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NONLINEAR BOUNDARY VALUE PROBLEMS FOR SECOND ORDER ELLIPTIC SYSTEMS

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1. Introduction. In [3] P. Habets and K. Schmitt, [4] H. W. Knobloch and K. Schmitt presented a unifying theory for existence of solutions of boundary value problems for systems of ordinary differential equations of the form

$$x'' = f(t, x, x'). \tag{1.1}$$

In this article we shall show that by using the same arguments as in [3], [4] the major results proved there hold for boundary value problems for elliptic systems of second order:

$$\mathscr{L}u_r = f_r(x, u, \partial u),\ r = 1, 2, \ldots, N,\ x \in \Omega, \tag{1.2}$$

$$B_r u_r(x) = \phi_r(x),\ x \in \partial\Omega, \tag{1.3}$$

where

$$\mathscr{L}u = -\sum_{i,j=1}^{m} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j},$$

$$a_{ij} \in C^{0,\alpha}(\bar{\Omega}),\ 0 < \alpha < 1,$$

Ω is a bounded domain with $C^{2,\alpha}$ boundary,

$$0 < \frac{1}{M}|\xi|^2 \leqq \sum_{i,j=1}^{m} a_{ij}(x)\xi_i\xi_j \leqq M|\xi|^2 \tag{1.4}$$

for all $\xi \in R^m$, $\xi \not\equiv 0$ and all $x \in \bar{\Omega}$, $B_r u = u$ or $B_r u = p_r u + q_r\, \partial u/\partial\nu$, $p_r q_r \in C^{0,\alpha}(\partial\Omega)$, $p_r > 0$, $q_r > 0$, (ν is the unit outward normal). In order to generalize results in [3], [4] we need an apriori estimate, which will be proved in §2. In §3 we prove an existence result for systems of elliptic boundary value problems.

2. An Apriori Estimate for Solutions Of Coupled Elliptic Systems.

Assumptions. Let Ω be a bounded domain in R^m with C^2 boundary $\partial\Omega$, define

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$$(2.1)\qquad (\mathscr{L}_r u)(x) = -\sum_{i,j=1}^{m} a_{ij}^r \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b_i^r(x) \frac{\partial u}{\partial x_i} + c_r(x)u$$

for all $u \in W^{2,2}(\Omega)$, where $a_{ij}^r \in C(\bar{\Omega})$, b_i^r, $c_r \in L^\infty(\Omega)$,

$$(2.2)\qquad 0 \leqq \frac{1}{M}|\xi|^2 \leqq \sum_{i,j=1}^{m} a_{ij}^r(x)\xi_i\xi_j \leqq M|\xi|^2$$

for all $x \in \bar{\Omega}$, $\xi = (\xi_1, \ldots, \xi_m) \in R^m$, $r = 1, 2, \ldots N$. Let $f\colon \Omega \times R^N \times R^{mN} \to R^N$ satisfy Carathéodory conditions ($f(x, \ldots)$ is continuous for almost all $x \in \Omega$, and $f(\cdot, u, p)$ is measurable for all $u \in R^N$, $p \in R^{mN}$) and let the following Nagumo condition hold:

For every Positive number U there exists a continuous, nondecreasing function $\psi_U\colon [0, \infty) \to (0, \infty)$ such that

$$(2.3)\qquad \lim_{s\to\infty} \frac{s^2}{\psi_U(s)} = \infty,$$

$$(2.4)\qquad |f(x, u, p)| \leqq \psi_U(|p|) \text{ for all } x \in \bar{\Omega},\ |u| < U,\ u \in R^m, p \in R^{mN}.$$

Let $B_r u_r = u_r$ or $B_r u_r = p_r(x)u_r + q_r(x)\partial u_r/\partial\nu$, where p_r, $q_r \in C^{0,\alpha}(\partial\Omega)$, $p_r(x) > 0$, $q_r(x) > 0$, $x \in \partial\Omega$.

LEMMA 2.1. *Let $\mathscr{L}$, f, B, Ω satisfy all assumptions above. Then the following holds. For every constant $P > 0$ there exists a constant Q such that: If $u \in W^{2,p}(\Omega)$, $p \geqq 3(m-1)$, $m \geqq 2$, is a solution of*

$$(2.5)\qquad (\mathscr{L}u)(x) = f(x, u, \partial u) \text{ a.e. in } \Omega$$

$$(2.6)\qquad Bu = 0,\ x \in \partial\Omega,\ |u(x)| \leqq P,\ x \in \bar{\Omega},$$

then $\quad |\partial u(x)| \leqq Q$ *for all* $x \in \bar{\Omega}$, *where*

$$(2.7)\qquad Q^2 \leqq C(\psi_p(Q) + 1).$$

The constant C depends on P, the bounding function ψ_p, the constant M from (1.2), *the modulus of continuity of a_{ij}^r, the norms $\|b_i^r\|_\infty$, $\|c_r\|_\infty$, the boundary $\partial\Omega$, which is assumed to be of class C^2 and* meas Ω.

