$ is an essentially harmonic polarization for φ.*

PROOF. The proof is a somewhat lengthy reduction argument. It is composed of five steps:

(i) $\mathfrak{n}$ is abelian;

(ii) It is no loss of generality to assume $\mathfrak{g} = \mathfrak{g}_\theta + \mathfrak{n}$, so that $\mathfrak{g} = \mathfrak{s} + \mathfrak{n}$, $\mathfrak{s} \cdot \theta = 0$;

(iii) It is no loss of generality to assume further that $\mathfrak{n}$ is Heisenberg;

(iv) $\mathfrak{g} = sp(n, \mathbf{R}) + \mathfrak{n}$, $\mathfrak{n}$ Heisenberg of dimension $2n + 1$;

(v) It is no loss of generality to assume $\mathfrak{s} = sp(n, \mathbf{R})$.

Let $\mathfrak{b}$ be a metric polarization for θ. Then the subspace $\mathfrak{c} = \mathfrak{a} + \mathfrak{b}$ is automatically Lagrangian for B_φ. This is because of Proposition 2.3 and the equality $\mathfrak{g}_\varphi + \mathfrak{n}_\theta = (\mathfrak{s}_\theta)_\xi + \mathfrak{n}_\theta$. In fact, in each of steps (i) – (v), we shall demonstrate the existence of a metric polarization $\mathfrak{b}$ for θ, so that $\mathfrak{c} = \mathfrak{a} + \mathfrak{b}$ also satisfies:

(a) $\mathfrak{c}$ is a solvable subalgebra;

(b) $\mathfrak{c} + \bar{\mathfrak{c}}$ is an algebra;

(c) $\exp \mathfrak{e}$ is closed;

(d) $\exp \mathfrak{d} \cdot \varphi = \varphi + \mathfrak{e}^\perp$;

(e) $\mathrm{Ad}_{\mathfrak{e}/\mathfrak{d}} \exp \mathfrak{d}$ is compact.

Now for the first step in the argument.

(i) We assume $\mathfrak{n}$ is abelian. Then $\mathfrak{n}_\theta = \mathfrak{n}$ and $\mathfrak{g}_\theta = \mathfrak{s}_\theta + \mathfrak{n}$. Let $\mathfrak{a}$ be an essentially harmonic polarization for $\xi = \varphi|_{\mathfrak{s}_\theta}$. It is clear that we must take $\mathfrak{b} = \mathfrak{n}_c$. Then $\mathfrak{c} = \mathfrak{a} + \mathfrak{b} = \mathfrak{a} + \mathfrak{n}_c$. We verify the properties (a)-(e) listed above.

(a) It is obvious that $\mathfrak{c}$ is a solvable subalgebra of $\mathfrak{g}_c$.

(b) $\mathfrak{c} + \bar{\mathfrak{c}} = (\mathfrak{a} + \mathfrak{n}_c) + (\mathfrak{a} + \mathfrak{n}_c)^- = (\mathfrak{a} + \bar{\mathfrak{a}}) + \mathfrak{n}_c$ is clearly an algebra. Note that $\mathfrak{d} = \mathfrak{d}_\mathfrak{a} + \mathfrak{n}$, $\mathfrak{e} = \mathfrak{e}_\mathfrak{a} + \mathfrak{n}$.

(c) Consider the projection $G \to G/N = S$. It is obvious that $\exp \mathfrak{e}$ is the inverse image of $\exp \mathfrak{e}_\mathfrak{a}$ under the projection; the latter is presumed closed, hence so is the former.

(d) Let $\lambda \in \mathfrak{e}^\perp$. Then $\lambda|_\mathfrak{n} = 0$, $\lambda|_{\mathfrak{e}_\mathfrak{a}} = 0$ and we may consider $\lambda_1 = \lambda|_{\mathfrak{s}_\theta} \in \mathfrak{e}_\mathfrak{a}^\perp$. By assumption $\mathfrak{a}$ is a polarization for $\xi + \lambda_1$. But we have $\mathfrak{g}_{\varphi+\lambda} \subseteq (\mathfrak{g}_\theta)_{\xi+\lambda_1} \subseteq \mathfrak{g}_\theta \subseteq \mathfrak{g}$. Therefore it's enough to show that $\dim (\mathfrak{g}_\theta)_{\xi+\lambda_1}/\mathfrak{g}_{\varphi+\lambda} = \dim \mathfrak{g}/\mathfrak{g}_\theta$. Now $(\mathfrak{g}_\theta)_{\xi+\lambda_1} = \mathfrak{g}_{\varphi+\lambda} + \mathfrak{n}$ (Prop. 2.3), so that $(\mathfrak{g}_\theta)_{\xi+\lambda_1}/\mathfrak{g}_{\varphi+\lambda} \cong \mathfrak{n}/\mathfrak{n}_\varphi$. By Formula (2.1) the dimension of the latter is $\dim \mathfrak{g}/\mathfrak{g}_\theta$ ($\mathfrak{n}$ is abelian here). This proves that $\mathfrak{c}$ satisfies the Pukanszky condition.

(e) This is obvious—N acts trivially on $\mathfrak{e}/\mathfrak{d}$ and the metric property carries over.

(ii) It is no loss of generality to assume $\mathfrak{g} = \mathfrak{g}_\theta + \mathfrak{n}$. We assume the lemma proven in that instance, and show that it would follow in general. So put $\mathfrak{h} = \mathfrak{g}_\theta + \mathfrak{n}$, $\eta = \varphi|_\mathfrak{h}$. Then we claim: $\mathfrak{h}_\eta = \mathfrak{g}_\varphi + \mathfrak{n}_\theta$. The inclusion $\supseteq$ is obvious since $\mathfrak{n}_\theta$ annihilates $\varphi|_\mathfrak{h}$ (Prop. 2.3). Conversely if $X \in \mathfrak{h}_\eta$, then $X \cdot \theta = 0 \Rightarrow X \in \mathfrak{g}_\theta$. In addition $X \cdot \xi = 0 \Rightarrow X \in (\mathfrak{g}_\theta)_\xi = \mathfrak{g}_\varphi + \mathfrak{n}_\theta$. It is also evident that $\mathfrak{h}_\theta = \mathfrak{g}_\theta$. Let $\mathfrak{a}$ be an essentially harmonic polarization for $\xi = \varphi|_{\mathfrak{s}_\theta}$. Now we are assuming the lemma is true for $\mathfrak{h} = \mathfrak{g}_\theta + \mathfrak{n} = \mathfrak{s}_\theta + \mathfrak{n}$. Hence there exists a metric polarization $\mathfrak{b}$ for θ, such that $\mathfrak{c} = \mathfrak{a} + \mathfrak{b}$ is an essentially harmonic polarization for η. We shall verify properties (a)-(e) for $\mathfrak{c}$ with respect to φ.

(a) It's solvable subalgebra by assumption. Furthermore $\mathfrak{g}_\varphi \subseteq \mathfrak{h}_\eta \subseteq \mathfrak{g}_\theta \subseteq \mathfrak{g}$. and $\mathfrak{h}_\eta/\mathfrak{g}_\varphi \cong \mathfrak{n}_\theta/\mathfrak{n}_\varphi$. Once again Prop. 2.3 does the trick.

(b) Obvious.

(c) If $\exp \mathfrak{e}$ is closed in $G_\theta N$—the latter being a closed subgroup of G—then it's closed in G.

(d) Let $\lambda \in \mathfrak{g}^*$, $\lambda(\mathfrak{e}) = 0$. It's enough to show $\varphi + \lambda = g \cdot \varphi$ for some $g \in G$. But $\mathfrak{e} \subseteq \mathfrak{h}$ and so if $\lambda_1 = \lambda|_\mathfrak{h}$, then $\lambda_1(\mathfrak{e}) = 0$. Therefore $(\varphi + \lambda)|_\mathfrak{h} = \eta + \lambda_1 = h \cdot \eta$ for some $h \in G_\theta N$. But then $\{h^{-1} \cdot (\varphi + \lambda) - \varphi\}|_\mathfrak{h} = 0$. By Prop. 2.3, $h^{-1} \cdot (\varphi + \lambda) = n \cdot \varphi$ for some $n \in N_\theta$. That proves the result.

(e) Everything is taking place inside $G_\theta N$, so the metric property is also clear.

