ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 14, Number 3, Summer 1984

MINIMAL EXISTENCE OF NONOSCILLATORY SOLUTIONS IN FUNCTIONAL DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

BHAGAT SINGH

ABSTRACT. For the equation

$$\text{(A)} \qquad L_n y(t) + F(t, y(g(t))) = f(t)$$

minimal sufficient conditions ensure the existence of a nonoscillatory solution of (A). L_n is a disconjugate differential operator of the form

$$L_n = \frac{1}{P_n(t)} \frac{d}{dt} \frac{1}{p_{n-1}(t)} \cdots \frac{1}{p_1(t)} \frac{d}{dt} \frac{\cdot}{p_0(t)}.$$

1. Introduction. It is well known from the works of Onose [3] and Singh [7] that, subject to the conditions

$$\text{(1)} \qquad \int^{\infty} t^{n-1}|q(t)|\, dt < \infty$$

and

$$\text{(2)} \qquad \int^{\infty} t^{n-1}|f(t)|dt < \infty,$$

an equation of the form

$$\text{(3)} \qquad y^{(n)}(t) + q(t)y(g(t)) = f(t)$$

has a nonoscillatory solution with a prescribed limit at ∞. However when the integral size in (1) or (2) is allowed to be unbounded, then the results of Singh [7], Onose [3], Lovelady [2] and Philos [4] do not indicate if a nonoscillatory solution still exists. Our purpose in this work is to prove the existence of a nonoscillatory solution of a much more general functional equation of the form

$$\text{(4)} \qquad L_n y(t) + F(t, y(g(t))) = f(t)$$

AMS (MOS) subject classification: 34K15
Key words: *Oscillatory, Nonoscillatory, Proper, Disconjugate, Canonical, Principal system*
Received by the editors November 19, 1982 and in revised form May 23, 1983.

where $n \geqq 2$ and L_n is a disconjugate operator of the form

(5)
$$L_n = \frac{1}{p_n(t)} \frac{d}{dt} \cdot \frac{1}{p_{n-1}(t)} \frac{d}{dt} \cdots \frac{d}{dt} \frac{\cdot}{p_0(t)}$$

under much less severe conditions, even when (4) is specialized to (3). In what follows, we assume that

(i) $p_i(t)$, $f(t)$ are continuous on $[a, \infty)$ for some $a > 0$, $p_i(t) > 0$ for $0 \leqq i \leqq n$, and

(6)
$$\int^{\infty} p_i(t)dt = \infty \text{ for } 1 \leqq i \leqq n - 1;$$

(ii) F: $R \times R \to R$ is continuous where R is the real line; $vF(u, v) > 0$; $F(u, v)$ is increasing in v;

(iii) $g(t)$: $R \to (0, \infty)$ is continuous, $g(t) \to \infty$ as $t \to \infty$, $g(t) \leqq t$.

Following our notations in [9] (which are generalized version of notations of Willett [12]), we let $i_k \in \{1, 2, \ldots, n - 1\}$, $1 \leqq k \leqq n - 1$ and $t, s \in [a, \infty)$. We define

(7)
$$I_0 = 1$$
$$I_k(t, s; p_{i_k}, \ldots, P_{i_l}) = \int_s^t p_{i_k}(r) I_{k-1}(r, s; p_{i_{k-1}}, \ldots, p_{i_l}) dr.$$

It can be easily verified that for $1 \leqq k \leqq n - 1$, we have the identities

(8)
$$I_k(t, s; p_{i_k}, \ldots, p_{i_l}) = (-1)^k I_k(s, t; p_{i_l}, \ldots, p_{i_k})$$

(9)
$$I_k(t, s; p_{i_k}, \ldots, p_{i_l}) = \int_s^t p_{i_1}(r) I_{k-1}(t, r; p_{i_k}, \ldots, p_{i_2}) dr.$$

For simplicity, we let

(10)
$$J_i(t, r) = p_0(t) I_i(t, r; p_1, \ldots, p_i)$$

(11)
$$J_i(t) = J_i(t, T) \text{ for any } T \geqq a.$$

Note that for any function $G(t)$

(12)
$$\int_T^t I_{n-1}(t, r; p_1, p_2, \ldots, p_{n-1}) G(r) p_n(r)\, dr$$
$$= \int_T^t p_1(s_1) \int_T^{s_1} p_2(s_2) \int_T^{s_2} \cdots \int_T^{s_{n-1}} p_n(r) G(r) dr\, ds_{n-1} \ldots ds_1.$$