PROOF. Let $u \in W^{2,p}(\Omega)$, $p \geqq 3(m-1)$ be a solution of (2.5) and (2.6), $|u(x)| \leqq p$ for all $x \in \Omega$. Then one can apply the inequality (11.8), page 193, [5], or the continuity of the operator $T\colon L^p(\Omega) \to W^{2,p}(\Omega)$ in the case $B_r u = p_r(x)u + q_r(x)\partial u/\partial\nu$ (see [1]) in order to obtain

$$(2.8)\qquad \begin{aligned} \|u\|_{2,p} = {} & C_p(\|f(x, u, \partial u)\|_p \\ & + P(\text{meas } \Omega)^{1/p}),\ p \geqq 3(m-1) (\text{note } u \in C^1(\bar{\Omega})), \end{aligned}$$

where C_p depends on M, the modulus of continuity of a_{ij}^r, the norms $\|b_i^r\|_\infty$, $\|c_r\|_\infty$ and the boundary $\partial\Omega$. Let $s_u = \|\partial u\|_\infty$, then

(2.9) $\|u\|_{2,p} \leqq d_p(\psi_p(s_u) + 1),\ d_p = 2C_p(P + 1)(\text{meas } \Omega)^{1/p}.$

Now we shall modify the proof from [9], where it was proved for ordinary differential equations.

First we have to prove an interior estimate for any subregion Ω' of Ω such that dist $(\partial\Omega', \partial\Omega) = \delta > 0$.

Let

$$K^t_{\pi/3}(v) = \{stw\colon w \in S^1_{\pi/3}(v),\ 0 \leqq s \leqq 1\},$$

where

$$S^t_{\pi/3}(v) = \{w\colon w \in R^m,\ |w| = t,\ w\ v \geqq (1/2)t|v|\},$$

v being a fixed nonzero vector in R^m. Note that $\text{meas}_{m-1} S^t_{\pi/3}(v)$ does not depend on v, $v \neq 0$.

Let s_0 be chosen in such a way that

$$\frac{(\text{meas}_{m-1} S^\delta_{\pi/3})^{1/3(m-1)p}}{d_{3(m-1)}(\psi_p(s_0) + 1)} < \delta^2$$

(assume $\psi_p(s)_{s\to\infty} \to \infty$, otherwise the assertion of the lemma is trivial). Pick a point $x_0 \in \bar{\Omega}'$ with $|\nabla u_r(x_0)| \neq 0$, put $v_r = \nabla u_r(x_0)/|\nabla u_r(x_0)|$, $\bar{\phi}(s) = u_r(x_0 + stv_r)$ and apply Taylor's Theorem in order to get

$$u_r(x_0 + tv_r) = u_r(x_0) + t\nabla u_r(x_0)\cdot v_r + t^2 \sum_{i,j=1}^m \int_0^1 (t - s) \frac{\partial u_r}{\partial x_i \partial x_j} (x_0 + stv_r)v_{ri}v_{rj}ds.$$

If one replaces tv_r by an arbitrary $w_r \in S^t_{\pi/3}(v_r)$, one can obtain

$$\frac{1}{2}|\nabla u_r(x_0)| \cdot t \leqq 2P + t^2 \sum_{i,j=1}^m \int_0^1 \left|\frac{\partial^2 u_r(x_0 + sw_r)}{\partial x_i x \partial_j}\right| ds.$$

Integrate over $S^t_{\pi/3}(v_r)$, then

$$|\nabla u_r(x_0)| \leqq \frac{4P}{t} + \frac{4t}{(\text{meas}_{m-1} S^t_{\pi/3})^{1/3}} \left(\int_0^1 \int_{S^t_{\pi/3}} s \cdot \left|\frac{\partial^2 u_r(x_0 + sw_r)}{\partial x_i \partial x_j}\right|^3 ds\, d_{w_r} S\right)^{1/3}.$$

Using the transformation of the coordinates:

$$x_0 + sw_r \to x_0 + s\,\omega_r,\ w_r = (t/\delta)\omega_r,\ d_{w_r} S = (t/\delta)^{m-1} d_{\omega_r} F,$$
$$\text{meas}_{m-1} S^t_{\pi/3}(v_r) = (t/\delta)^{m-1} \text{meas}_{m-1} S^\delta_{\pi/3}(v_r)$$

one gets

$$|\nabla u_r(x_0)| = \frac{4P}{t} \tag{2.10}$$
$$+ \frac{4t}{(\text{meas}_{m-1} S^\delta_{\pi/3})^{1/3(m-1)}} \left(\int_0^1 \int_{S^\delta_{\pi/3}} s^{m-1} \left|\frac{\partial^2 u_r(x_0 + s\,\omega_r)}{\partial x_i \partial x_j}\right|^{3(m-1)} ds\, d\omega_r S\right)^{1/3(m-1)},$$

or

$$(2.11)\qquad |\nabla u_r(x_0)| \leqq \frac{4P}{t} + \frac{4t}{(\text{meas}_{m-1} S^{\delta}_{\pi/3})^{1/3(m-1)}} \cdot d_{3(m-1)} \cdot (\psi_p(s_u) + 1).$$

Note that (2.10) holds for any $u \in C^2(\bar{\Omega})$ and therefore also for any $u \in W^{2,p}(\Omega)$, $p \geqq 3(m-1)$.