This concludes the proof of step (ii) and shows that in the proof of the lemma, there is no loss of generality if we assume $\mathfrak{g} = \mathfrak{g}_\theta + \mathfrak{n} = \mathfrak{s}_\theta + \mathfrak{n}$. Thus we have a semidirect product $\mathfrak{g} = \mathfrak{s} + \mathfrak{n}$, where $\mathfrak{s} \cdot \theta = 0$. Next comes step three.

(iii) It is no loss of generality to assume $\mathfrak{n}$ is Heisenberg. We assume the lemma proven in that case, and show that it follows in general. In order to achieve that we need the following auxiliary result on polarizations of nilpotent Lie algebras. It is reminiscent of certain well-known results in the subject, but I could not find it in the literature.

LEMMA 3.4. *Let* $\mathfrak{n}$ *be a nilpotent Lie algebra,* $\mathfrak{z} = \mathrm{Cent}\,\mathfrak{n}$. *Suppose* $\dim \mathfrak{n} > 1$, $\dim \mathfrak{z} = 1$, $\theta \in \mathfrak{n}^*$ *and* $\theta|_{\mathfrak{z}} \neq 0$. *Suppose also that* $\mathfrak{n}$ *is not Heisenberg. Let* $\mathfrak{m} = \mathfrak{z}^{(2)} \cap [\mathfrak{n}, \mathfrak{n}]$, *where* $\mathfrak{z}^{(2)}$ *is the second center. Also let* $\mathfrak{n}_1 =$ *centralizer of* $\mathfrak{m}$ *in* $\mathfrak{n}$ *and* $\theta_1 = \theta|_{\mathfrak{n}_1}$. Then:

(i) $\mathfrak{z} \subsetneqq \mathfrak{m}$ and $\mathfrak{m}$ is a characteristic abelian ideal of $\mathfrak{n}$;

(ii) $\mathfrak{m} \subseteq \mathfrak{n}_1 \subsetneqq \mathfrak{n}$ and $[\mathfrak{m}, \mathfrak{n}] = \mathfrak{z}$;

(iii) Any polarization for θ_1 is a polarization for θ.

PROOF. $\mathfrak{m}$ and $\mathfrak{n}_1$ are obviously characteristic ideals. We have $[\mathfrak{m}, \mathfrak{m}] \subseteq [\mathfrak{z}^{(2)}, [\mathfrak{n}, \mathfrak{n}]] \subseteq [\mathfrak{n}, [\mathfrak{n}, \mathfrak{z}^{(2)}]] \subseteq [\mathfrak{n}, \mathfrak{z}] = \{0\}$. Therefore $\mathfrak{m}$ is abelian. Now we have $0 \subsetneqq [\mathfrak{n}, \mathfrak{n}] \cap \mathfrak{z} \subsetneqq [\mathfrak{n}, \mathfrak{n}]$. This is because, on the one hand the last non-vanishing algebra in the descending central series is contained in $[\mathfrak{n}, \mathfrak{n}] \cap \mathfrak{z}$, and on the other hand $[\mathfrak{n}, \mathfrak{n}]$ cannot be inside $\mathfrak{z}$ since $\mathfrak{n}$ is not Heisenberg. In particular $\mathfrak{z} \subseteq [\mathfrak{n}, \mathfrak{n}]$, since $\dim \mathfrak{z} = 1$. Now apply Engel's Lemma to the action of $\mathfrak{n}$ on $[\mathfrak{n}, \mathfrak{n}]/([\mathfrak{n}, \mathfrak{n}] \cap \mathfrak{z})$. We obtain $X \in [\mathfrak{n}, \mathfrak{n}]$, $X \notin [\mathfrak{n}, \mathfrak{n}] \cap \mathfrak{z}$ such that $[X, \mathfrak{n}] \subseteq [\mathfrak{n}, \mathfrak{n}] \cap \mathfrak{z} \subseteq \mathfrak{z}$. In particular $X \in \mathfrak{z}^{(2)}$. So $X \in \mathfrak{m}$, but $X \notin \mathfrak{z}$. This proves (i) and the first part of (ii). The rest of (ii) is clear since $[\mathfrak{m}, \mathfrak{n}] \subseteq [\mathfrak{z}^{(2)}, \mathfrak{n}] \subseteq \mathfrak{z}$, $\dim \mathfrak{z} = 1$ and $\mathfrak{m} \supsetneqq \mathfrak{z}$.

Now let us prove (iii). We shall first demonstrate the inclusions $\mathfrak{n}_\theta \subseteq (\mathfrak{n}_1)_{\theta_1} \subseteq \mathfrak{n}_1 \subseteq \mathfrak{n}$. In fact $\theta[\mathfrak{n}_\theta, \mathfrak{m}] = 0 \Rightarrow [\mathfrak{n}_\theta, \mathfrak{m}] = 0$ (by (ii) and $\theta|_{\mathfrak{z}} \neq 0$) $\Rightarrow \mathfrak{n}_\theta \subseteq \mathfrak{n}_1 \Rightarrow (\mathfrak{n}_\theta) \subseteq (\mathfrak{n}_1)_{\theta_1}$. Therefore it is enough to prove $\dim \mathfrak{n}/\mathfrak{n}_1 = \dim (\mathfrak{n}_1)_{\theta_1}/\mathfrak{n}_\theta$. We may use Prop. 2.3 since $\mathfrak{n}_1$ is an ideal. It gives $(\mathfrak{n}_1)_{\theta_1} \cdot \theta = (\mathfrak{n}_1 + \mathfrak{n}_\theta)^\perp \Rightarrow \dim (\mathfrak{n}_1)_{\theta_1}/(\mathfrak{n}_1)_\theta = \dim \mathfrak{n}/(\mathfrak{n}_1 + \mathfrak{n}_{\theta_1})$. But $(\mathfrak{n}_1)_\theta = \mathfrak{n}_\theta$ because $\mathfrak{n}_\theta \subseteq \mathfrak{n}_1$; and $\mathfrak{n}_{\theta_1} \subseteq \mathfrak{n}_1$ since $\theta[\mathfrak{n}_{\theta_1}, \mathfrak{m}] = \theta_1[\mathfrak{n}_{\theta_1}, \mathfrak{m}] \subseteq \theta_1[\mathfrak{n}_{\theta_1}, \mathfrak{n}_1] = 0 \Rightarrow [\mathfrak{n}_{\theta_1}, \mathfrak{m}] = 0 \Rightarrow \mathfrak{n}_{\theta_1} \subseteq \mathfrak{n}_1$. This finishes the proof of Lemma 2.4.

We reason now by induction. By the hypothesis we may assume $\mathfrak{n}$ is not Heisenberg. We may also assume it's not abelian (since that case is covered in (i)). Let $\mathfrak{z} = \mathrm{Cent}\,\mathfrak{n}$. Consider the ideal $\mathfrak{p} = \mathrm{Ker}\,\theta|_{\mathfrak{z}}$ of $\mathfrak{g} = \mathfrak{s} + \mathfrak{n}$, $\mathfrak{s} = \mathfrak{s}_\theta$. If $\dim \mathfrak{p} > 0$, then $\varphi|_{\mathfrak{p}} = 0$ and we may divide it out. The induction hypothesis applies and we obtain the desired result. Otherwise $\mathfrak{p} = \{0\}$, $\dim \mathfrak{z} = 1$ and $\theta|_{\mathfrak{z}} \neq 0$. We apply Lemma 3.4. The algebra $\mathfrak{n}_1$ is characteristic so we may consider $\mathfrak{g}_1 = \mathfrak{s} + \mathfrak{n}_1$. By induction there exists $\mathfrak{b}$, a metric polarization for $\theta_1 = \theta|_{\mathfrak{n}_1}$, such that $\mathfrak{c} = \mathfrak{a} + \mathfrak{b}$ is an essentially harmonic polarization for $\varphi_1 = \varphi|_{\mathfrak{g}_1}$. (Note: it is not needed here, but even though $\mathfrak{g}_1$ is not an ideal in $\mathfrak{g}$, we can use [6, Ch. IV] to show that $\varphi_1 \in \mathscr{A}\mathscr{P}((SN_1))$.) By Lemma 3.4, $\mathfrak{b}$ is also a polarization for

θ and it is clearly metric. We verify that $\mathfrak{c}$ is also essentially harmonic for φ.

