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COEFFICIENT BOUNDS FOR QUOTIENTS OF STARLIKE FUNCTIONS

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ABSTRACT. For functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ whose coefficients satisfy the inequality $\sum_{n=2}^{\infty}(n - \alpha)|a_n| \leqq 1 - \alpha$, $0 \leqq \alpha \leqq 1$, we investigate bounds for the coefficients of $F(z) = wf(z)/(w - f(z))$ when $w \notin f(|z|<1)$. A sharp upper bound for the second coefficient independent of w is obtained, along with a conjecture on the bounds for the remaining coefficients.

Denote by S the family of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analytic and univalent in $\Delta = \{z: |z| < 1\}$. Such functions are said to be in K if they map Δ onto convex domains and in $S^*(\alpha)$, the family of functions starlike of order α, if they satisfy $\mathrm{Re}\{zf'(z)/f(z)\} > \alpha$ for $z \in \Delta$. If $f \in S$ and $w \notin f(\Delta)$ it is known that $F(z) = wf(z)/(w - f(z))$ is also in S. In [1] Hall proved that $F(z)$ has bounded coefficients if $f \in K$ by first showing that $|z^2 f'(z)/f^2(z)| > 4/\pi^2$ for f in K. In fact, he essentially showed that $F(z)$ will have bounded coefficients whenever $z^2 f'(z)/f^2(z)$ is bounded away from zero. We state this as a Lemma.

LEMMA 1. *Let G be a subfamily of S with $|z^2 f'(z)/f^2(z)| \geqq B > 0$ for all $f \in G$, and set $F(z) = wf(z)/(w - f(z))$ for $w \notin f(\Delta)$. Then there exists a constant A, independent of f and w, such that the modulus of the coefficients of F are bounded above by A.*

PROOF. A computation shows $z^2 F'(z)/F^2(z) = z^2 f'(z)/f^2(z)$, so that $|z^2 F'(z)/F^2(z)| \geqq B$ or, equivalently, $|F(z)| \leqq (r/B)|(zF'(z)/F(z)|$. By the Koebe distortion theorem, $|F(z)| \leqq (r/B)((1 + r)/(1 - r)) \leqq (2/B)(1 - r)$. But Spencer has shown [3] that a function in S has bounded coefficients if its modulus is bounded above by $K/(1 - r)$ for some absolute constant K, which completes the proof.

A function f is said to be in $S^*(\alpha, M)$ if $f \in S^*(\alpha)$ and $|f| \leqq M$ in Δ.

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We show that such functions satisfy the conditions of Lemma 1 if α is positive.

THEOREM 1. *If* $f \in S^*(\alpha, M)$, $\alpha > 0$, *and* $w \notin f(\Delta)$, *then for* $F(z) = wf(z)/(w - f(z))$ *there exists a constant* A, *independent of* w *and* f, *such that the modulus of the coefficients of* F *are bounded above by* A.

PROOF. Since

$$\left|\frac{z^2 f'(z)}{f^2(z)}\right| = \left|\frac{z}{f(z)}\right|\left|\frac{zf'(z)}{f(z)}\right| \geqq \frac{\alpha}{M} > 0,$$

the result follows from Lemma 1.

REMARK. Theorem 1 cannot be improved to allow $\alpha = 0$. If we take $f(z) = z - z^2/2$ and $w = 1/2$, then

$$F(z) = z + \sum_{n=2}^{\infty}\left(\frac{n+1}{2}\right)z^n. \tag{1}$$

We will investigate a special family of bounded starlike functions $z + \sum_{n=2}^{\infty} a_n z^n$, those for which $\sum_{n=2}^{\infty} n|a_n| \leqq 1$. It is known [2] that such functions are in $S^*(\alpha)$ if

$$\sum_{n=2}^{\infty} (n - \alpha)\,|a_n| \leqq 1 - \alpha. \tag{2}$$

Lemma 2. *If* $f \in S$ *and*

$$F_w(z) = \frac{wf(z)}{w - f(z)} = z + \sum_{n=2}^{\infty} c_n(w) z^n,$$

then the $w \notin f(\Delta)$ *for which* $|c_n(w)|$ *is maximal must be a boundary point of* $f(\Delta)$.

PROOF. For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, we have $c_2(w) = a_2 + (1/w)$ and

$$c_n(w) = a_n + \sum_{k=1}^{n-1} a_k c_{n-k}(w)/w \quad (a_1 = c_1 = 1). \tag{3}$$

An induction shows that we may write $c_n(w)$ as $c_n(w) = a_n + P_{n-2}(w)/w^{n-1}$, where $P_{n-2}(w)$ is a polynomial of degree at most $n - 2$ whose coefficients depend only on $a_2, a_3, \ldots, a_{n-1}$. Either $C - f(|z| \leqq 1)$ is empty, in which case every $w \notin f(\Delta)$ is a boundary point, or $c_n(w)$ is an analytic function of w in the domain $C - f(|z| \leqq 1)$. In the latter case $c_n(w)$ cannot attain a maximum in the domain, and so must attain its maximum on the boundary.

We now find the maximum of the second coefficient of F when f satisfies (2).

THEOREM 2. *If the coefficients of* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *satisfy the in-*

equality $\sum_{n=2}^{\infty}(n-\alpha)|a_n| \leqq 1-\alpha$, $0 \leqq \alpha \leqq 1$, *then for any* $w \notin f(\Delta)$ *the function* $F(z) = wf(z)/(w-f(z)) = z + \sum_{n=2}^{\infty} c_n z^n$ *will satisfy* $|c_2| \leqq (3-\alpha)/2$. *This result is sharp, with equality for* $f(z) = z-(1-\alpha)z^3/(3-\alpha)$ *and* $w = 2/(3-\alpha)$.