The right hand side in (2.11) takes on a local minimum for

$$t^2 = \frac{P \cdot (\text{meas}_{m-1} S^{\delta}_{\pi/3})^{1/3(m-1)}}{d_{3(m-1)} (\psi_p(s_u) + 1)}.$$

Then either $s_u \leqq s_0$, or $t^2 < \delta^2$, since ψ_p is nondecreasing and

$$|\nabla u_r(x_0)|^2 \leqq 64\, P\, d_{3(m-1)}(\psi_p(s_u) + 1) \cdot (\text{meas } S^{\delta}_{\pi/3})^{-1/3(m-1)},$$

i.e.,

$$(2.12)\qquad \begin{aligned} &\max_{x_0 \in \bar{\Omega}'} |\nabla u(x_0)|^2 \\ &\leqq 2N^2(s_0^2 + 64 P d_{3(m-1)}(\psi_p(s_u) + 1)(\text{meas}_{m-1} S^{\delta}_{\pi/3})^{-1/3(m-1)}). \end{aligned}$$

Now we need an estimate near the boundary $\partial\Omega$. Let us take $x_0 \in \partial\Omega$ an arbitrary point and assume that $x_0 = (0, 0, \ldots 0)$, $x_0 \in \mathcal{O}$, $\mathcal{O} = \{x \in R^m: |x_i| \leqq \gamma_1, 0 \leqq x_m \leqq \gamma_2, i = 1, 2, \ldots n - 1\}$, $\mathcal{O} \in \bar{\Omega}$. Otherwise we take a neighborhood U of x_0 and a C^2-function h, $x_i = h(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_m)$ and we transform the entire region in such a way that in new coordinates $y_1, \ldots y_m$, $y_m = 0$ describes the boundary in small neighborhood $\mathcal{O}$ of $x_0 \in \partial\Omega$. In the new coordinates $y_1, \ldots y_m$ our equations will have the same form and the same properties as the original system, provided the functions $y_i = y_i(x)$, $i = 1, 2, \ldots m$ have bounded first and second derivatives; but this is satisfied locally for each point $x_0 \in \partial\Omega$ in our case (for details see [5]).

Let

$$(2.13)\qquad \mathcal{O}_1 = \{x \in R^m: |x_i| \leqq \gamma_1/2, 0 \leqq x_m \leqq \gamma_2/2, j = 1, 2, \ldots m-1\},$$

then $\mathcal{O}_1$ is a neighborhood of x_0 relative to $\bar{\Omega}$. For any $y \in \mathcal{O}$, with $|\nabla u_r(y)| \neq 0$, either $y - tv_r$ or $y + tv_r$ does not intersect the hyperplane $x_m = (1/2)y_m$ for all $t > 0$, where $v_r = \nabla u_r(y)/|\nabla u_r(y)|$.

Suppose the former and define:

$$\begin{aligned} S^+(y) &= \{z \in R^M, z_m \geqq (1/2)y_m\}, \\ K^t_{\pi/3}(y, v) &= \{y + stw: w \in S^1_{\pi/3}(y, v), 0 \leqq s \leqq 1\}, \\ S^t_{\pi/3}(y, v) &= \{y + w: w \in S^t_{\pi/3}(v)\}, \\ {}^+K^t_{\pi/3}(y, v) &= K^t_{\pi/3}(y, v) \cap S^+(y), \\ {}^+S^t_{\pi/3}(y, v) &= S^t_{\pi/3}(y, v) \cap S^+(y). \end{aligned}$$

Then ${}^{+}K^{t}_{\pi/3}(y, v_r) \subset \mathcal{O}$ for all $t \in (0, \delta)$, $y \in \mathcal{O}_1$, with $|\nabla u_r(y)| \neq 0$, $v_r = \nabla u_r(y)/|\nabla u_r(y)|$, and $\text{meas}_{m-1}\, {}^{+}S^{t}_{\pi/3}(y, v_r) \geqq (1/2)$ meas $S^{t}_{\pi/3}(v_r)$, since the axis of symmetry of $S^{t}_{\pi/3}(y, v_r)$ is $y + tv_r \in S^{+}(y)$ for all $t > 0$. Hence one may obtain similarly as for an interior estimate by using Taylor's Theorem on ${}^{+}K^{t}_{\pi/3}(y, v_r)$ that:

$$|\nabla u_r(y)|^2$$
$$\leqq 2N^2\{s_0^2 + 64\, P\, d_{3(m-1)} \cdot [\psi_p(s_u) + 1] \cdot [\text{meas}_{m-1}\, {}^{+}S^{\delta}_{\pi/3}(y, v_r(y))]^{-1/3(m-1)}\}$$

where $v_r(y) = (1, 0, \ldots 0)$ for $\nabla u_r(y) = 0$ and $v_r(y) = \nabla u_r(y)/|\nabla u_r(y)|$ otherwise, and $\text{meas}_{m-1}\, {}^{+}S^{\delta}_{\pi/3}(y, v_r(y)) \geqq (1/2)\, \text{meas}_{m-1}\, S^{\delta}_{\pi/3}(v_r(y))$, where $\text{meas}_{m-1}\, S^{\delta}_{\pi/3}(v_r(y))$ is independent of $v_r(y)$.

