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CENTRALIZERS IN DIFFERENTIAL, PSEUDO-DIFFERENTIAL, AND FRACTIONAL DIFFERENTIAL OPERATOR RINGS

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The size and structure of centralizers in three related types of rings arising from formal linear differential operators are studied. Each case is built from a coefficient ring R which is a commutative ring equipped with a derivation δ, to obtain the ring S of all formal linear differential operators over R, the ring T of all formal linear pseudo-differential operators over R, and, in case R is a field, the division ring Q of all fractional linear differential operators over R (i.e., fractions of operators from S). The main structural results are that in quite general circumstances, centralizers in S, T, and Q are commutative. For example, given an operator a in S, T, or Q whose order and whose leading coefficient are both non-zero-divisors in R, the centralizer of a must be commutative. This result is proved first in T (because S and Q are subrings of T), following a method introduced by Schur in 1904. As an application, we find exactly which differential operators over the ring of complex-valued C^∞ functions on the real line have commutative centralizers.

The sizes of centralizers are studied relative to subrings built from the subring F of constants of R, i.e., elements $r \in R$ satisfying $\delta(r) = 0$. Namely, the centralizer of a differential operator $a \in S$ always contains the polynomial ring $F[a]$, and the centralizer of a fractional operator $a \in Q$ always contains the field $F(a)$, while the centralizer of a pseudo-differential operator $a \in T$ with positive order and invertible leading coefficient always contains the Laurent series field $F((a^{-1}))$. In each case, the size of the centralizer of a, relative to the appropriate natural subring, is considered for the case when F is a field. For instance, if R is semiprime, $a \in S$ is a differential operator of positive order n, and the leading coefficient of a is invertible in R, then the centralizer of a in S is a free $F[a]$-module of rank at most n^2; moreover, if n is invertible in F, then the rank of the centralizer is a divisor of n. Similarly, if R is semiprime, $a \in T$ is a pseudo-differential operator of positive order n, and n and the leading coefficient of a are invertible in R, then the centralizer of a in T is a

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finite-dimensional field extension of $F((a^{-1}))$ of dimension a divisor of n. The corresponding results for Q are less complete, covering mainly the case of a fractional differential operator $a = bc^{-1}$ where $b, c \in S$ and b, c commute. For example, if b and c have distinct positive orders n and k, and $n, k, n - k$ are all invertible in F, then the centralizer of a in Q is a finite-dimensional field extension of $F(a)$ of dimension less than $n + k$.

We present this material in a semi-expository style, for several reasons. First, several of the results derived here for differential and pseudo-differential operator rings have evolved fairly directly from various precursors in the literature, as discussed in the introductions to the particular sections of the paper. In these cases, we have tried to simplify the existing methods and apply them in as wide a context as reasonable. Second, the present literature in this area is mostly directed at just one of two relatively disjoint audiences, namely differential operator theorists and ring theorists, whose vocabularies are developing toward mutual unintelligibility. In the interest of trying to preserve a useful line of communication, we have aimed at presenting sufficient detail for readers of either persuasion to follow. Finally, we wish to bring the historical record a little more up-to-date by emphasizing just how far back these ideas and methods really originated. In particular, the construction and effective use of the formal pseudo-differential operator ring dates back to an obscure 1904 paper of Schur, although a number of recent authors have re-discovered the technique. This ring construction also appears to have some potential as a source of examples of interest to ring theorists, hence we have taken care to present the construction in reasonable detail.

I. Differential operator rings, Part 1. In this section, we investigate the sizes of centralizers in rings of formal linear differential operators. The commutativity of such centralizers will be considered in §IV, after the machinery of the ring of formal linear pseudo-differential operators has been developed.

Specifically, we consider a commutative ring R with no nonzero nilpotent elements, equipped with a derivation δ, and we assume that the subring $F = \{r \in R \mid \delta(r) = 0\}$ of constants of R is a field. We construct the ring S of formal linear differential operators over (R, δ), and we study the centralizer $C_S(a) = \{b \in S \mid ab = ba\}$ for an operator $a \in S$ of positive order whose leading coefficient is invertible in R. Then a is transcendental over F, and the polynomial ring $F[a]$ is a subring of $C_S(a)$. We prove that $C_S(a)$ is a free $F[a]$-module of finite rank, and good upper bounds for this rank are found. Namely, if the order n of a is invertible in F, then the rank of $C_S(a)$ as a free $F[a]$-module is a divisor of n, while in general the rank is at most n^2. A consequence of the fact that $C_S(a)$

has finite rank over $F[a]$ is that given any $b \in C_S(a)$, there exists a non-constant polynomial q over F in two commuting indeterminates such that $q(a, b) = 0$.