In the foregoing analysis, the quasi derivatives will be used. We define

(13)
$$L_0 y(t) = \frac{y(t)}{p_0(t)},$$

(14)
$$L_i y(t) = \frac{1}{p_i(t)} (L_{i-1}\, y(t))', \; 1 \leqq i \leqq n.$$

The domain of L_n is defined to be the set of all functions y: $[a, \infty) \to R$ such that $L_i y(t)$ exist and are continuous on $[a, \infty)$. By a solution of equation (4) is meant a function y in the domain of L_n which satisfies (4) on $[a, \infty)$. By a proper solution of equation (4) and its type is meant a solution $y(t)$ which satisfies

$$\sup\{|y(t)|: t \geqq T_y\} > 0 \tag{15}$$

for every $T_y \geqq a$. A proper solution $y(t)$ of (4) is said to be *oscillatory* if it has arbitrarily large zeros on the interval $[a, \infty)$, otherwise $y(t)$ is called *nonoscillatory*.

There is not much known about the asymptotic behavior of the solutions of functional equations involving disconjugate operators such as L_n. Quite often such solutions are assumed to exist, and oscillation criteria obtained. For sufficiency type results insuring oscillation of the solutions of (4), we refer the reader to Singh [**7**], Lovelady [**2**], Philos [**4**] and Trench [**10**]. For asymptotic boundedness of the solutions of equation (4), the excellent sources are Philos and Staikos [**5**] and Kusano and Naito [**1**]. For asymptotic limits of nonoscillatory solutions and other related results the reader is referred to Singh and Kusano [**8**].

2. Main Results. In this section we shall establish the existence of a nonoscillatory solution of (4) under the most minimal conditions. We shall consider the case when the operator L_n is in canonical form. L_n is said to be *in canonical form* when (6) holds. According to Trench [**10**], any operator of the type of L_n which is not in canonical form can be represented uniquely with a different set of p_i, $1 \leqq i \leqq n$.

THEOREM 1. *Suppose* (i)–(iii) *hold. Further suppose that for each* $T \geqq a$ *the limit*

$$\lim_{t\to\infty}\left[(J_{n-1}(t))^{-1}p_0(t)\int_T^t I_{n-1}(t, r; p_1, \ldots, p_{n-1})\cdot p_n(r)|f(r)|dr < \infty\right] \tag{16}$$

exists, and for each finite constant $b > 0$ *and* $T \geqq a$, *the limit*

$$\lim_{t\to\infty}\left[\begin{matrix} p_0(t)\,(J_{n-1}(t))^{-1}\int_T^t I_{n-1}(t,\, r;\, p_1,\, \ldots,\, p_{n-1})\cdot p_n(r) \\ \cdot F(r,\, bJ_{n-1}(g(r)))\, dr \end{matrix}\right] = h(b,\, T) \tag{17}$$

exists and $h(b, T) \to 0$ *as* $T \to \infty$. *Then equation* (4) *has a nonoscillatory proper solution* $y(t)$ *with the property* $y(t) = O(J_{n-1}(t))$.

PROOF. Let D be the locally convex space of all continuous functions S: $[T, \infty) \to N$, $T \geqq 8a$, superimposed with the topology of uniform convergence on compact subsets of $[T, \infty)$. The members of D satisfy the additional property that for each $S \in D$, $|S(t)|/J_{n-1}(t) \leqq C$, where

$C > 0$ is the same for all $S \in D$. We consider the set $X \subset D$ defined, for a finite constant $c > 0$, $8c < C$, as

$$X = \{y \in D: ((c/2)J_{n-1}(t)) \leqq y(t) \leqq (3c\, J_{n-1}(t)),\ t \geqq T'\}, \tag{18}$$

where $T' > T$ is large enough so that for $t \geqq T'$, $g(t) > T$,

$$(J_{n-1}(t))^{-1}P_0(t) \int_{T'}^{t} P_n(r)I_{n-1}(t, r; p_1, \ldots, p_{n-1})\,|f(r)|dr < c/8 \tag{19}$$

and

$$\begin{aligned}(J_{n-1}(t))^{-1}p_0(t) \int_{T'}^{t} p_n(r)I_{n-1}(t, r; p_1, \ldots, p_{n-1})& \\ \cdot F(r, cJ_{n-1}(g(r)))dr &< c/8.\end{aligned} \tag{20}$$