Now if we combine both kinds of estimates together with the compactness of $\bar{\Omega}$, we may conclude

$$s_u^2 \leqq C(\psi_p(s_u) + 1), \tag{2.14}$$

where C depends only on these quantities: M from (1.2), the modulus of continuity of the a^r_{ij}, $\|b^r_i\|_\infty$, $\|c_r\|_\infty$, meas Ω, $\partial\Omega$. Hence there exists a constant $Q > 0$ such that: $s_u \leqq Q < \infty$ for any solution u of

$$(\mathcal{L}u)(x) = f(x, u, \partial u),\ x \in \Omega,$$
$$(Bu)(x) = \phi(x),\ x \in \partial\Omega,$$
$$\|u\|_\infty \leqq P,$$

since $\lim_{s\to\infty} s^2/\psi_p(s) = \infty$. It is clear that Q can be chosen in such a way that (2.14) holds for Q instead of s_u.

REMARK. The last lemma is in its various forms due to Bernstein **[2]**, Nagumo **[7, 9]**, Tomi **[12]**, Schmitt and Thompson **[9]**, Sindler **[10]**, for a detailed discussion, see references **[5, 6]**.

REMARK 2.2. Q-estimate (2.7) for partial differential systems is new (case of ordinary differential systems is in **[9]**) and will be needed in §3.

REMARK 2.3. Let $\phi \in C^{i,\alpha}(\partial\Omega)$, $i = 1$ or 2 depending on the form of B. Then one can assume $Bu = \phi$, $x \in \partial\Omega$ in (2.6) and Lemma 2.1 stays true.

3. Nonlinear Boundary Value Problems for Systems of Second Order Elliptic Equations.

ASSUMPTIONS. Let Ω be a bounded domain in R^m with $C^{2,\alpha}$ boundary $\partial\Omega$, define

$$\mathcal{L}u = -\sum_{i,j=1}^{m} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} \tag{3.1}$$

for all $u \in C^2(\bar{\Omega})$, where $a_{ij} \in C^{\gamma}(\bar{\Omega})$ and

$$0 \leqq \frac{1}{M}|\xi|^2 \leqq \sum_{i,j=1}^{m} a_{ij}(x)\, \xi_i \xi_j \leqq M|\xi|^2, \tag{3.2}$$

for all $x \in \bar{\Omega}$, all $\xi \in R^m$.

Let $f\colon \bar{\Omega} \times R^N \times R^{mN} \to R^N$ be locally γ-Holder continuous satisfying the Nagumo condition: For every bounded set $U \subset R^N$ there exists a nondecreasing, continuous function ψ_U such that

$$\lim_{s\to\infty} \frac{s^2}{\psi_U(s)} = \infty,$$

$$|f(x, u, p)| \leqq \psi_U(|p|),\ x \in \bar{\Omega},\ u \in U,\ p \in R^{mN}. \tag{3.3}$$

Let $B_r u = u$ or $B_r u = p_r(x)u + q_r(x)\partial u/\partial \nu$ for each $r = 1, 2, \ldots N$, where $p_r, q_r \in C^{0,\alpha}(\partial\Omega)$, $p_r > 0$, $q_r > 0$.

LEMMA 3.1. *Let E be a real Banach space and let $\mathcal{O}$ be a bounded neighborhood of $0 \in E$. Let $H\colon \bar{\mathcal{O}} \times [0, 1] \to E$ be a completely continuous operator such that for all $\lambda \in [0, 1]$ and $u \in \partial\mathcal{O}$, $u \neq H(\lambda, u)$. Then $d_{LS}(H(\cdot, 0), \mathcal{O}, 0) = d_{LS}(H(\cdot, 1), \mathcal{O}, 0)$.*

PROOF. See [11].

THEOREM 3.2. *Let Ω be a bounded domain in R^m with $C^{2,\alpha}$ boundary, $\mathcal{L}$, f satisfy all assumptions from §3. $\phi_r \in C^{i,\alpha}(\partial\Omega)$ ($i = 1$ or 2 depending on the form of B_r) $r = 1, 2, \ldots N$, $g\colon \bar{\Omega} \to R^N$, $g \in C^{2,\alpha}(\bar{\Omega})$ and let Σ be a bounded, open subset of $\bar{\Omega} \times R^N$ such that*

$$B_r g_r(x) = \phi_r(x),\ x \in \partial\Omega,\ r = 1, 2, \ldots N, \tag{3.4}$$

$$g(x) \in \Sigma_x = \{u\colon (x, u) \in \Sigma\},\ x \in \bar{\Omega}. \tag{3.5}$$