The pattern of these results was clear starting with the work of Flanders [**8**], who remarked, in the case when R is the ring of complex-valued C^∞ functions on the real line, that $C_S(a)$ must be a free module of finite rank over $\mathbf{C}[a]$. This result was transferred to an algebraic setting by Amitsur [**3**], who studied the case when R is a differential field of characteristic zero, showing that $C_S(a)$ must be a free $F[a]$-module of finite rank dividing the order of a [**3**, Theorem 1]. Using Amitsur's method of proof, Carlson and the author extended this result to certain commutative differential rings (still in characteristic zero) having invertible solutions for differential equations of the form $\delta(r) = sr$ [**6**, Theorem 1.2]. With a slight change of perspective, this differential equation hypothesis may be dropped in favor of the algebraic hypothesis of no nonzero nilpotent elements, and this is the version of the theorem that we present here.

In positive characteristic, centralizer results of this type seem not to have been studied before. However, a somewhat related result does appear in Ore's well-known 1932 paper [**20**, Satz 3, p. 236]: assuming that R is a differential field (of any characteristic), and given an operator $a \in S$ of positive order n, the vector space

$$\{c \in S \mid ac = c'a \text{ for some } c' \in S\}/\{c''a \mid c'' \in S\}$$

has dimension at most n^2 over F. Using this result, we prove that $C_S(a)$ must be a free $F[a]$-module of rank at most n^2.

The final result of the section, that commuting pairs of differential operators (with at least one of positive order) must satisfy a non-constant polynomial in two variables with constant coefficients, goes back to a 1923 paper of Burchnall and Chaundy [**5**], who proved the result for some unspecified case in which R is a ring of "functions of a single variable" [**5**, pp. 421, 422]. For R the ring of germs of complex-valued C^∞ functions on the real line, this was rediscovered by Krichever [**13**, Lemma 1.5]. A more general case, in which R is a commutative differential ring and certain bounds are placed on the dimensions of solution spaces of systems of homogeneous linear differential equations in R, was covered by Carlson and the author in [**6**, Corollary 2.3].

DEFINITION. A *differential ring* is a ring R (assumed associative, with identity, but not necessarily commutative) equipped with a specified *derivation* δ, that is, an additive map $\delta\colon R \to R$ satisfying the product rule: $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in R$. To save naming the derivation each time we refer to a differential ring, all derivations in this paper will be denoted δ. We shall need the fact that any derivation on a ring R satisfies *Leibniz' Rule*:

$$\delta^n(xy) = \sum_{i=0}^{n} \binom{n}{i} \delta^i(x)\delta^{n-i}(y) = \sum_{i=0}^{n} \binom{n}{i} \delta^{n-i}(x)\delta^i(y)$$

for all $x, y \in R$ and $n \in \mathbf{N}$, which is easily established by induction on n. The elements $r \in R$ such that $\delta(r) = 0$ are called *constants* of R, and the collection of these constants forms a subring of R.

Various ring-theoretic adjectives will be attached to the term "differential ring" as needed. For instance, most of the time we shall deal with a commutative differential ring, meaning a differential ring which, as a ring, is commutative. A differential ring which happens to be a field (i.e., a commutative division ring) is called a *differential field.*

DEFINITION. Given a differential ring R, we form the *formal linear differential operator ring* over R (also called the *Ore extension* of R), denoted $R[\theta; \delta]$, as follows. Additively, $R[\theta; \delta]$ is the abelian group of all polynomials over R in an indeterminate θ. Multiplication in $R[\theta; \delta]$ is performed by using the given multiplication in R and the obvious rules for powers of θ together with the rule $\theta r = r\theta + \delta(r)$ for all $r \in R$. [This is just the product rule in disguise, provided one thinks of these expressions as operators acting on the left of elements of R. Thus θr corresponds to the operation "multiply on the left by r, then differentiate", $r\theta$ to the operation "differentiate, then multiply on the left by r", and $\delta(r)$ to the operation "multiply on the left by $\delta(r)$".] An easy induction establishes a useful analog of Leibniz' Rule:

$$\theta^n r = \sum_{i=0}^{n} \binom{n}{i} \delta^i(r)\theta^{n-i} = \sum_{i=0}^{n} \binom{n}{i} \delta^{n-i}(r)\theta^i$$

for all $r \in R$ and $n \in \mathbf{N}$.

