ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 14, Number 4, Fall 1984

SEPARATING HYPERPLANES FOR CONVEX SETS OVER ORDERED FIELDS

JONATHAN LEE MERZEL

Dedicated to the memory of Gus Efroymson

We work with ordered fields $(F, <)$, the ordering being fixed. For an extension K of F, we may use the same symbol $<$ for an extension of the ordering on F to K. F^n denotes the space of n-tuples of elements of F, whose elements in turn may be designated by vector notation $\mathbf{x}$. For $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$, $\mathbf{x} \cdot \mathbf{y}$ is the "usual" dot product $x_1 y_1 + \cdots + x_n y_n$.

DEFINITION. A subset A of F^n is called convex iff for all $\mathbf{a}, \mathbf{b} \in A$ and all $\lambda \in F$: $0 < \lambda < 1$ implies $(1 - \lambda)\mathbf{a} + \lambda \mathbf{b} \in A$.

DEFINITION. A hyperplane in F^n with equation $\mathbf{a} \cdot \mathbf{x} = b$ is said to strongly separate subsets A, B of F^n if either

$$\mathbf{a} \cdot \mathbf{y} < b \text{ for all } \mathbf{y} \in A \text{ and } \mathbf{a} \cdot \mathbf{y} > b \text{ for all } \mathbf{y} \in B \text{ or}$$

$$\mathbf{a} \cdot \mathbf{y} > b \text{ for all } \mathbf{y} \in A \text{ and } \mathbf{a} \cdot \mathbf{y} < b \text{ for all } \mathbf{y} \in B.$$

(Weaker versions of separation allow points of A, B, or possibly both to lie on the hyperplane.)

The question to be studied is this: given disjoint convex sets A, B in F^n, is there an ordered field extension $(K, <)$ of $(F, <)$ such that A, B are (weakly or strongly) separated by a hyperplane in K^n? Of course, if K exists it may be taken to have transcendence degree at most n over F (for we need only adjoin to F the coefficients in the equation of the hyperplane, one of which can be taken to be 1).

In fact, we have the following theorem.

THEOREM. *If $(F, <)$ is an ordered field and A, B are disjoint convex sets in F^n, then there is an ordering on $K = F(t_1, \ldots, t_n)$ $(t_1, \ldots, t_n$ independent transcendentals over $F)$ extending that on F such that A, B are strongly separated by a hyperplane in K^n (with respect to the extended ordering).*

The gist of the argument is to first replace A, B with a maximal disjoint pair of convex sets, and then to identify the intercepts of the separating hyperplane via cuts on the coordinate axes. Some care is needed, however, in the way these cuts are defined.

In general, one cannot do better than the theorem above, as is shown by the following:

THEOREM/EXAMPLE. *Let* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ *be an n-tuple of real numbers algebraically independent over the field* $\mathbf{Q}$ *of rationals. Let* $A = \{\mathbf{y} \in \mathbf{Q}^n\colon \boldsymbol{\alpha}\cdot\mathbf{y} < 1\}$ *and* $B = \{\mathbf{y} \in \mathbf{Q}^n\colon \boldsymbol{\alpha}\cdot\mathbf{y} > 1\}$. *Then any ordered field extension of* $\mathbf{Q}$ *over which* A, B *can be (even weakly) separated by a hyperplane has transcendence degree at least n over* $\mathbf{Q}$.

DEPARTMENT OF MATHEMATICS, HOLY NAMES COLLEGE, OAKLAND, CA 94619