In fact properties (a), (b), (c), (e) all follow trivially from the fact that $\mathfrak{c}$ satisfies them as a polarization for φ_1. The only item not obvious is property (d), the Pukanszky condition. For that we reason as follows. Let $\lambda \in \mathfrak{g}^*$, $\lambda(\mathfrak{e}) = 0$. We need to show $\phi + \lambda \in G \cdot \varphi$. Let $\varphi_1 = \varphi|_{\mathfrak{g}_1}$, $\lambda_1 = \lambda|_{\mathfrak{g}_1}$. Then since $\mathfrak{e} \subseteq \mathfrak{g}_1$ we have that $\varphi_1 + \lambda_1 = g_1 \cdot \varphi_1$ for some $g_1 \in G_1 = SN_1$. Set $\mu = \lambda|_{\mathfrak{n}}$, $\mu_1 = \lambda|_{\mathfrak{n}_1} = \mu|_{\mathfrak{n}_1}$. Then $g_1^{-1} \cdot (\theta + \mu)|_{\mathfrak{n}_1} = g_1^{-1} \cdot (\theta_1 + \mu_1) = \theta_1 = \theta|_{\mathfrak{n}_1}$. Hence there exists $n \in (N_1)_{\theta_1}$ such that $g_1^{-1} \cdot (\theta + \mu) = n \cdot \theta$. That is

$$\theta + \mu = g_1 n \cdot \theta = n_1 g_1 \cdot \theta,\ n_1 = g_1 n g_1^{-1} \in (N_1)_{\theta_1}.$$

So

$$(\varphi + \lambda)|_{\mathfrak{n}} = n_1 g_1 \cdot \varphi|_{\mathfrak{n}}.$$

And

$$(\varphi + \lambda)|_{\mathfrak{s}} = (\varphi_1 + \lambda_1)|_{\mathfrak{s}} = g_1 \cdot \varphi_1|_{\mathfrak{s}} = n_1 g_1 \cdot \varphi|_{\mathfrak{s}}$$

because, since $\mathfrak{s} = \mathfrak{s}_{\theta_1}$, $(N_1(_{\theta_1}$ fixes φ_1 or $g_1 \cdot \varphi_1$ on $\mathfrak{s}$. Hence $\varphi + \lambda$ and $n_1 g_1 \cdot \varphi$ have identical restrictions to both $\mathfrak{n}$ and $\mathfrak{s}$. That is, they coincide. So $\varphi + \lambda \in G \cdot \varphi$ and the Pukanszky condition is established.

We have now arrived at the situation where to prove Lemma 3.2′ it is enough to consider the setup $\mathfrak{g} = \mathfrak{s} + \mathfrak{n}$, $\mathfrak{n}$ Heisenberg, $\theta|_{\mathfrak{z}} \neq 0$. $\mathfrak{s} = \mathfrak{s}_\theta$. We consider first the special case

(iv) $\mathfrak{n}$ Heisenberg of dimension $2n + 1$, $\mathfrak{s} = sp(n, \mathbf{R}) \cong sp(\mathfrak{n}/\mathfrak{z}, B_\theta)$. We have $\varphi = \xi + \theta$ and $\xi \in \mathscr{A}\mathscr{P}(Sp(n, \mathbf{R}))$. That means that $\mathfrak{s}_\xi$ is a Cartan subalgebra of $\mathfrak{s}$ and that $\mathfrak{a}$ is a Borel subalgebra of $\mathfrak{s}_c = sp(n, \mathbf{C})$, maximal totally isotropic for B_ξ. It's an easy fact to check that one can find a metric polarization $\mathfrak{b}$ for θ which is $\mathfrak{a}$-invariant. The point is to show that it can be chosen so that $\mathfrak{c} = \mathfrak{a} + \mathfrak{b}$ has the remaining properties (a)-(e).

So $\mathfrak{s}_\xi = \mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{s}$. Let $\Phi(\mathfrak{s}_c, \mathfrak{h}_c)$ be the roots. Then $\mathfrak{a}$ determines some set $\Phi^+(\mathfrak{s}_c, \mathfrak{h}_c)$ of positive roots. Consider the action of the solvable algebra $\mathfrak{a}$ on $\mathfrak{n}_c$. Then it is a well-known fact—which is easy to serive by direct computation—that there exists a subset $R = \{\alpha_1, \ldots, \alpha_n\} \subseteq \Phi$, consisting of strongly orthogonal roots, such that the set of weights for the action of $\mathfrak{a}$ on $\mathfrak{n}_c$ is precisely $W = \{\pm(1/2)\alpha_1, \ldots, \pm (1/2)\alpha_n\}$. Now it is possible to use the data $(\mathfrak{h}_c, R)$ to classify the collection of conjugacy classes of real Cartan subalgebras of $\mathfrak{s}$ [16] (see also [18, pp. 93–05]). Furthermore from that work we see that every root $\alpha_i \in R$ has the property that it's either real or purely imaginary. That is if we write $\mathfrak{h} = \mathfrak{t} + \mathfrak{h}_+$ so that $\Phi \subseteq i\mathfrak{t}_{\mathbf{R}}^* \cup \mathfrak{h}_+^*$ (e.g., $\mathfrak{t}$ = unique maximal compact subalgebra of $\mathfrak{h}$, $\mathfrak{h}_+ = \mathfrak{t}^\perp$ with respect to the Killing form), then every α_i is identically zero on exactly one of $\mathfrak{t}$ or $\mathfrak{h}_+$.

We must be more specific about how the weights in W arise. By Lie's Theorem, there exists a sequence of subspaces $\mathfrak{n}_c^i$ of $\mathfrak{n}_c$ and weights $\lambda_i \in \mathfrak{a}_c^*$ such that

$$\mathfrak{z}_c = \mathfrak{n}_c^0 \subseteq \mathfrak{n}_c^1 \subseteq \cdots \subseteq \mathfrak{n}_c^{2n-1} \subseteq \mathfrak{n}_c^{2n} = \mathfrak{n}_c$$

$\dim \mathfrak{n}_c^i/\mathfrak{n}_c^{i-1} = 1$, $X \cdot Y \equiv \lambda_i(X)Y \bmod \mathfrak{n}_c^{i-1}$, $X \in \mathfrak{a}$, $Y \in \mathfrak{n}_c^i$. The functionals λ_i are Lie mappings, so vanish on $[\mathfrak{a}, \mathfrak{a}]$. Moreover there is $Y_i \in \mathfrak{n}_c^i - \mathfrak{n}_c^{i-1}$ so that $X \cdot Y_i = \lambda_i(X)Y_i$, $X \in \mathfrak{h}_c$. In fact this can be set up so that $\lambda_i = -\lambda_{2n+1-i}$. For example, apply Lie's Theorem to get $\mathfrak{n}_c^1$, then apply it to $\mathfrak{n}_c/(\mathfrak{n}_c^1)^\theta$, $(\mathfrak{n}_c^1)^\theta =$ orthogonal complement of $\mathfrak{n}_c^1$ with respect to B_θ. Then from $\mathfrak{a} \cdot \theta = 0$, it's easy to see that $\lambda_1 = -\lambda_{2n}$. One continues in this fashion. Call λ_i real (or imaginary) iff α_i is real (imaginary). We choose $\mathfrak{b} = \mathfrak{n}_c^n$.

(a) That $\mathfrak{c} = \mathfrak{a} + \mathfrak{b}$ is a solvable algebra is clear.