Any nonzero operator $a \in R[\theta; \delta]$ can be uniquely written in the form $a = a_0 + a_1\theta + a_2\theta^2 + \cdots + a_n\theta^n$ with the $a_i \in R$ and $a_n \neq 0$. The integer n is called the *order* of a, denoted $\operatorname{ord}(a)$. We also adopt the convention that the zero operator has order $-\infty$. [Here we are following differential equation terminology; many algebraists prefer to call n the *degree* of a rather than the order.] The element a_n is called the *leading coefficient* of a. [The leading coefficient of the zero operator is defined to be the zero element of R.] The operator a can also be uniquely written in the form $a = b_0 + \theta b_1 + \theta^2 b_2 + \cdots + \theta^n b_n$ with the $b_i \in R$. In general $b_i \neq a_i$ for $i < n$, unless all the coefficients are constants, but $b_n = a_n$ always.

DEFINITION. A ring R is said to be *semiprime* provided R has no nonzero nilpotent two-sided ideals. When R is commutative, this happens if and only if R has no nonzero nilpotent elements. In fact, to check that a commutative ring R is semiprime, it suffices to show that R has no nonzero elements r satisfying $r^2 = 0$.

DEFINITION. Given a ring S and an element $a \in S$, the *centralizer of a in S* is the set $C_S(a)$ of all those b in S which commute with a. Note that $C_S(a)$ is a (unital) subring of S.

LEMMA 1.1. *Let R be a semiprime commutative differential ring, such that the subring F of constants of R is a field, and set $S = R[\theta; \delta]$. Let $a \in S$ be an operator with positive order n and leading coefficient a_n, such that n is invertible in F and a_n is invertible in R. Let b and c be nonzero operators in $C_S(a)$ with orders k and m and leading coefficients b_k and c_m.*

(a) *There is a nonzero constant $\beta \in F$ such that $b_k^n = \beta a_n^k$, and b_k is invertible in R.*

(b) *We have $bc \neq 0$, and* $\operatorname{ord}(bc) = \operatorname{ord}(b) + \operatorname{ord}(c)$.

(c) *If $k = m$, then $b_k = \alpha c_m$ for some nonzero constant $\alpha \in F$.*

PROOF. Write out the operators $a = a_0 + a_1\theta + \cdots + a_n\theta^n$; $b = b_0 + b_1\theta + \cdots + b_k\theta^k$; $c = c_0 + c_1\theta + \cdots + c_m\theta^m$ with coefficients from R.

(a) Comparing coefficients of θ^{n+k-1} in the equation $ab = ba$, we find that

$$na_n\delta(b_k) + a_nb_{k-1} + a_{n-1}b_k = kb_k\delta(a_n) + b_ka_{n-1} + b_{k-1}a_n,$$

hence $na_n\delta(b_k) = kb_k\delta(a_n)$. By assumption, n and a_n are both invertible in R, whence $\delta(b_k) = (k/n)(\delta(a_n)a_n^{-1})b_k$. Setting $\beta = b_k^na_n^{-k}$, we compute that

$$\delta(\beta) = nb_k^{n-1}\delta(b_k)a_n^{-k} - kb_k^na_n^{-k-1}\delta(a_n) = 0,$$

so that $\beta \in F$ and $b_k^n = \beta a_n^k$. As $b_k \neq 0$ and R is semiprime, $b_k^n \neq 0$, and so $\beta \neq 0$. Now β is invertible (in F, hence in R), and consequently b_k is invertible in R.

(b) This follows immediately from (a).

(c) Set $s = (k/n)\delta(a_n)a_n^{-1}$. As in (a), we obtain $\delta(b_k) = sb_k$, and since $m = k$, we obtain $\delta(c_m) = sc_m$ as well. Setting $\alpha = b_kc_m^{-1}$, we compute that $\delta(\alpha) = sb_kc_m^{-1} - b_kc_m^{-2}sc_m = 0$. Thus $\alpha \in F$ and $b_k = \alpha c_m$; moreover, $\alpha \neq 0$ because $b_k \neq 0$.