Furthermore assume that for every $(x_0, u_0) \in \partial\Sigma$ there exists a twice differentiable function $r\colon \bar{U} \to R$, where U is some neighborhood of (x_0, u_0) in R^{m+N}, and constants $\gamma_1 > 0$, $\gamma_2 > 0$ which are such that:

(i) $$\bar{\Sigma} \cap U \subseteqq \{(x, u)\colon r(x, u) \leqq 0\},\ r(x_0, u_0) = 0,$$

(ii) $$\frac{\partial r}{\partial u}(x_0, u_0) \cdot (u_0 - g(x_0)) \geqq \gamma_1 > 0,$$

(iii) $$\left|\frac{\partial^2 r}{\partial x^2}(x_0, u_0)\right|, \left|\frac{\partial^2 r(x_0, u_0)}{\partial x \partial u}\right|, \left|\frac{\partial^2 r(x_0, u_0)}{\partial u^2}\right|, \left|\frac{\partial r}{\partial u}(x_0, u_0)\cdot \mathcal{L}g(x_0)\right| \leqq \gamma_2,$$

(iv) $$\sum_{i,j=1}^{m} a_{ij}(x_0)\frac{\partial^2 r}{\partial x_i \partial x_j}(x_0, u_0) + 2\sum_{i,j=1}^{m}\sum_{s=1}^{N} a_{ij}(x_0)\cdot\frac{\partial^2 r}{\partial x_i \partial u_s}(x_0, u_0) y_{sj} + \sum_{i,j=1}^{m}\sum_{s,\ell=1}^{N} a_{ij}(x_0)\cdot\frac{\partial^2 r(x_0, u_0)}{\partial u_s \partial u_\ell} y_{si} y_{\ell j} - \frac{\partial r}{\partial u}(x_0, u_0)\cdot f(x_0, u_0, u) \geqq 0$$

for all $y = (y_1, \ldots y_N)$, $y_i = (y_{i1}, y_{i2}, \ldots y_{im})$, $i = 1, 2, \ldots N$ such that

$$\frac{\partial r}{\partial x_j}(x_0, u_0) + \sum_{\ell=1}^{N} \frac{\partial r(x_0, u_0)}{\partial u_\ell} y_{\ell j} = 0, j = 1, 2, \ldots m.$$

Moreover in the case $B_r u = p_r(x)u + q_r(x)\partial u/\partial \nu$ *and* $x_0 \in \partial\Omega$ *suppose also*

(v) $$\frac{\partial r(x_0, u_0)}{\partial x} \cdot \nu + \sum_{r=1}^{N} \frac{\partial r}{\partial u_r}\left(\frac{1}{q_r}\phi_r(x_0) - \frac{p_r(x_0)}{q_r(x_0)} u_{0r}\right) < 0.$$

Then the boundary value problem

(3.6) $$(\mathscr{L}u)(x) = f(x, u, \partial u), x \in \Omega,$$

(3.7) $$(Bu)(x) = \phi(x), x \in \partial\Omega,$$

has a solution $u \in C^2(\bar{\Omega})$ *such that* $u(x) \in \bar{\Sigma}_x$, $x \in \bar{\Omega}$.

PROOF. For simplicity assume $B_r u_r = p_r u_r + g_r \partial u_r/\partial\nu$(the other case is even simpler). Consider the problem

(3.8) $$(\mathscr{L} u_r)(x) - \lambda f_r(x, u, \partial u) = (1-\lambda)[(\mathscr{L} g_r)(x) - k(u_r - g_r(x))], x \in \Omega,$$

(3.9) $$(B_r u_r)(x) = \phi_r(x), x \in \partial\Omega, r = 1, 2, \ldots N, 0 \leqq \lambda \leqq 1,$$

where $k > 0$ is to be chosen. If u is a solution of (3.8) and (3.9), $u(x) \in \bar{\Sigma}_x$, $x \in \bar{\Omega}$, then

$$|\lambda f_r(x, u, \partial u) + (1 - \lambda)((\mathscr{L} g_r)(x) - k(u_r - g_r(x)))| \leqq \psi_K(|\partial u|) + T,$$

where $K = \max\{|u|: u \in \bar{\Sigma}_x, x \in \bar{\Omega}\}$, and

$$T = \max\{|(\mathscr{L} g_r)(x) - k(u_r - g_r(x))|: |u| \leqq K, x \in \Omega, r = 1, 2, \ldots N\}.$$

Let $\Phi_K(s) = \psi_K(s) + T$, then Φ_K is also nondecreasing and continuous such that $\lim_{s\to\infty} \Phi_K(s)/s^2 = 0$, hence, there exists a constant N_k such that