(b) That $\mathfrak{c} + \bar{\mathfrak{c}}$ is an algebra is not so clear. First observe that λ_i real $\Rightarrow$ $\mathbf{C}\bar{Y}_i = \mathbf{C}Y_i$. In fact $\bar{X} \cdot \bar{Y}_i = \overline{X \cdot Y_i} = \overline{\lambda_i(X)}\,\bar{Y}_i$. If we take $X \in \mathfrak{h}$, then $X \cdot \bar{Y}_i = \overline{\lambda_i(\bar{X})}\bar{Y}_i = \overline{\lambda_i(X)}\bar{Y}_i$. If moreover $X \in \mathfrak{h}_+$, then $X \cdot \bar{Y}_i = \lambda_i(X)\bar{Y}_i$. Hence $\bar{Y}_i = \omega Y_i$ for some $\omega \in \mathbf{C}$. Next we observe that λ_i imaginary $\Rightarrow \mathbf{C}\bar{Y}_i = \mathbf{C}Y_{2n+1-i}$. In fact, reasoning as above with $X \in \mathfrak{t}$ instead of $\mathfrak{h}_+$, we get $X \cdot \bar{Y}_i = -\lambda_i(X)\bar{Y}_i$, which is enough to deduce the desired fact. Combining these observations, we obtain that

$$\mathfrak{b} \cap \bar{\mathfrak{b}} = (\mathfrak{d}_\mathfrak{b})_c = \mathfrak{z}_c + \sum_{\substack{\lambda_i \text{ real} \\ 1 \leq i \leq n}} \mathbf{C}Y_i$$

$$\mathfrak{b} + \bar{\mathfrak{b}} = (\mathfrak{e}_c) = (\mathfrak{d}_\mathfrak{b})_c + \sum_{\substack{\lambda_i \text{ imaginary} \\ 1 \leq i \leq 2n}} \mathbf{C}Y_i.$$

Now in order to deduce (b) we shall prove that $[\mathfrak{a}, \mathfrak{b} + \bar{\mathfrak{b}}] \subseteq \mathfrak{b} + \bar{\mathfrak{b}}$. First we know that $\mathfrak{a} \cap \bar{\mathfrak{a}}$ leaves $\mathfrak{b} + \bar{\mathfrak{b}}$ invariant. But

$$\mathfrak{a} = (\mathfrak{a} \cap \bar{\mathfrak{a}}) + \sum_{\substack{\beta \in \Phi^+ \\ \beta \text{imag}}} \mathfrak{g}_c^\beta.$$

For any Y_i and $X \in \mathfrak{g}_c^\beta$, we know $X \cdot Y_i \equiv \sum_{j<i} y_j Y_j$, $y_j \in \mathbf{C}$. It's enough to prove that Y_i imaginary, Y_j real $\Rightarrow y_j = 0$. Indeed by the usual additivity of weights argument, we have $y_j \neq 0 \Rightarrow \beta + \lambda_i = \lambda_j$. This is impossible if β, λ_i are imaginary and λ_j is real.

(c) Note that $\mathfrak{e} = \mathfrak{e}_\mathfrak{a} + \mathfrak{e}_\mathfrak{b}$, $\mathfrak{d} = \mathfrak{d}_\mathfrak{a} + \mathfrak{d}_\mathfrak{b}$, $\exp \mathfrak{e} = \exp \mathfrak{e}_\mathfrak{a} \exp \mathfrak{e}_\mathfrak{b}$. But $\exp \mathfrak{e}_\mathfrak{a}$ is the identity component of a parabolic subgroup of $Sp(n, \mathbf{R})$ and $\exp \mathfrak{e}_\mathfrak{b}$ is a closed subgroup of N. Therefore the product group is closed.

(d) In this case the orbit is closed. Indeed

$$G \cdot \varphi = SN \cdot \varphi = SN \cdot (\xi + \theta) = S \cdot \xi + N \cdot \theta$$

is closed. It's standard that closed orbits $\Rightarrow$ the Pukanszky condition (see, e.g., [9, Rem. 3(*d*), p. 265]).

(e) This condition requires a little work. Since by hypothesis $Ad_{\mathfrak{e}_a/\mathfrak{d}_a}$ exp $\mathfrak{d}_a$ is compact, it's enough to prove separately: $Ad_{\mathfrak{e}/\mathfrak{d}}\exp \mathfrak{d}_{\mathfrak{b}} = 1$, $Ad_{\mathfrak{e}_{\mathfrak{b}}/\mathfrak{d}_{\mathfrak{b}}}$ exp $\mathfrak{d}_a$ is compact. To prove the first of these we shall show $[\mathfrak{e}, \mathfrak{d}_{\mathfrak{b}}] \subseteq \mathfrak{d}$. In fact $[\mathfrak{e}_{\mathfrak{b}}, \mathfrak{d}_{\mathfrak{b}}] \subseteq \mathfrak{d}_{\mathfrak{b}}$ because any polarization for a Heisenberg algebra is relatively ideal. (The notion of relatively ideal—i.e., $\mathfrak{d}$ is an ideal of $\mathfrak{e}$—was introduced in [13]. For nilpotent groups it is equivalent to metric.) Now we show $[\mathfrak{e}_a, \mathfrak{d}_{\mathfrak{b}}] \subseteq \mathfrak{d}_{\mathfrak{b}}$. First of all $[\mathfrak{d}_a, \mathfrak{d}_{\mathfrak{b}}] \subseteq \mathfrak{d}_{\mathfrak{b}}$ (clearly). Next $\mathfrak{a}$ cuspidal implies that $\mathfrak{d}_a = \mathfrak{t} + \mathfrak{h}_+ + \mathfrak{u}$ where $\mathfrak{u}$ is the unipotent radical of the cuspidal parabolic $\mathfrak{e}_a$; and

$$\mathfrak{e}_a = \mathfrak{d}_a + \sum_{\substack{\beta\in\Phi \\ \beta \text{ imaginary}}} \mathfrak{s}_c^\beta.$$

But if β is positive imaginary and Y_i is real, then $[\mathfrak{s}_c^\beta, Y_i] \subseteq (\mathfrak{d}_{\mathfrak{b}})_c$. Indeed the same argument as above shows that for any $X \in \mathfrak{s}_c^\beta\, [X, Y_i] = \sum_{j<i} y_j Y_j$; $y_j \neq 0 \Rightarrow \lambda_i + \beta = \lambda_j$. But β imaginary and λ_i real is impossible. Thus $[\mathfrak{a}, (\mathfrak{d}_{\mathfrak{b}})_c] \subseteq (\mathfrak{d}_{\mathfrak{b}})_c$. Taking the conjugate, we get the desired result.

It remains to show the second property. Once again we invoke $\mathfrak{d}_a = \mathfrak{t} + \mathfrak{h}_+ + \mathfrak{u}$. But we know exp $\mathfrak{t}$ is compact. So the result would follow from $[\mathfrak{h}_+ + \mathfrak{u}, \mathfrak{e}_{\mathfrak{b}}] \subseteq \mathfrak{d}_{\mathfrak{b}}$. Now $X \in \mathfrak{h}_+$, Y_i imaginary $\Rightarrow X \cdot Y_i = \lambda_i(X) Y_i = 0$ since $\lambda_i|_{\mathfrak{h}_+} \equiv 0$. Thus $[\mathfrak{h}_+, \mathfrak{e}_{\mathfrak{b}}] \subseteq \mathfrak{d}_{\mathfrak{b}}$. Finally to show $[\mathfrak{u}, \mathfrak{e}_{\mathfrak{b}}] \subseteq \mathfrak{d}_{\mathfrak{b}}$ we argue as follows. Let $\mathfrak{s}^\gamma$ be any positive restricted root space for $(\mathfrak{s}, \mathfrak{h}_+)$. It's enough to show $[\mathfrak{s}^\gamma, \mathfrak{e}_{\mathfrak{b}}] \subseteq \mathfrak{d}_{\mathfrak{b}}$. If the claim is false, then by the unipotence of the action we know there exist $X \in \mathfrak{s}^\gamma$, $Y, Z \in \mathfrak{e}_{\mathfrak{b}}$, $Y, Z \notin \mathfrak{d}_{\mathfrak{b}}$, such that $X \cdot Y \equiv 0 \bmod \mathfrak{d}_{\mathfrak{b}}$ and $X \cdot Z \equiv Y \bmod \mathfrak{d}_{\mathfrak{b}}$. Let $H \in \mathfrak{h}_+$ be such that $H \cdot X = X$ (i.e., $\gamma(H) = 1$). Then $H \cdot (X \cdot Z) \equiv H \cdot Y \equiv 0 \bmod \mathfrak{d}_{\mathfrak{b}}$ (because $[\mathfrak{h}_+, \mathfrak{e}_{\mathfrak{b}}] \subseteq \mathfrak{d}_{\mathfrak{b}}$). But $H \cdot (X \cdot Z) = (H \cdot X) \cdot Z + X \cdot (H \cdot Z) \equiv X \cdot Z \equiv Y \bmod \mathfrak{d}_{\mathfrak{b}}$ (again using $[\mathfrak{h}_+, \mathfrak{e}_{\mathfrak{b}}] \subseteq \mathfrak{d}_{\mathfrak{b}}$). This contradiction completes the proof of (*e*).