THEOREM 1.2. *Let R be a semiprime commutative differential ring, such that the subring F of constants of R is a field, and set $S = R[\theta; \delta]$. Let $a \in S$ be an operator of positive order n, such that n is invertible in F and the leading coefficient of a is invertible in R.*

Let X be the set of those i in $\{0, 1, \ldots, n-1\}$ for which $C_S(a)$ contains an operator of order congruent to i modulo n. For each $i \in X$, choose $b_i \in$

$C_S(a)$ such that $\mathrm{ord}(b_i) \equiv i \pmod{n}$ *and b_i has minimal order for this property.* [*In particular,* $0 \in X$, *and we may choose* $b_0 = 1$.]

Then $C_S(a)$ is a free $F[a]$-module with basis $\{b_i | i \in X\}$. Moreover, the rank of $C_S(a)$ as a free $F[a]$-module is a divisor of n.

PROOF. For nonzero operators in $C_S(a)$, Lemma 1.1 shows that products are nonzero, and that the order of a product is the sum of the orders of the factors. In particular, if $Y = \{\mathrm{ord}(c) \mid c \in C_S(a) \text{ and } c \neq 0\}$, then Y is closed under addition. Consequently, the set $\bar{Y} = \{i + n\mathbf{Z} \mid i \in Y\}$ is a nonempty additively closed subset of the finite abelian group $\mathbf{Z}/n\mathbf{Z}$. It follows that $\bar{Y}$ must be a subgroup of $\mathbf{Z}/n\mathbf{Z}$, and that $\mathrm{card}(\bar{Y})$ divides n [see any introduction to group theory]. Since there exists a bijection of X onto $\bar{Y}$ (given by the rule $i \mapsto i + n\mathbf{Z}$), we have thus proved that $\mathrm{card}(X)$ divides n.

It remains to prove that the b_i are linearly independent over $F[a]$, and that they span $C_S(a)$ as an $F[a]$-module.

Suppose that $\sum_{i \in X} c_i b_i = 0$ for some $c_i \in F[a]$. For each $i \in X$, either $c_i = 0$ or else c_i has order divisible by n, in which case

$$\mathrm{ord}(c_i b_i) = \mathrm{ord}(c_i) + \mathrm{ord}(b_i) \equiv \mathrm{ord}(b_i) \equiv i \pmod{n}.$$

Thus for distinct $i, j \in X$, if $c_i \neq 0$ and $c_j \neq 0$, then $\mathrm{ord}(c_i b_i) \not\equiv \mathrm{ord}(c_j b_j) \pmod{n}$, hence $\mathrm{ord}(c_i b_i) \neq \mathrm{ord}(c_j b_j)$. Since a sum of nonzero operators in S cannot vanish unless at least two of the operators have the same order, we conclude that each $c_i b_i = 0$, whence each $c_i = 0$. Therefore the b_i are linearly independent over $F[a]$.

Let W denote the $F[a]$-module spanned by $\{b_i \mid i \in X\}$. We show by induction on order that any c in $C_S(a)$ must belong to W. If $\mathrm{ord}(c) = 0$, then $c \in R$, and c is its own leading coefficient. As $C_S(a)$ also contains the nonzero operator b_0 of order 0, Lemma 1.1 shows that $c = \alpha b_0$ for some $\alpha \in F$, hence $c \in W$.

Now let c have positive order j, and assume that W contains all operators in $C_S(a)$ having order less than j. Some $i \in X$ is congruent to j modulo n. If $m = \mathrm{ord}(b_i)$, then $m \equiv i \equiv j \pmod{n}$, and the minimality of m implies that $j = m + qn$ for some nonnegative integer q. We can thus construct the operator $a^q b_i$, which lies in $C_S(a)$ and has order j, the same as c. According to Lemma 1.1, the leading coefficient of c must equal α times the leading coefficient of $a^q b_i$, for some $\alpha \in F$. Then $c - \alpha a^q b_i$ has order less than j. Since $c - \alpha a^q b_i$ lies in $C_S(a)$, the induction hypothesis shows that $c - \alpha a^q b_i$ is in W, and therefore $c \in W$.

This completes the induction step, proving that $C_S(a) = W$, that is, the b_i span $C_S(a)$ as an $F[a]$-module. Therefore $\{b_i \mid i \in X\}$ is indeed a basis for $C_S(a)$ as a free $F[a]$-module.