(3.10) $$\|\partial u\|_\infty \leqq N_k$$

for any solution u of (3.8) and (3.9) with $u(x) \in \bar{\Sigma}_x$, $x \in \bar{\Omega}$.
Let

$$\mathcal{O} = \{u \in C^1(\bar{\Omega}): u(x) \in \Sigma_x, |\partial u(x)| < N_k + 1, x \in \bar{\Omega}\},$$

the $\mathcal{O}$ is a bounded, open neighborhood of $0 \in C^1(\bar{\Omega})$ and (3.8) and (3.9) is equivalent to the operator equation

$$u = \mathscr{L}_k^{-1}(\lambda f(\cdot, u, \partial u) + \lambda k u + (1 - \lambda)[(\mathscr{L} g)(\cdot) + k g(\cdot)])$$

where $\mathscr{L}_k u = \mathscr{L}u + ku$ subject to the boundary conditions (3.9). Since $k > 0$, $\mathscr{L}_k^{-1}$ is a compact, linear operator on $C^1(\bar{\Omega})$. If now there exists $u \in \partial\mathcal{O}$ which is a solution of (3.8) and (3.9) for some $\lambda \in [0, 1)$, then it must be the case that $|\partial u(x)| \leqq N_k < N_k + 1$, $x \in \bar{\Omega}$, $u(x_0) \in \partial\Sigma_{x_0}$, $x_0 \in \bar{\Omega}$. First assume $x_0 \in \partial\Omega$, then there exists a twice differentiable function r on some neighborhood (x_0, u_0) in R^{m+N} such that (i)–(v) hold. Therefore

$\partial r(x_0, u_0)/\partial \nu \geqq 0$ since $r(x, u(x)) \leqq 0$, $|x - x_0| \leqq \varepsilon$, $x \in \bar{\Omega}$, $r(x_0, u_0) = 0$. But

$$\frac{\partial r(x_0, u_0)}{\partial \nu} = \frac{\partial r}{\partial x}(x_0, u_0) \cdot \nu + \frac{\partial r}{\partial u} \cdot \frac{\partial u}{\partial \nu}$$
$$= \frac{\partial r(x_0, u_0)}{\partial x} \cdot \nu + \sum_{r=1}^{N} \frac{\partial r}{\partial u_r}\left(\frac{1}{q_r}\phi_r(x_0) - \frac{p_r}{q_r} u_{0r}\right) < 0,$$

and this is a contradiction. Thus $x_0 \in \Omega$ and there exists a function r, $r(x_0, u_0) = 0$, $r(x, u(x)) \leqq 0$ for $|x - x_0| \leqq \varepsilon$ for some $\varepsilon > 0$. It follows that

$$\frac{\partial r}{\partial x_j}(x_0, u_0) + \sum_{s=1}^{N} \frac{\partial r}{\partial u_s}(x_0, u_0)\frac{\partial u_s(x_0)}{\partial x_j} = 0, \; j = 1, 2, \ldots N, \text{ and} \tag{3.11}$$

$$\begin{aligned} \mathscr{L}(r(x, u(x))\,|_{x=x_0} = &- \sum_{i,j=1}^{m} a_{ij}(x_0)\frac{\partial^2 r(x_0, u_0)}{\partial x_i \partial x_j} \\ &- 2\sum_{i,j=1}^{m}\sum_{s=1}^{N} a_{ij}(x_0)\frac{\partial^2 r}{\partial x_i \partial u_s}\frac{\partial u_s(x_0)}{\partial x_j} \\ &- \sum_{i,j=1}^{m}\sum_{s,\ell=1}^{N} a_{ij}(x_0)\cdot\frac{\partial^2 r(x_0,u_0)}{\partial u_s \partial u_\ell}\frac{\partial u_s(x_0)}{\partial x_i}\frac{\partial u_\ell(x_0)}{\partial x_j} \\ &+ \frac{\partial r}{\partial u}(x_0, u_0)\,(\lambda f(x_0, u_0, u(x_0)) \\ &+ (1 - \lambda)[(\mathscr{L} g)(x_0) - k(u_0 - g(x_0)]). \end{aligned} \tag{3.12}$$

on the other hand

$$\frac{\partial r}{\partial u}(x_0, u_0)(u_0 - g(x_0)) \geqq \gamma_1 > 0, \text{ and} \left|\frac{\partial r}{\partial u}(x_0, u_0)\,\mathscr{L}\,g(x_0)\right| \leqq \gamma_2.$$

We therefore obtain that (3.12) is negative if we can show that

$$k\frac{\partial r}{\partial u}(x_0, u_0)\,(u_0 - g(x_0)) - \frac{\partial r}{\partial u}(x_0, u_0)\,\mathscr{L} g(x_0) + \frac{\partial r}{\partial u}(x_0, u_0) f(x_0, u_0, \partial u(x_0))$$

is positive, it shall be in the case if

$$k\gamma_1 - \gamma_2 - m_0\Phi_k(|\partial u(x_0)|) > 0 \tag{3.13}$$

where $m_0 = \sup|\partial r(x, u)/\partial u|$. It is enough to show

$$\lim_{k\to\infty}\frac{\Phi_k(s_k)}{k} = 0, \text{ where } s_k = N_k, \tag{3.14}$$

see (3.10). Assume for a while that (3.14) is satisfied, then from (iv) and (3.13) one can conclude that (3.12) is negative and this is a contradiction to $r(x, u(x))$ having a local maximum at $x = x_0 \in \Omega$, since $\mathscr{L} r(x, u(x))|_{x=x_0}$