Notice that (19) and (20) are possible in view of (16) and (17) for a sufficiently large T. We now define an operator $\phi: X \to D$ as

$$\phi y(t) = cJ_{n-1}(t),\ t \leqq T' \tag{21}$$

and

$$\begin{aligned}\phi y(t) &= c\, J_{n-1}(t) \\ &+ (p_0(t)) \int_{T'}^{t} p_n(r)I_{n-1}(t, r; p_1, \ldots, p_{n-1})[f(r) - F(r, y(g(r)))]dr,\ t \geqq T'.\end{aligned} \tag{22}$$

We shall show that ϕ is continuous and $\phi X \subset X$. To prove continuity of ϕ, we choose a sequence $\{y_m(t)\}$ of functions from X converging to $y \in X$ as $m \to \infty$. We only need to consider the case when $t \geqq T'$ since for $t \leqq T'$ the conclusion is obvious. Now

$$\begin{aligned}&|\phi y_m(t) - \phi y(t)| \\ &\leqq p_0(t) \int_{T'}^{t} p_n(r) \cdot I_{n-1}(t, r; p_1, \ldots, p_{n-1})\,|F(r, y_m(g(r))) - F(r, y(g(r)))|dr,\end{aligned} \tag{23}$$

which yields

$$\begin{aligned}&|\phi y_m(t) - \phi y(t)|(J_{n-1}(t))^{-1} \\ &\leqq p_0(t)(J_{n-1}(t))^{-1} \int_{T}^{t} p_n(r)I_{n-1}(t, r; p_1, \ldots, p_{n-1})G_m(r)dr,\end{aligned}$$

where $G_m(r) = [F(r, y_m(g(r))) - F(r, y(g(r)))]$. Since

$$|G_m(r)| \leqq 2F(r, CJ_{n-1}(g(r)))$$

and in view of (17), $G_m(r) \to 0$ as $m \to \infty$ for $r \geqq T'$, by Lebesgue dominated convergence theorem, we have $\phi y_m(t) \to \phi y(t)$ as $m \to \infty$ (in the topology induced on D earlier). Hence $\phi: X \to D$ is continuous. Now we will show that $\phi X \subset X$. From equation (22), in view of (19) and (20), it is obvious that

$$\phi y(t) \geqq J_{n-1}(t)[c - c/8 - c/8] \geqq (c/2)J_{n-1}(t),$$

and

$$\begin{aligned}\phi y(t) &\leqq cJ_{n-1}(t) \\ &\quad + J_{n-1}(t)\Big[(J_{n-1}(t))^{-1}p_0(t)\int_{T'}^{t} p_n(r)I_{n-1}(t, r; p_1, \ldots, p_{n-1}(r)) \\ &\qquad \cdot(|f(r)| + F(r, y(g(r)))|)dr\Big] \\ &\leqq J_{n-1}(t)(c + c/8 + 3c/8) \leqq 3c\, J_{n-1}(t)\end{aligned}$$

in view of (19) and (20) and the fact that $t \geqq T'$. Hence $\phi X \subset X$. Next we shall show that ϕX is precompact. Differentiating (21) and (22) we get

$$\left|\frac{d}{dt}\,\phi(y(t))/p_0(t))\right| = c\cdot\left|\frac{d}{dt}\,J_{n-1}(t)/p_0(t))\right|$$

for $t \leqq T'$ and

$$\begin{aligned}\left|\frac{d}{dt}(\phi y(t)/p_0(t))\right| &\leqq \left|c\cdot\frac{d}{dt}\left(\frac{J_{n-1}(t)}{p_0(t)}\right)\right| \\ &\quad + p_1(t)\int_{T'}^{t} p_n(r)\cdot I_{n-2}(t.\, r; p_2, \ldots, p_{n-1}) \\ &\quad \cdot|f(r) - F(r, y(g(r)))|dr, \text{ for } t \geqq T'.\end{aligned} \tag{24}$$

Since $y(g(r))) \leqq C\, J_{n-1}(g(r))$, (24) reveals that the family of functions $\{(d/dt)(\phi y(t))/p_0(t)) : y(t) \in X\}$ is uniformly bounded on any finite subinterval of $[T, \infty)$. Thus the family $\{\phi y(t)/p_0(t) : y \in X\}$ is equicontinuous at each point of $[T, \infty)$. Now for any points $t_1, t_2 \in [T, \infty)$ we have

$$\phi y(t_2) - \phi y(t_1) = (p_0(t_2) - p_0(t_1))\frac{\phi y(t_2)}{p_0(t_2)} + p_0(t_1)\left(\frac{\phi y(t_2)}{p_0(t_2)} - \frac{\phi y(t_1)}{p_0(t_1)}\right)$$

which yields

$$|\phi y(t_2) - \phi y(t_1)| \leqq |p_0(t_2) - p_0(t_1)|\left|\frac{\phi y(t_2)}{p_0(t_2)}\right| + p_0(t_1)\left|\frac{\phi y(t_2)}{p_0(t_2)} - \frac{\phi y(t_1)}{p_0(t_1)}\right|. \tag{25}$$