It is easy to construct examples of the situation analyzed in Theorem

1.2 in which $C_S(a)$ has the maximum possible rank as an $F[a]$-module, namely n. For instance, $C_S(\theta^n)$ is a free $F[\theta^n]$-module of rank n, with basis $\{1, \theta, \ldots, \theta^{n-1}\}$. For a less trivial example, let R be the ring of complex-valued C^∞ functions on the open interval $(0, 1)$, with δ being ordinary differentiation, and let $a = \theta^2 - 2x^{-2}$ (where x^{-2} is of course the function $x \mapsto x^{-2}$). Setting $b = \theta^3 - 3x^{-2}\theta + 3x^{-3}$, we compute that $ab = ba$. On the other hand, it is easily checked that $C_S(a)$ contains no first-order operators. Therefore $C_S(a)$ is a free $\mathbf{C}[a]$-module of rank 2, with basis $\{1, b\}$. This example also shows the necessity of using congruence classes of orders in Theorem 1.2.

For an example with small rank, let R be the ring of complex-valued C^∞ functions on the real line, and let $a = \theta^2 + x$. As shown in [**6**, p. 346], $C_S(a)$ contains no operators of odd order. [This was proved independently by M. Giertz, M. K. Kwong, and A. Zettl.] Thus $C_S(a)$ is a free $\mathbf{C}[a]$-module of rank 1 with basis $\{1\}$, that is, $C_S(a) = \mathbf{C}[a]$.

As these examples indicate, it is relatively easy to find examples of differential operators whose centralizers have maximal rank, but less easy to find and check the details for examples where the centralizer has less than maximal rank. In general, we can reduce the upper bound on the rank of a centralizer for a differential operator whose leading coefficient fails to have certain fractional powers, as in the following theorem. While the method only gives a bound rather than completely specifying the rank, it previews the method to be used for the centralizer of a pseudo-differential operator, where the fractional powers of the leading coefficient do specify the rank.

THEOREM 1.3. *Let R be a semiprime commutative differential ring, such that the subring F of constants of R is a field, and set $S = R[\theta; \delta]$. Let $a \in S$ be an operator with positive order n and leading coefficient a_n, such that n is invertible in F and a_n is invertible in R.*

Let t be the smallest positive integer such that αa_n^t has an n-th root in R for some nonzero constant $\alpha \in F$. Then n is divisible by t, all operators in $C_S(a)$ have order divisible by t, and the rank of $C_S(a)$ as a free $F[a]$-module is a divisor of n/t. In particular, if $t = n$, then $C_S(a) = F[a]$.

PROOF. Since a_n^n has an n-th root in R, we see that t exists, and $t \leqq n$. By hypothesis, $\alpha a_n^t = u^n$ for some $u \in R$. Since α and a_n are invertible in R, so is u.

We claim that if $s \in \mathbf{Z}$ and $\beta \in F$ such that $\beta \neq 0$ and βa_n^s has an n-th root in R, then s must be divisible by t.

Write $s = qt + r$ with $q, r \in \mathbf{Z}$ and $0 \leqq r \leqq t - 1$. There is some $v \in R$ such that

$$v^n = \beta a_n^s = \beta(a_n^t)^q a_n^r = \beta\alpha^{-q}u^{nq}a_n^r,$$

whence $(vu^{-q})^n = (\beta\alpha^{-q})a_n^r$. As $\beta\alpha^{-q}$ is a nonzero element of F, the minimality of t forces $r = 0$, and so t divides s, as claimed.

In particular, since a_n^n has an n-th root in R, it follows that n is divisible by t.

We next show that any nonzero operator $b \in C_S(a)$ has order divisible by t. If k is the order of b and b_k is its leading coefficient, then by Lemma 1.1, $b_k^n = \beta a_n^k$ for some nonzero $\beta \in F$. Thus βa_n^k has an n-th root in R and so k is divisible by t (by the claim above), as desired.

The rank of $C_S(a)$ as a free $F[a]$-module is card(X), using the notation of the proof of Theorem 1.2. Continuing with that notation, what we have just proved above shows that $Y \subseteq t\mathbf{Z}$, so that $\bar{Y}$ is contained in the set $W = \{kt + n\mathbf{Z} \mid k \in \mathbf{Z}\}$. Now W is a subgroup of $\mathbf{Z}/n\mathbf{Z}$, of order n/t. Since $\bar{Y}$ is a subgroup of W, it follows that card$(\bar{Y})$ must divide n/t. But card$(\bar{Y})$ = card(X), so that card(X) is a divisor of n/t.