The proof of (iv) is also complete. Before we go on to the last case (v), we list more precisely, what was proven in (iv)—namely, we showed:

(α) $[\mathfrak{a}, \mathfrak{b} + \bar{\mathfrak{b}}] \subseteq \mathfrak{b} + \bar{\mathfrak{b}}$;

(β) $[\mathfrak{e}, \mathfrak{d}_{\mathfrak{b}}] \subseteq \mathfrak{d}_{\mathfrak{b}}$;

(γ) $Ad_{\mathfrak{e}_{\mathfrak{b}}/\mathfrak{d}_{\mathfrak{b}}}$ exp $\mathfrak{d}_a$ is compact.

(v) $\mathfrak{g} = \mathfrak{s} + \mathfrak{n}$, $\mathfrak{n}$ Heisenberg, $\theta|_{\mathfrak{z}} \neq 0$, $\mathfrak{s} \cdot \theta = 0$. In part (iv) we have done the case $\mathfrak{s} = sp(\mathfrak{n}/\mathfrak{z}, B_\theta)$. Now we have the obvious natural map $\mathfrak{s} \to sp(\mathfrak{n}/\mathfrak{z}, B_\theta)$. $\mathfrak{s}$ splits as the Lie algebra direct sum of the kernel and the image. It is clear that the kernel is playing no role in the proof and so we may identify $\mathfrak{s}$ to its image, that is we assume $\mathfrak{s} \subseteq sp(\mathfrak{n}/\mathfrak{z}, B_\theta)$. Now given the Cartan subalgebra $\mathfrak{h} = \mathfrak{s}_\xi$ of $\mathfrak{s}$ and a Borel subalgebra $\mathfrak{a}$ of $\mathfrak{s}_c$ which is metric for ξ, I claim that we may find $\mathfrak{h}_1 \supseteq \mathfrak{h}$ and $\mathfrak{a}_1 \supseteq \mathfrak{a}$ such that: $\mathfrak{h}_1$ is a Cartan subalgebra of $\mathfrak{s}_1 = sp(\mathfrak{n}/\mathfrak{z}, B_\theta)$, $\mathfrak{a}_1$ is a Borel subalgebra of $sp((\mathfrak{n}/\mathfrak{z})_c, B_\theta)$ containing $\mathfrak{h}_1$, and $\mathfrak{a}_1 + \bar{\mathfrak{a}}_1$ is a cuspidal parabolic corresponding to $\mathfrak{h}_1$. (We choose compatible Cartan involutions on $\mathfrak{s}, \mathfrak{s}_1$.) Well, $\mathfrak{h}$ is an abelian semisimple subalgebra of $\mathfrak{s}_1$. Write $\mathfrak{h} = \mathfrak{t} + \mathfrak{h}_+$ according to the involution. Then we may extend, perhaps non-uniquely, to a Cartan subalgebra $\mathfrak{h}_1 = \mathfrak{t}_1 + \mathfrak{h}_{1,+}$, $\mathfrak{t}_1 \supseteq \mathfrak{t}$, $\mathfrak{h}_{1,+} \supseteq \mathfrak{h}_+$. Now take the two sets of roots Φ, Φ_1. $\mathfrak{a}$ defines a positive system Φ^+. Look at the positive imaginary roots Φ_i^+. We can choose an element $Y \in \mathfrak{t}$ so that $\Phi_i^+ = \{\alpha \in \Phi$: α imaginary, $\alpha(Y) > 0\}$. Consider $\mathfrak{m} = Z_{\mathfrak{s}}(\mathfrak{h}_+)$ and $\mathfrak{m}_1 = Z_{\mathfrak{s}_1}(\mathfrak{h}_{1,+})$. If Y is regular in $\mathfrak{m}_1$, define $\Phi_{1,i}^+$ by means of it. Otherwise, choose $Y_j \in \mathfrak{t}$, regular in $\mathfrak{m}_1$, all in the same $(\mathfrak{m}_1, \mathfrak{t}_1)$ chamber, $Y_j \to Y$. Set $\Phi_{1,i}^+ = \{\alpha \in \Phi_{1,i}: \alpha(Y_j) > 0\}$, independent of j. Then $\mathfrak{a} \cap \mathfrak{m}_c \subseteq \sum_{\beta \in \Phi_{1,i}^+} \mathfrak{m}_{1,c}^\beta$. In fact let $X \in \mathfrak{m}_c^\alpha$, $\alpha \in \Phi_i^+$. Write $X = \sum X_\beta$ and take $H \in \mathfrak{t}$. Then $[H, X] = \sum \alpha(H) X = \sum [H, X_\beta] = \sum \beta(H) X_\beta \Rightarrow \beta|\mathfrak{t} = \alpha \forall \beta \in \Phi_{1,c}$. Thus $\beta(Y_j) \to \beta(Y) = \alpha(Y) > 0 \Rightarrow \beta \in \Phi_{1,c}^+$. We can reason similarly with the split part. Namely there exists $Y \in \mathfrak{h}_+$ such that $\alpha \in \Phi^+(\mathfrak{s}, \mathfrak{h}_+) \Rightarrow \alpha(Y) > 0$. Extend as above to Φ_1^+, a set of non-purely imaginary roots. Continuing, we get a set of positive roots defining a Borel subalgebra $\mathfrak{a}_1$ of $\mathfrak{s}_{1,c}$ such that $\mathfrak{a} \subseteq \mathfrak{a}_1$, and $\mathfrak{a}_1$ is metric by construction. Note: we may not be able to extend $\xi \in \mathfrak{s}^*$ to an admissible $\xi_1 \in \mathfrak{s}_1^*$ (e.g., if $\mathfrak{s} = \mathfrak{h} = \mathfrak{h}_1 \subsetneqq \mathfrak{s}_1$, $\xi = 0$), but we don't need that for the argument.

Now apply the reasoning of (iv). There exists $\mathfrak{b} \subseteq \mathfrak{n}_c$ metric for θ such that $\mathfrak{c}_1 = \mathfrak{a}_1 + \mathfrak{b}$ satisfies properties (α), (β), (γ) for $\mathfrak{s}_1 + \mathfrak{n}$. Set $\mathfrak{c} = \mathfrak{a} + \mathfrak{b}$. Then properties (α), (β), (γ) hold for $\mathfrak{g}$. Indeed all are clear since $\mathfrak{a}$, $\mathfrak{d}_\mathfrak{a}$, $\mathfrak{e}_\mathfrak{a}$ are subalgebras of $\mathfrak{a}_1$, $\mathfrak{d}_{\mathfrak{a}_1}$, $\mathfrak{e}_{\mathfrak{a}_1}$, respectively. But it was explained in (iv) how properties (a)-(e) follow from these three. Thus the proof of (v) is finished.

The proof of Lemma 3.2′ is at last complete. By Lemma 3.3 that means that Lemma 3.2 is also proven. The proof of Theorem 3.1 will be completed in the next section.

REMARK 3.5. It is clear by tracing through the proof of Lemma 3.2 that if $\mathfrak{a}$ is real, then $\mathfrak{b}$ can be chosen real also. It is not at all clear whether if $\mathfrak{a}$ is positive, then $\mathfrak{b}$ can also be chosen positive. We expect to return to such matters when discussing square-integrable representations in future work.

4. Reductive data and essentially harmonic polarizations. In this section we show that associated to any $(\varphi, \tau) \in \mathscr{B}(G)$, there is canonically assigned a reductive group S_φ and a tempered irreducible unitary (perhaps projective) representation $\sigma_{\varphi,\tau}$ of S_φ. We call this pair $(S_\varphi, \sigma_{\varphi,\tau})$ the reductive data corresponding to (φ, τ) (see Definition 4.2 below). We shall also complete the proof of Theorem 3.1. For this we need to recall the recursive procedure of Duflo for associating a representation class $\pi_{\varphi,\tau}$ of G to the data (φ, τ). Later we shall develop an alternative prodecure.