PROOF. In view of Lemma 2, it suffices to set $w = f(e^{i\theta})$ so that $c_2 = c_2(w) = a_2 + 1/f(e^{i\theta})$. With $|a_2| = p \leqq (1-\alpha)/(2-\alpha)$ we see that

$$(3-\alpha)\sum_{n=3}^{\infty}|a_n| \leqq \sum_{n=3}^{\infty}(n-\alpha)|a_n| \leqq (1-\alpha)-(2-\alpha)p,$$

and $\sum_{n=3}^{\infty}|a_n| \leqq ((1-\alpha)-(2-\alpha)p)/(3-\alpha)$. Since $|\sum_{n=3}^{\infty} a_n e^{in\theta}| \leqq \sum_{n=3}^{\infty}|a_n|$, we may write

$$|c_2| = \left|a_2 + \frac{1}{e^{i\theta}+a_2e^{2i\theta}+Re^{ig(\theta)}}\right| = \left|\frac{1+a_2e^{i\theta}+a_2^2e^{2i\theta}+a_2Re^{ig(\theta)}}{1+a_2e^{i\theta}+Re^{i(g(\theta)-\theta)}}\right|,$$

where $R \leqq [(1-\alpha)-(2-\alpha)p]/(3-\alpha)$ and $g(\theta)$ is a real function of θ. Setting $h(\theta) = g(\theta)-\theta$, we obtain

$$|c_2| = \left|1+\frac{a_2^2e^{2i\theta}+Re^{ih(\theta)}(a_2e^{i\theta}-1)}{1+a_2e^{i\theta}+Re^{ih(\theta)}}\right| \leqq 1+\frac{p^2+(1+p)R}{1-p-R}$$

$$\leqq 1+\frac{p^2+(1+p)[(1-\alpha)-(2-\alpha)p]/(3-\alpha)}{(1-p)-[(1-\alpha)-(2-\alpha)p]/(3-\alpha)} = \frac{3-\alpha-2p+p^2}{2-p}.$$

This last expression attains a maximum for $0 \leqq p \leqq (1-\alpha)/(2-\alpha)$ when $p = 0$, and the theorem is proved.

When $\alpha = 0$, the bound in Theorem 2 is also attained for $f(z) = z - z^2/2$ and $w = 1/2$, and we conjecture that the coefficients of $F(z)$ defined by (1) are extremal for all functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that satisfy the inequality $\sum_{n=2}^{\infty} n|a_n| \leqq 1$.

The bounds on the coefficients of $F(z)$ when the coefficients for $f(z)$ satisfy the more general inequality (2), we believe, are more complicated. For $f_k(z) = z - ((1-\alpha)/(k-\alpha))z^k$ and $w_k = (k-1)/(k-\alpha)$ $(k = 2, 3, \ldots)$, set

$$F_k(z) = \frac{w_k f_k(z)}{w_k - f_k(z)} = z + \sum_{n=2}^{\infty} c_n(\alpha, k) z^n.$$

From (3) we see that

$$c_n(\alpha, k) = 1/w_k^{n-1} = \left(\frac{k-\alpha}{k-1}\right)^{n-1} \text{ for } n = 2, 3, \ldots, k-1,$$
(4)
$$c_k(\alpha, k) = \left(\frac{k-\alpha}{k-1}\right)^{k-1} - \left(\frac{1-\alpha}{k-\alpha}\right),$$

and for $m = 1, 2, 3, \ldots$ we have, recursively,

$$c_{k+m}(\alpha, k) = c_{k+m-1}(\alpha, k) + \left(\frac{1-\alpha}{k-\alpha}\right)(c_{k+m-1}(\alpha, k) - c_m(\alpha, k)).$$

Note that $c_2(\alpha, 2) = (3 - 3\alpha + \alpha^2)/(2 - \alpha)$ and for $n = 3, 4, \ldots,$

$$c_n(\alpha, 2) = (2 - 2\alpha + \alpha^2) + \frac{(1-\alpha)^4}{2-\alpha}\left(\frac{1-(1-\alpha)^{n-3}}{\alpha}\right), \tag{5}$$

where $c_n(0, 2) = \lim_{\alpha\to 0} c_n(\alpha, 2) = (n + 1)/2$.

If $d_n(\alpha)$ is the maximum modulus of the n-th coefficient of $wf(z)/(w - f(z))$ taken over all f whose coefficients satisfy (2) and all $w \notin f(\Delta)$, we see from (5) that $d_n(\alpha) \geqq c_n(\alpha, 2) \geqq K/\alpha$ for some positive constant K. On the other hand, for large n we have form (4) that $c_n(\alpha, n) \approx e^{1-\alpha}$, which is greater than $c_n(\alpha, 2)$ when α is sufficiently close to 1. We believe that $d_n(\alpha) = c_n(\alpha, k)$ for some $k = 2, 3, \ldots, n + 1$, the choice of k being a nondecreasing function of α, with $d_n(0) = c_n(0, 2) = (n + 1)/2$.

We close with a question about the lower bounds on the coefficients of F when $f \in S$.

CONJECTURE. *If $f \in S$, then there exists a $w \notin f(\Delta)$ such that the coefficients for $F(z) = wf(z)/(w - f(z)) = z + \sum_{n=2}^{\infty} c_n z^n$ satisfy $|c_n| \geqq 1$. Equality holds for $f(z) = z$ and $w = 1$.*

REFERENCES

1. R.R. Hall, *On a conjecture of Clunie and Sheil-Small*, Bull. London Math. Soc. **12** (1980), 25–28.

2. H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. **51** (1975), 109–116.

3. D.C. Spencer, *On finitely mean valent functions*, Proc. London Math. Soc. (2) **47** (1941), 201–211.

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