From (25) we obtain that the family $\{\phi y : y \in X\}$ is equicontinuous and uniformly bounded at each point of $[T, \infty)$. We conclude that ϕX is precompact. By the Schauder-Tychonoff theorem, ϕ has a fixed point $y(t)$ in X which is obviously the nonoscillatory solution of (4) satisfying

$$\limsup_{t\to\infty}\left(\frac{y(t)}{J_{n-1}(t)}\right) = \lambda < \infty,\ c/2 \leqq \lambda \leqq 3c. \tag{26}$$

This completes the proof of Theorem 1.

REMARK 1. In relation to equation (3), we have improved and extended sufficiency conditions of Onose [3] and Singh [6, 7] for nonoscillation. In fact we have the following corollary.

COROLLARY 1. *Suppose that the limits*

$$\lim_{t\to\infty}\left(t^{1-n}\int_T^t (t-\mu)^{n-1}(g(\mu))^{n-1}\,|q(\mu)|d\mu\right) < \infty \tag{27}$$

and

$$\lim_{t\to\infty}\left(t^{1-n}\int_T^t (t-\mu)^{n-1}|f(\mu)|d\mu\right) < \infty \tag{28}$$

exist for each $T \geqq a$. *Then* (3) *has a proper nonoscillatory solution* $y(t)$ *which satisfies* $y(t) = O(t^{n-1})$.

EXAMPLE 1. Consider the equation

$$y''(t) + \frac{1}{t^2}y(\sqrt{t}) = \frac{6}{t^4} + \frac{1}{t^2} + \frac{1}{t^3},\ t > 0. \tag{29}$$

It is easily verified that conditions (27) and (28) hold even though (1) fails. This equation has $y(t) = 1 + 1/t^2$ as a nonoscillatory solution satisfying the conclusion of Corollary 1.

EXAMPLE 2. For the equation

$$y''(t) + \frac{1}{t^2}y(\sqrt{t}) = \frac{1}{t^{3/2}},\ t > 0, \tag{30}$$

conditions (1) and (2) fail but (27) and (28) are easily verified. This equation has $y(t) = t$ as a solution satisfying the conclusion of Corollary 1.

REMARK 2. We would like to point out that whereas conditions (27) and (28) suffice for equation (3) to have a nonoscillatory solution which is asymptotic to (t^{n-1}), conditions (1) and (2) guarantee the existence of a nonoscillatory solution with any finite limit at ∞.

We have the following partial converse of Theorem 1.

THEOREM 2. *Suppose* (i)–(iii) *hold. Further suppose that for each* $T \geqq a$, *condition* (16) *of Theorem* 1 *holds. Let* $y(t)$ *be a proper nonoscillatory solution of equation* (4) *satisfying*

$$\lim_{t\to\infty} (|y(t)|/J_{n-1}(t)) \leqq \beta \tag{31}$$

for some $0 < \beta < \infty$. *Then there exists a* δ, $0 < \delta < 1$ *such that*

$$\limsup_{t\to\infty}\Big(p_0(t)(J_{n-1}(t))^{-1}\int_T^t I_{n-1}(t, r; p_1, \ldots, p_{n-1})\cdot p_n(r) \cdot F(r, \delta\beta J_{n-1}(g(r)))dr\Big) < \infty \tag{32}$$

for each $T \geqq a$.

PROOF. Without any loss of generality, suppose there exists a $T > a$ large enough so that $y(t) > 0$, $y(g(t)) > 0$, for $t \geqq T$ and (16) holds. Choose $T' > T$ large enough so that $g(t) \geqq T$ and

$$(y(t)/J_{n-1}(g(t))) \geqq \delta\beta \tag{33}$$

for $t \geqq T'$. From the condition on F, we have

$$F(t, y(g(t))) \geqq F(t, \delta\beta J_{n-1}(g(t))) \tag{34}$$

for $t \geqq T'$. Integrating equation (4), for $t \geqq T'$ we have

$$\begin{aligned}\frac{y(t)}{p_0(t)} &= \sum_{i=0}^{n-1} L_i(T)I_i(t, T; p_1, \ldots, p_i)\\ &\quad + \int_T^t I_{n-1}(t, r; p_1, \ldots, p_{n-1})p_n(r)f(r)dr\\ &\quad - \int_T^t I_{n-1}(t, r; p_1, \ldots, p_{n-1})\cdot p_n(r)F(r, y(g(r)))dr,\end{aligned} \tag{35}$$