< 0, and $\mathscr{L}$ is a uniformly elliptic operator. We hence conclude that the Leray-Schauder degree

$$d_{LS}(\mathrm{id} - \mathscr{L}_k^{-1}(\lambda f(\cdot, \cdot, \cdot) - \lambda k \cdot -(1 - \lambda) \mathscr{L} g(\cdot) + kg(\cdot)), \mathcal{O}, 0)$$

is independent of λ [0, 1), i.e., it equals $d_{LS}(\mathrm{id} - \mathscr{L}_k^{-1}(\mathscr{L} g(\cdot) + kg(\cdot), \mathcal{O}, 0)$, see lemma 3.1. If on the other hand $u = \mathscr{L}_k^{-1}(\mathscr{L} g(\cdot) + kg(\cdot))$, then $\mathscr{L}u + ku = \mathscr{L}g + kg$, $u(x) = g(x)$, thus $u = g \in \mathcal{O}$. Therefore the above degree equals 1 and (3.8) and (3.9) has a solution $u_\lambda \in \mathcal{O}$ for all $\lambda \in [0, 1)$ and also for $\lambda = 1$, i.e., (3.6) and (3.7) has a solution $u \in \bar{\mathcal{O}}$ completing the proof provided we show that (3.14) holds. From lemma 2.1 one can get $s_k^2 \leqq C(\phi(s_k) + ak + b + 1)$, for all $k = 1, 2, \ldots$. Then either $s_k \leqq D < \infty$, $k = 1, 2, \ldots$, or $\lim\inf_{k\to\infty} k/s_k^2 > 0$, or $\lim\sup s_k^2/k < \infty$, and

$$(3.15) \qquad 0 \leqq \lim_{k\to\infty} \frac{\Phi_K(s_k)}{k} \leqq \lim\sup \frac{s_k^2}{k} \cdot \lim_{k\to\infty} \frac{\Phi_k(s_k)}{s_k^2} = 0,$$

hence (3.14) holds.

COROLLARY 3.3. *Let Ω be a bounded domain with $C^{2,\alpha}$ boundary, $\mathscr{L}$, f, B, ϕ satisfy all assumptions from Theorem* 3.2. *Moreover assume there exist twice differentiable functions α, β: $\bar{\Omega} \to R^N$ such that*

$$(3.16) \qquad \alpha_i(x) < 0 < \beta_i(x),\ x \in \Omega,\ i = 1, 2, \ldots N,$$

$$(\mathscr{L}\alpha_i)(x) = f_i(x, u_1, \ldots u_{i-1}, \alpha_i, u_{i+1}, \ldots u_N, p_1, \ldots \partial\alpha_i, \ldots p_N),$$
$$(\mathscr{L}\beta_i)(x) \geqq f_i(x, u_1, \ldots u_{i-1}, \beta_i, u_{i+1}, \ldots u_N, p_1, \ldots \partial\beta_i, \ldots p_N)$$

for all $u = (u_1, \ldots u_N)$ with $\alpha_i(x) \leqq u_i \leqq \beta_i(x)$, $p_i \in R^m$, $1 \leqq i \leqq N$.

$$(3.17) \qquad B_i\alpha_i(x) < 0 < B_i\beta_i(x),\ x \in \partial\Omega,\ 1 \leqq i \leqq N.$$

Then there exists a solution u of

$$(\mathscr{L} u)(x) = f(x, u, \partial u),\ x \in \partial\Omega),\ x \in \partial\Omega,$$
$$Bu(x) = 0,\ x \in \partial\Omega,$$

such that $\alpha_i(x) \leqq u_i(x) \leqq \beta_i(x)$, $1 \leqq i \leqq N$, $x \in \bar{\Omega}$.

PROOF. For Σ we take in Theorem 3.2 the following set:

$$\Sigma = \{(x, u): \alpha_i(x) < u_i < \beta_i(x),\ x \in \bar{\Omega},\ 1 \leqq i \leqq N\}.$$

If $u_0 \in \partial\Sigma_{x_0}$, $x_0 \in \bar{\Omega}$, then there exists j such that either $u_{0j} = \alpha_j(x_0)$ or $u_{0j} = \beta_j(x_0)$. Assume the former and put $r(x, u) = u_j - \beta_j(x)$, then r satisfies all assumptions in Theorem 3.2 and we can conclude the existence of a solution $u \in C^2(\bar{\Omega})$ with $u(x) \in \bar{\Sigma}_x$, $x \in \bar{\Omega}$.