Let $\varphi \in \mathscr{A}\mathscr{P}(G)$, $\tau \in \mathfrak{X}_G(\varphi)$. As usual put $\theta = \varphi|_{\mathfrak{n}}$. Then if $\gamma = \gamma(\theta)$ is any realization of the Kirillov class in $\hat{N}$ determined by $N \cdot \theta$, one knows [5] there is a canonical (perhaps projective) representation $\tilde{\gamma}$ of G_θ on the space of γ such that $\tilde{\gamma}(g_\theta)\gamma(g_\theta^{-1}ng_\theta) = \gamma(n)\tilde{\gamma}(g_\theta)$, $n \in N$, $g_\theta \in G_\theta$. Let $\mathfrak{q}_\theta = \operatorname{Ker} \theta|_{\mathfrak{n}_\theta}$, Q_θ the corresponding analytic subgroup, $\tilde{G}_\theta$ the canonical 2-fold cover (determined by B_θ), $G_1 = \tilde{G}_\theta/Q_\theta$. It's possible that G_1 is not algebraic, but of course it is a real Lie group. We denote by φ_1 the functional obtained from $\varphi|_{\mathfrak{g}_\theta}$ by passage to the quotient. Then $\varphi_1 \in \mathscr{A}\mathscr{P}(G_1)$ [6, Ch. IV]. Furthermore τ determines canonically an element $\tau_1 \in \mathfrak{X}_{G_1}(\varphi_1)$ [6, Lemma IV.6]. Duflo's definition of $\pi_{\phi,\tau}$ is then

$$\pi_{\varphi,\tau} = \operatorname{Ind}_{G_\theta N}^G \pi_{\varphi_1,\tau_1} \otimes \tilde{\gamma} \times \gamma.$$

Here π_{φ_1,τ_1} is defined inductively if $\dim G_1 < \dim G$. Otherwise G is reductive and the tempered class $\pi_{\varphi,\tau}$ is defined in [6, Ch. III].

Now we set $G^1 = G_\theta N$, $\mathfrak{g}^1 = \mathfrak{g}_\theta + \mathfrak{n}$, $\varphi^1 = \varphi|_{\mathfrak{g}^1}$.

PROPOSITION 4.1. (i) $\varphi^1 \in \mathscr{A}\mathscr{P}(G^1)$, *and any* $\tau \in \mathfrak{X}_G(\varphi)$ *canonically determines and element* $\tau^1 \in \mathfrak{X}_{G^1}(\varphi^1)$ *such that*

$$\pi_{\varphi,\tau} \otimes \tilde{\gamma} \times \gamma \cong \pi_{\varphi^1,\tau^1}. \tag{4.1}$$

(ii) *Any (essentially) harmonic polarization for* φ^1, *admissible for* $\mathfrak{n}$, *is also (essentially) harmonic for* φ.

PROOF. (i) By [6, Ch. IV] φ well-polarizable $\Rightarrow$ θ and $\xi = \varphi|_{\mathfrak{g}_\theta}$ are well-polarizable. Hence there exists a solvable subalgebra $\mathfrak{b} = (\mathfrak{b} \cap \mathfrak{n}_c) + (\mathfrak{b} \cap (\mathfrak{g}_\theta)_c)$ which is Lagrangian for B_ϕ. But we have

$$\mathfrak{g}_\varphi \subseteq (\mathfrak{g}_\theta + \mathfrak{n})_{\varphi^1} = \mathfrak{g}_\varphi + \mathfrak{n}_\theta \subseteq \mathfrak{g}_\theta + \mathfrak{n} = \mathfrak{g}^1 \subseteq \mathfrak{g}.$$

Furthermore $\dim \mathfrak{g}/\mathfrak{g}^1 = \dim(\mathfrak{g}_\varphi + \mathfrak{n}_\theta)/\mathfrak{g}_\varphi$—as was already proven in part (ii) of the proof of Lemma 3.2′. Next let φ be admissible. Then $(G^1)_{\varphi^1} = G_\varphi N_\theta$. Let L be Lagrangian for φ^1. Then L is Lagrangian for φ and there exists a representation σ such that $s\sigma(g_\varphi, s) = r\tau(g_\varphi, r)$, where r is defined on $\Lambda(\mathfrak{g}/\mathfrak{g}_\varphi)$, s on $\Lambda(\mathfrak{g}^1, \mathfrak{g}^1_{\varphi^1})$ (here we use the terminology of [6, Chs. I & IV]). Clearly σ is defined on $(G^1)^{\sim}_{\varphi^1} \cong (\tilde{G}_\varphi \times N_\theta)/N_\theta$ by the above formula, and equals χ_θ on N_θ. Thus $\varphi^1 \in \mathscr{A}\mathscr{P}(G^1)$. The equivalence in formula (4.1) is also clear by Duflo's result [7, Thm. III.19] that the representations $\pi_{\varphi,\tau}$ can be defined via any closed normal connected nilpotent Lie subgroup.

(ii) Suppose $\mathfrak{a}$ is (essentially) harmonic for φ^1 and admissible for $\mathfrak{n}$. It is obvious from the inclusions

$$\mathfrak{g}_\varphi \subseteq \mathfrak{g}^1_{\varphi^1} = \mathfrak{g}_\varphi + \mathfrak{n}_\theta \subseteq \mathfrak{g}^1 = \mathfrak{g}_\theta + \mathfrak{n} \subseteq \mathfrak{g}$$

that $\mathfrak{a}$ has all the properties (a), (b), (c), (e) of §3. In order to demonstrate property (d), we'll show that $\exp \mathfrak{d} \cdot \varphi$ is closed, using that $\exp \mathfrak{d} \cdot \varphi^1$ is closed. Let $p: \mathfrak{g}^* \to (\mathfrak{g}^1)^*$ be the canonical projection. Then it's enough to show $p^{-1}(\exp \mathfrak{d} \cdot \varphi^1) = \exp \mathfrak{d} \cdot \varphi$. Clearly, since $\mathfrak{d} \subseteq \mathfrak{g}^1$, we know that the left side includes the right. Conversely suppose $\psi \in \mathfrak{g}^*$ and $p(\psi) = \psi|_{\mathfrak{g}^1} = d \cdot \varphi^1$ for some $d \in \exp \mathfrak{d}$. Then $\psi|_{\mathfrak{g}_\theta + \mathfrak{n}} = d \cdot \varphi^1 = (d \cdot \varphi)|_{\mathfrak{g}_\theta+\mathfrak{n}}$. Hence by Prop. 2.3 $\psi \in N_\theta \cdot d \cdot \varphi$. That does it since $N_\theta \subseteq \exp \mathfrak{d}$. This completes the proof for essentially harmonic polarizations. Finally suppose $\mathfrak{a}$ is actually harmonic for φ^1. Since $G_\varphi \subseteq (G^1)_{\varphi^1}$, invariance is clear. Also by assumption $E^1 = G^1_{\varphi^1} \cdot \exp \mathfrak{e}$ is closed and $\mathrm{Ad}_{\mathfrak{e}/\mathfrak{d}} D^1$ is compact. But $G^1_{\varphi^1} = G_\varphi N \Rightarrow E = G_\varphi \exp \mathfrak{e} = E^1$ must be closed. Similarly $D = D^1$ so $\mathrm{Ad}_{\mathfrak{e}/\mathfrak{d}} D$ is compact. That completes the proof.