where $L_i(T)$, $0 \leqq i \leqq n-1$ are constants as defined in (13) and (14). Now (9) implies

$$\lim_{t\to\infty} \frac{I_i(t, T; p_1, \ldots, p_i)}{I_{n-1}(t, T; p_1, \ldots, p_{n-1})} = 0,$$

for $0 \leqq i \leqq n-2$. Using (33), (34) in (35) we have

$$\limsup_{t\to\infty}\Big[p_0(t)\cdot(J_{n-1}(t))^{-1}\cdot\int_T^t I_{n-1}(t, r; p_1, \ldots, p_{n-1}) \cdot p_n(r)F(r, \delta\beta J_{n-1}(g(t)))dr\Big] < \infty \tag{36}$$

which concludes the proof of Theorem 2.

3. The Equation $L_n y(t) + F(r, y(g(t))) = 0$. In view of Theorem 1 and Theorem 2, we have the following theorem which gives a necessary condition for all proper solutions of the equation

$$L_n y(t) + F(t, y(g(t))) = 0 \tag{37}$$

to be oscillatory.

THEOREM 3. *Suppose* (i)–(iii) *hold. Then a necessary condition for all proper solutions of equation* (37) *to be oscillatory is that the limit*

$$\lim_{t\to\infty}\Big[p_0(t)\,(J_{n-1}(t))^{-1}\int_T^t I_{n-1}(t, r; p_1, \ldots, p_{n-1}) \cdot p_n(r)\, F(r, cJ_{n-1}(g(r)))dr\Big] = \infty \tag{38}$$

exists for some $c > 0$.

REMARK 3. Recently Yeh [13] has shown that all proper solutions of the equation

$$y''(t) + p(t)f_1(y(t), y(g(t))) = 0, p(t) \geqq 0 \tag{39}$$

are oscillatory if

$$\limsup_{t\to\infty}\Big(t^{1-n}\int_T^t (t - \mu)^{n-1}p(\mu)d\mu\Big) = \infty \tag{40}$$

for some $n > 2$. The conditions imposed upon p, g and f_1 in Yeh [13] are more severe but compatible with ours. Yeh's Theorem 1 in [13] indicates that if the left hand side in (40) is finite for some $n > 2$ then equation (39) has a nonoscillatory solution. Our condition (27) in Corollary 1 supports this claim under less restrictive conditions. More precisely we have the following theorem in regard to equation (39).

THEOREM 4. *Suppose $p, g \in C[a, \infty)$, $f_1 \in C(R \times R)$, $f_1(u, v)$ has the sign of u and v when they have the same sign, and the limit*

$$\lim_{t\to\infty}\Big(t^{-1}\int_T^t (t - \mu)g(\mu)p(\mu)d\mu\Big) < \infty \tag{41}$$

exists for each $T \geqq a$. Further suppose that

$$\liminf_{v\to\infty}\left|\frac{f_1(u, v)}{v}\right| \geqq C \tag{42}$$

for some constant $C > 0$. Then equation (39) *has a nonoscillatory solution.*

PROOF. This follows in the manner of Theorem 1.

Our next theorem gives a necessary and sufficient condition for the oscillation of all bounded solutions of equation (39) and significantly strengthen's Yeh's criterion for bounded solutions of (39).

THEOREM 5. *Suppose all conditions except* (41) *of Theorem* 4 *hold. Further suppose $g'(t) \geqq 0$ and $g''(t) \leqq 0$ for large t. Then*

$$\lim_{t\to\infty}\Big(t^{-1}\int_T^t (t - \mu)p(\mu)g(\mu)d\mu\Big) = \infty. \tag{43}$$

is necessary and sufficient for all proper bounded solutions of equation (39) *to be oscillatory.*

PROOF. In view of Theorem 4, all we need to show is that (43) is a sufficient condition to cause all proper bounded solutions of (39) to oscillate.