REMARK 3.4. In Corollary 3.3 instead of (3.16) and (3.17) one might assume

(3.16′) $\alpha_i(x) \leqq 0 \leqq \beta_i(x),\ x \in \Omega,\ i = 1, 2, \dots N,$

(3.17′) $\beta_i\alpha_i(x) \leqq 0 \leqq \beta_i\beta_i(x),\ x \in \partial\Omega,\ i = 1, 2, \dots N.$

To see that we take $U_i^\varepsilon = U_i + \varepsilon$, $L_i^\varepsilon = L_i - \varepsilon$,

$$f_i^\varepsilon(x, u, p) = \begin{cases} f_i(x, U, p) + \dfrac{U_i - u_i}{1 + u_i^2}, & u_i \geqq U_i \\ f_i(x, u, p), & L_i \leqq u_i \leqq U_i \\ f_i(x, L, p) + \dfrac{L_i - u_i}{1 + u_i^2}, & u_i \leqq L_i, \end{cases}$$

apply Theorem 3.2 with U^ε, L^ε, f^ε and by a standard limiting argument one can conclude the assertion.

EXAMPLE 3.5. Let Ω be a bounded domain in R^m with boundary $\partial\Omega$. Consider the following system:

(3.18) $$\Delta u = u^3 + \mathrm{v}\nabla u\ (0.25, 0.75)$$

(3.19) $$\Delta \mathrm{v} = \mathrm{v}^3 - (u^2 + 1)(\mathrm{v} + 1) + u\nabla \mathrm{v}\ (0.75, 0.25)$$

subject to the boundary conditions:

(3.20) $$u + \frac{\partial u}{\partial \nu} = 0,\ \mathrm{v} + \frac{\partial \mathrm{v}}{\partial \nu} = 0,\ x \in \partial\Omega.$$

Let $\Sigma = \{(x, u, \mathrm{v}): x \in \bar{\Omega}, (u, \mathrm{v}) \in R^2, u^2 + \mathrm{v}^2 < d^2\}$. Assume that for some $x \in \bar{\Omega}$, $(u_0, \mathrm{v}_0) \in \partial\Sigma_x$. Then $u_0^2 + \mathrm{v}_0^2 = d_0^2$. Put $r(u, \mathrm{v}) = (u^2 + \mathrm{v}^2 - d^2)/2$, $g(x) = (0, 0)$, note that $r(u_0, \mathrm{v}_0) = 0$, $r(u, \mathrm{v}) \leqq 0$ for all $(x, u, \mathrm{v}) \in \Sigma$,

$$-\frac{\partial r}{\partial u}(u_0, \mathrm{v}_0)u_0 - \frac{\partial r}{\partial \mathrm{v}}(u_0, \mathrm{v}_0)\cdot \mathrm{v}_0 = -u_0^2 - \mathrm{v}_0^2 = -d^2 < 0.$$

Let $u_0\xi + \mathrm{v}_0\eta = 0$, then

$$\frac{\partial^2 r}{\partial u^2}\xi^2 + \frac{\partial^2 r}{\partial \mathrm{v}^2}\eta^2 + \left(\frac{\partial r}{\partial u}, \frac{\partial r}{\partial \mathrm{v}}\right)\cdot(f_1(u_0, \mathrm{v}_0, \xi, \eta), f_2(u_0, \mathrm{v}_0, \xi, \eta))$$
$$= \xi^2 + \eta^2 + u_0^4 + u_0\mathrm{v}_0\xi + \mathrm{v}^4 - \mathrm{v}_0(u_0^2 + 1)(\mathrm{v}_0 + 1) + u_0\mathrm{v}_0\eta$$
$$\geqq \frac{1}{9}\xi^2 + \frac{1}{9}\eta^2 + \frac{1}{5}\left(\frac{d^2 - 5}{2}\right)^2 - \frac{7}{2} \geqq 0, \text{ provided } d \geqq \sqrt{5 + \sqrt{70}}.$$

Therefore all assumptions of Theorem 3.2 are satisfied and one may conclude the existence of a solution $(u, \mathrm{v}): \bar{\Omega} \to R^2$ of (3.18)–(3.20) such that $u^2(x) + \mathrm{v}^2(x) \leqq 5 + \sqrt{70}$, for all $x \in \bar{\Omega}$.

REMARK 3.6. One can replace (3.20) by

(3.21) $$u(x) = 0,\ \mathrm{v}(x) = 0,\ x \in \partial\Omega.$$

Then there exists a solution (u, v) of (3.18), (3.19) and (3.21) such that

$$u^2(x) + v^2(x) \leqq 5 + \sqrt{70},\ x \in \bar{\Omega}.$$

REMARK 3.7. Let

$$\frac{\partial u(x)}{\partial \nu} = 0, \frac{\partial v(x)}{\partial \nu} = 0,\ x \in \partial\Omega. \tag{3.22}$$

Then there exists a solution (u, v) of (31.8), (3.19) and (3.22) such that $u^2(x) + v^2(x) \leqq 5 + \sqrt{70}$. To see that we consider, instead of (3.22), these boundary conditions:

$$\varepsilon u(x) + \frac{\partial u}{\partial \nu} = 0 = \varepsilon v(x) + \frac{\partial v}{\partial \nu},\ x \in \partial\Omega,\ \varepsilon > 0,$$

and apply a limiting argument.

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