We are now ready for the alternate recursive procedure for setting up the representations. Instead of treating $G \hookrightarrow G_1 \hookrightarrow G_2 \hookrightarrow \ldots$ as in Duflo, we do the following. Note that G^1 is an algebraic group. Let N^1 be its unipotent radical. It may be larger than N. We can apply Prop. 4.1 to the setup $\varphi^1 \in \mathscr{AP}(G^1)$, $\theta^1 = \varphi^1|_{\mathfrak{n}^1}$, $\tau^1 \in \mathfrak{X}_{G^1}(\varphi^1)$. In this way we arrive at a finite sequence of algebraic subgroups

$$G \supseteq G^1 \supseteq G^2 \supseteq \cdots \supseteq G^r \supseteq N^r = N^{r-1} \supseteq \cdots \supseteq N^1 \supseteq N,$$

where N^i is the unipotent radical of G^i, $\pi_{\varphi,\tau} \cong \mathrm{Ind}_{G^i}^G \pi_{\varphi^i,\tau^i}$, $i = 1, \ldots, r$ (using induction in stages), $N^r = N^{r-1}$ is the unipotent radical of G^r, and $G^r = (G^r)_{\theta^r} N^r$. In particular then $\varphi^r \in \mathscr{AP}(G^r)$, $\theta^r = \varphi^r|_{\mathfrak{n}^r}$, $\mathfrak{g}^r = (\mathfrak{g}^r)_{\theta^r} + \mathfrak{n}^r$ and $\pi_{\varphi,\tau} \cong \mathrm{Ind}_{G^r}^G \pi_{\varphi^r,\tau^r}$. Of course N^j, $j \geqq 1$, will not be normal in G, but that doesn't matter. Let S^r be a Levi factor of N^r in G^r. S^r is unique only up to conjugacy by elements of N^r. Also we may assume S^r is a subgroup of the fixed Levi subgroup S (of G) that we began with. Then $G^r = S^r N^r$ and the representation π_{φ^r,τ^r} is given by $\pi_{\varphi^r,\tau^r} = \omega \otimes \tilde{\gamma}_{\theta^r} \times \gamma_{\theta^r}$, where ω is a unitary representation (perhaps projective) of S^r, uniquely determined by the original data φ, τ.

Definition 4.2. We call (S^r, ω) the reductive data of φ, τ. Sometimes we refer to S^r—which only depends on φ—as the reductive data of φ. Also if π is a representation in the class associated to (φ, τ), we sometimes say (S^r, ω) is the reductive data of π.

It is clear from Lemmas 3.2, 3.2′ and Prop. 4.1 that Theorem 3.1 is proven once we have demonstrated the next lemma.

Lemma 4.3. *Let G be reductive, $\varphi \in \mathscr{AP}(G)$. Then there exists an essentially harmonic polarization for φ.*

Proof. We know $\mathfrak{h} = \mathfrak{g}_\varphi$ is a Cartan subalgebra [6, Lemma II.7]. Let Φ be the roots of $(\mathfrak{g}_c, \mathfrak{h}_c)$ and write $\mathfrak{h} = \mathfrak{t} + \mathfrak{h}_+$ so that $\Phi \subseteq i\mathfrak{t}^* \cup \mathfrak{h}_+^*$. Then we can always choose a Borel subalgebra $\mathfrak{a}$ of $\mathfrak{g}_c$ so that $\mathfrak{e} = (\mathfrak{a} + \bar{\mathfrak{a}}) \cap \mathfrak{g}$

is a cuspidal parabolic subalgebra corresponding to $\mathfrak{h}$. Here's how: Choose any set Φ_i^+ of positive imaginary roots. Then let Φ_c^+ be a set of positive complex roots closed under conjugation. For example, let $X \in \mathfrak{h}_+$ be chosen such that $\alpha \in \Phi$, $\alpha(X) = 0 \Rightarrow \alpha(\mathfrak{h}_+) = 0$. Then α positive complex can be taken to mean $\alpha(X) > 0$. Hence $\Phi^+ = \Phi_i^+ \cup \Phi_c^+$ is a set of positive roots and $\mathfrak{a} = \mathfrak{h}_c + \sum_{\alpha \in \Phi^+} \mathfrak{g}_c^\alpha$ has the desired property. Of course $\mathfrak{a}$ may or may not be G_φ-invariant, but that won't concern us until the next section. The Borel $\mathfrak{a}$ has properties (a)–(e): the first three are evident, (d) follows because $G \cdot \varphi$ is closed and (e) comes by [**11**, Lemma 4.2].

The proof of the main Theorem 3.1 is now complete.

REMARK 4.4. Our work here renders the additional two properties required of polarizations in [**11**] unnecessary. Why is that? Let $(\varphi, \tau) \in \mathscr{B}(G)$ and suppose that $\mathfrak{a}$ is a harmonic polarization for φ^r (and so also for φ—Prop. 4.1). The first of the two properties was endless admissibility [**11**, Def. 3.2]. But it is clear for the polarizations constructed above that $\mathfrak{e} \subseteq \mathfrak{g}^r$. Thus (e.g., by [**11**, Lemma 2.10]) we have

$$\pi_G(\varphi, \tau, \mathfrak{a}) \cong \operatorname{Ind}_{G^r}^G \pi_{G^r}(\varphi^r, \tau^r, \mathfrak{a}). \tag{4.2}$$

The polarization $\mathfrak{a}$ is automatically endlessly admissible for φ^r (since on $\mathfrak{g}^r$ that's equivalent to admissibility). The second property was that of obeying the Satake condition [**11**, Def. 3.4]. But once again it is enough to know that $\mathfrak{a}$ satisfies the Satake condition relative to φ^r. Now we know that $G^r = S^r N^r$, where S^r is reductive, N^r is unipotent, S^r fixes $\theta^r \in \mathscr{A}\mathscr{P}(N^r)$ and $\pi^{G^r}(\varphi^r, \tau^r, \mathfrak{a}) \cong \omega \otimes \tilde{\gamma} \times \gamma_{\theta^r}$. By the usual kind of reasoning [**11**, §3] or [**15**], we may assume that γ is square integrable mod the center of N. Then we may apply the basic result of [**15**]. (Rosenberg only deals with square-integrable ω, but the reduction to that case is also easy since ω is tempered—see, e.g., [**11**, §3].) It yields that $\pi_{G^r}(\varphi^r, \tau^r, \mathfrak{a}) \in \pi_{\varphi^r, \tau^r} \in (G^r)^\wedge$. Therefore, combining with formula (4.2), we have that $\pi_G(\varphi, \tau, \mathfrak{a}) \in \pi_{\varphi, \tau} \in \hat{G}$, the main result of [**11**].

So to summarize: Let G be a real algebraic group. Then for any $\varphi \in \mathscr{A}\mathscr{P}(G)$, there exists an essentially harmonic polarization $\mathfrak{a}$ for φ. The (weak) harmonically induced representation ${}^\circ\pi(\varphi, \mathfrak{a})$ is defined. It yields a finite direct sum of irreducible representations, parameterized by the irreducible elements of $\mathfrak{X}_G(\varphi)$. Nevertheless we still desire harmonic polarizations. The corresponding harmonically induced representation will then give a realization of the Duflo class $\pi_{\varphi, \tau} \in \hat{G}$, $\tau \in \mathfrak{X}_G(\varphi)$. But it's not true that for every $\varphi \in \mathscr{A}\mathscr{P}(G)$ a harmonic polarization must exist. In fact I believe it's true generically—that is, the set of φ which don't possess harmonic polarizations is of Plancherel measure zero. We shall take that up in a future publication. For now, we shall describe some situations in which harmonic polarizations do exist—and then derive a general sufficient condition.

5. Harmonic Polarizations. So we have shown that for any $\varphi \in \mathscr{AP}(G)$, G real algebraic, there exists an essentially harmonic polarization $\mathfrak{a}$ for φ. What additional facts must be established to guarantee that $\mathfrak{a}$ is actually harmonic? They are three:

(a) $\mathfrak{a}$ is G_φ-invariant;

(b) $E = G_\varphi \exp \mathfrak{e}$ is closed; and

(c) $\mathrm{Ad}_{\mathfrak{e}/\mathfrak{d}} D$ is compact, $D = G_\varphi \exp \mathfrak{d}$.

But in fact, since we are dealing here with algebraic groups, the component group $G_\varphi/G_\varphi^\circ$ is finite. Moreover $G_\varphi^\circ \subseteq \exp \mathfrak{d} \subseteq \exp \mathfrak{e}$. Thus if $\mathfrak{a}$ is essentially harmonic, then the closure of E follows from that of $E^\circ = \exp \mathfrak{e}$ and the compactness of $\mathrm{Ad}_{\mathfrak{e}/\mathfrak{d}} D$ follows from that of $\mathrm{Ad}_{\mathfrak{e}/\mathfrak{d}} D^\circ$. In short of we have the following proposition.

PROPOSITION 5.1. *Let G be real algebraic, $\varphi \in \mathscr{AP}(G)$, $\mathfrak{a}$ an essentially harmonic polarization for φ. Then $\mathfrak{a}$ is harmonic iff it is invariant.*

Let us indicate some situations where invariance is either obviously automatic or well-known to hold.