Suppose to the contrary that (39) has a proper nonoscillatory and bounded solution $y(t)$. Without any loss of generality suppose there exists a constant $T \geqq a$ such that $y(t) > 0$ and $y(g(t)) > 0$ for $t \geqq T$. From equation (39) we have $y(t) > 0$, $y'(t) > 0$ and $y''(t) \leqq 0$ for $t \geqq T_1 \geqq T$. Integrating (39) we have

$$(44) \quad y'(t)g(t) - y'(T_1)g(T_1) - \int_{T_1}^{t} y'(s)g'(s)ds + C\int_{T_1}^{t} g(s)y(g(s)p(s)ds \leqq 0.$$

Since $y'(t)g(t) > 0$ and $y(g(t))$ is nondecreasing, (44) yields

$$(45) \quad \begin{aligned} &- y'(T_1)g(T_1) - y(t)g'(t) + y(T_1)g'(T_1) \\ &+ \int_{T_1}^{t} y(s)g''(s)dt + Cy(g(T_1)) \int_{T_1}^{t} g(s)p(s)ds \leqq 0 \end{aligned}$$

which gives

$$(46) \qquad P_0 - y(t)g'(t) + P_1 \int_{T_1}^{t} g(s)p(s)ds \geqq 0,$$

where $P_0 = y(T_1)g'(T_1) - y'(T_1)g(T_1) + \int_{T_1}^{\infty} y(t)g''(t)dt$, $0 < P_1 = C\, y(g(T_1))$. Notice that $-\int_{T_1}^{\infty} y(t)g''(t)dt < \infty$ since $g'(t) \geqq 0$, $g''(t) \leqq 0$ and $y(t)$ is bounded for $t \geqq T_1$. Integrating (46) and dividing by t we have

$$(47) \quad P_0(t-T_1)/t - \frac{1}{t}\int_{T_1}^{t} y(\mu)g'(\mu)d\mu + P_1\Big(\frac{1}{t}\int_{T_1}^{t}(t-\mu)g(\mu)p(\mu)d\mu\Big) \leqq 0.$$

Now let $y(t) \leqq P_2$ for $t \geqq T_1$. Since $g(t) > 0$, $g'(t) \geqq 0$ and $g''(t) \leqq 0$ for $t \geqq T_1$, we have $g(t)/t$ eventually bounded. Since (43) holds, a contradiction is immediately apparent in (47), and the proof is complete.

ACKNOWLEDGEMENT. The author is thankful to the referee for some valuable suggestions.

REFERENCES

1. Kusano and Naito, *Boundedness of solutions of a class of higher order ordinary differential equations*, J. Differential Equations, (to appear).

2. D.L. Lovelady, *On the oscillatory behavior of bounded solutions of higher order differential equations*, J. Differential Equations, **19** (1975), 167–175.

3. H. Onose, *Oscillatory properties of ordinary differential equations of arbitrary order*, J. Differential Equations, **7** (1970), 454–458.

4. Ch. G. Philos, *On the oscillatory and asymptotic behavior of the bounded solutions of differential equations with deviating arguments*, Ann. Mat. Pura Appl., **119** (1979), 25–40.

5. ——— and V.A. Staikos, *Boundedness and oscillation of solutions of differential equations with deviating argument*, Tech. Rep. Univ. Ioannina No. 37, 1980.

6. Bhagat Singh, *Necessary and sufficient condition for maintaining oscillations and nonoscillations in general functional equations and their asymptotic properties*, SIAM J. Math. Anal. **10** (1979), 18–31.

7. ———, *A necessary and sufficient sondition for the oscillation of an even order nonlinear delay differential equation*, Canad. J. Math. **25** (1973), 1078–1089.

8. ——— and T. Kusano, *On asymptotic limits of nonoscillations in functional equations with retarded arguments*, Hiroshima Math. J. **10** (1980), 557–565.

9. ———, *Asymptotic behavior of oscillatory solutions of a differential euqation with deviating arguments*, J. Math. Anal. Appl. **83** (1981), 395–407.

10. W.F. Trench, *Canonical forms and principal systems for general disconjugate equations*, Trans. Amer. Math. Soc. **189** (1974), 319–327.

11. ———, *Oscillation properties of perturbed disconjugate equations*, Proc. Amer. Math. Soc. **52** (1975), 147–155.

12. D. Willett, *Asymptotic behavior of disconjugate nth order differential equations*, Canad. J. Math. **23** (1977), 293–314.

13. Cheh C. Yeh, *An oscillation criterion for second order nonlinear differential equations with functional arguments*, J. Math. Anal. Appl. **76** (1980), 72–75.

Department of Mathematics, University of Wisconsin Center, 705 Viebahn Street, Manitowoc, WI 54220