(1) If $G_\varphi = G_\varphi^\circ Z_G$, $Z_G = \mathrm{Cent}\, G$, then of course invariance is automatic. This condition is somewhat special. In a future publication we shall see that it is relevant to the study of spuare-integrable representations. The two special cases $G_\varphi = Z_G$ and $G_\varphi = G_\varphi^\circ$ are respecitvely the conditions of H-groups [**2**] and full orbits [**12**, Def. 2.1]. The question of existence of metric polarizations for square-integrable representations is a fascinating problem that merits further attention.

(2) If G is solvable and connected, then one can choose $\mathfrak{a}$ not only invariant, but also positive [**3**].

(3) If G is reductive and connected—more generally of the Harish-Chandra class—then harmonic polarizations can always be found. The polarizations are constructed as in Lemma 4.3 and the Harish-Chandra class condition mandates a structure on the Cartan subgroup that guarantees invariance [8, §2].

In both reductive and solvable groups G, one can give examples $\varphi \in \mathscr{AP}(G)$ to which there do not exist invariant polarizations if the component group G/G° is sufficiently nasty. But, as indicated earlier, I expect that the set of such bad orbits is "negligible".

We now develop a nice sufficient condition for invariance.

DEFINITION 5.2. Let G be a reductive Lie group. We say G is of class HCA if it satisfies the two conditions:

(i) $\mathrm{Ad}_{\mathfrak{g}_c} G \subseteq \mathrm{Ad}\mathfrak{g}_c$;

(ii) If $\mathfrak{h}$ is any Cartan subalgebra of $\mathfrak{g}$, then the Cartan subgroup $H = Z_G(\mathfrak{h})$ is abelian.

Property (i) is the usual key defining property of the Harish-Candra class. Property (ii) is also a common structural assumption (see [17]).

THEOREM 5.3. *Let G be real algebraic, $\varphi \in \mathscr{AP}(G)$, S_φ the reductive data for φ. Then if S_φ is of class* HCA, *there exists a harmonic polarization $\mathfrak{a}$ for φ.*

PROOF. We recall the construction of the reductive data:

$$G \supseteq G^1 \supseteq G^2 \supseteq \cdots \supseteq G^r = S^r N^r \supseteq N^r = N^{r-1} \supseteq \cdots \supseteq N^1 \supseteq N.$$

The group $G^r = S^r N^r$ is such that $S_\varphi \equiv S^r$ fixes θ^r. Furthermore we have seen (in Prop. 4.1) that any harmonic polarization for φ^r is harmonic for φ. Thus it is enough to prove the theorem under the assumption that $G = G^r$. So we place ourselves in the situation:

$$G = SN,\ \varphi \in \mathscr{AP}(G),\ \theta = \varphi|_{\mathfrak{n}} \in \mathscr{AP}(N),\ S = S_\theta$$
$$\text{reductive of class HCA};\ \xi = \varphi|_{\mathfrak{s}} \in \mathscr{AP}(\tilde{S}),\ G_\varphi = S_\xi N_\theta.$$

Now because of the first property of HCA groups, we know there exists a Borel subalgebra $\mathfrak{a}$ of $\mathfrak{s}_c$ which is a metric polarization for ξ. According to Lemma 3.2′, there exists $\mathfrak{b}$, a metric polarization for θ, $\mathfrak{a}$-invariant, so that $\mathfrak{c} = \mathfrak{a} + \mathfrak{b}$ is an essentially harmonic polarization for φ. By Prop. 5.1 and the equation $G_\varphi = S_\xi N_\theta$, we need only prove that $\mathfrak{b}$ can be chosen S_ξ-invariant. For that we shall use property (ii) in the definition of the HCA class. In fact, we accomplish this by working, step-by-step, through the proof of Lemma 3.2′ and seeing that the polarization $\mathfrak{b}$ can be chosen S_ξ-invariant at each stage if S is of class HCA.

(i) $\mathfrak{n}$ abelian. Since $\mathfrak{b} = \mathfrak{n}_c$ this case is obvious.

(ii) We are already in the situation $G = G_\theta N$ since $G = SN$, $S = S_\theta$.

(iii) It is no loss of generality to assume $\mathfrak{n}$ is Heisenberg. In fact the proof of that portion of Lemma 3.2′ respects the HCA conditions. The ideal $\mathfrak{p} = \operatorname{Ker} \theta|_{\mathfrak{z}}$, $\mathfrak{z} = \operatorname{Cent} \mathfrak{n}$, is normalized by S since S fixes θ. The induction argument, using Lemma 3.4, shows that the proof reduces to the case of Heisenberg $\mathfrak{n}$.

(iv) $G = SN$ where N is Heisenberg of dimension $2n + 1$ and $S = \mathrm{Sp}(n, \mathbf{R})$. The subgroup $H = S_\xi$ is a Cartan subgroup of S. It may not be connected, however it is abelian. In fact if $\mathfrak{h} = \mathfrak{s}_\xi = \mathfrak{t} + \mathfrak{h}_+$ as usual, then $H = H^\circ \Gamma$ where $\Gamma = K \cap \exp i\,\mathfrak{h}_+ (K = \exp \mathfrak{k},\ \mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a Cartan decomposition such that $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{k}$, $\mathfrak{h}_+ = \mathfrak{h} \cap \mathfrak{p})$. It follows that the metric polarization $\mathfrak{b}$ constructed in part (iv) of the proof of Lemma 3.2′ is invariant by S_ξ.

(v) Finally let $G = SN$, N Heisenberg of dimension $2n + 1$, $S = S_\theta$,

$\theta|_{\mathfrak{z}} \neq 0$. Then there is a natural homomorphism $S \xrightarrow{\Psi} \mathrm{Sp}(\mathfrak{n}/\mathfrak{z}, B_\theta) \cong \mathrm{Sp}(n, \mathbf{R})$. It may be that $\xi \neq 0$ on the Lie algebra $\mathfrak{s}_1$ of the kernel $S_1 = \mathrm{Ker}\, \Psi$. But that is not relevant since S_1 will leave any polarization $\mathfrak{b}$ invariant. Hence it is no loss of generality to replace S by its image under Ψ in the symplectic group. Therefore we have $S \subseteq \mathrm{Sp}(n, \mathbf{R})$, $\xi \in \mathscr{AP}(S)$, $\mathfrak{a}$ a metric polarization for $\xi \in \mathfrak{s}^*$ regular semisimple. Next I claim that S_ξ is a Cartan subgroup. Clearly $s \in S_\xi$ normalizes the Cartan subalgebra $\mathfrak{h} = \mathfrak{s}_\xi$. But $\mathrm{Ad}_{\mathfrak{s}_c} S \subseteq \mathrm{Ad}_{\mathfrak{s}_c}$ insures that $\mathrm{Ad}_{\mathfrak{h}}(s)$ is in the Weyl group. And the Weyl group permutes the chambers in a simply transitive fashion. But since $s \cdot \xi = \xi$, s must act by the identity element of the Weyl group. That is $s \in Z_G(\mathfrak{h})$. Thus, by property (ii) of HCA, S_ξ is abelian. That is S_ξ is an abelian semisimple subgroup of the symplectic group $\mathrm{Sp}(n, \mathbf{R})$. It is routine to see that any such subgroup must be contained in a Cartan subgroup H' of $\mathrm{Sp}(n, \mathbf{R})$. Then we may choose a Borel subalgebra $\mathfrak{a}'$ of $sp(n, \mathbf{C})$ which contains both $\mathfrak{a}$ and $\mathfrak{h}'$, and so that $(\mathfrak{a}' + \mathfrak{a}) \cap sp(n, \mathbf{R})$ is a cuspidal parabolic corresponding to $\mathfrak{h}'$. Applying case (iv) to the pair $(\mathfrak{a}', H')$, we know there exists a metric polarization $\mathfrak{b}$ of θ, invariant by H' and $\mathfrak{a}'$. Then $\mathfrak{b}$ is invariant by S_ξ and $\mathfrak{a}$, and the proof is completed.

REMARK 5.4. Theorem 5.3 is somewhat weaker than the result asserted in [**11**, §5]. In fact, it seems that Theorem 5.3 is false without property (ii) of the HCA class—although I have not constructed an explicit example. In some sense it shouldn't matter since I expect (as indicated earlier) that one can cut down the set $\mathscr{AP}(G)$ to a smaller, but still generic subset, all of whose functionals will have harmonic polarizations.

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