Finally, suppose that $t = n$, so that $C_S(a)$ is a free $F[a]$-module of rank 1. Then Theorem 1.2 shows that $\{1\}$ is a basis for $C_S(a)$ over $F[a]$, whence $C_S(a) = F[a]$.

Using Theorem 1.3, it is not hard to find differential operators whose centralizers have minimal rank, as follows.

EXAMPLE 1.4. Let R be the field $\mathbf{C}(x)$ of complex rational functions in an indeterminate x, with the usual derivation. Then

$$C_{R[\theta;\delta]}(x\theta^n) = \mathbf{C}[x\theta^n]$$

for any positive integer n.

Theorem 1.2 depends crucially on the invertibility of the leading coefficient of the differential operator a, as the following example shows.

EXAMPLE 1.5. Let R be the ring of complex-valued C^∞ functions on the real line, and set $S = R[\theta;\ \delta]$. There exists a non-zero-divisor $f \in R$ such that the centralizer $C_S(f\theta)$ is not even countably generated as a module over $\mathbf{C}[f\theta]$.

PROOF. For each $n \in \mathbf{Z}$, choose $f_n \in R$ such that f_n is nonzero on the open interval $(n,\ n + 1)$ but is zero everywhere else. [The existence of such functions is an easy exercise.] We may define a function $f \in R$ as the sum of all the f_n. Since f is nonzero except on $\mathbf{Z}$, we see that $fg \neq 0$ for any nonzero $g \in R$, that is, f is a non-zero-divisor in R.

As a complex vector space, $\mathbf{C}[f\theta]$ has countable dimension. Thus if $C_S(f\theta)$ were to be countably generated as a $\mathbf{C}[f\theta]$-module (or even as a $\mathbf{C}[f\theta]$-algebra), it would have countable dimension over $\mathbf{C}$. We shall show that this is not the case.

Let A denote the set of all functions from $\mathbf{Z}$ to $\{\pm 1\}$. For each $\alpha \in A$, define

$$f_\alpha = \sum_{n \in \mathbf{Z}} \alpha(n) f_n,$$

and note that $f_\alpha \in R$. As A spans an uncountable-dimensional vector space over $\mathbf{C}$, so does the set $\{f_\alpha \theta \mid \alpha \in A\}$. Thus it suffices to show that each $f_\alpha \theta$ lies in $C_S(f\theta)$.

Within any of the open intervals $(n, n+1)$ (where $n \in \mathbf{Z}$), we have either $f_\alpha = f$ or $f_\alpha = -f$, hence $f_\alpha \delta(f)$ and $f\delta(f_\alpha)$ agree on $(n, n+1)$. On the other hand, f_α and f both vanish at all integers. Therefore $f_\alpha \delta(f) = f\delta(f_\alpha)$, from which we conclude that $(f_\alpha \theta)(f\theta) = (f\theta)(f_\alpha \theta)$, as desired.

We present one further example, to show that the semiprime hypothesis may not be dropped from Theorem 1.2. However, it is possible to replace this hypothesis by the hypothesis that for any $s \in R$, the set $\{r \in R \mid \delta(r) = sr\}$ either equals $\{0\}$ or contains an invertible element, as in [**6**, Theorem 1.2].

EXAMPLE 1.6. There exists a commutative differential ring R, whose subring F of constants is a field of characteristic zero, such that there is an invertible element $a \in R$ for which the centralizer $C_{R[\theta;\delta]}(a\theta^2)$ is not finitely generated as a module over $F[a\theta^2]$.

PROOF. Choose a field F of characteristic zero, let K denote the field $F(t)$ of rational functions over F in an indeterminate t, and let δ be the usual derivation on K, so that δ is F-linear and $\delta(t) = 1$. Note that F equals the subfield of constants of K.

Now let P be a polynomial ring $K[x_1, x_2, \ldots]$ over K in infinitely many independent indeterminates $x_1, x_2, \ldots$, let I be the ideal of P generated by all products $x_i x_j$, and set $R = P/I$. If y_i denotes the image of x_i in R, then R is a K-algebra with basis $\{1, y_1, y_2, \ldots\}$, such that $y_i y_j = 0$ for all i, j.

We extend δ to a derivation on P by defining $\delta(x_i) = t^{-1}x_i$ for all i, j. Since $\delta(x_i x_j) = 2t^{-1}x_i x_j$ for all i, j, we see that $\delta(I) \subseteq I$, hence δ induces a derivation on R. Thus we have a well-defined derivation on R, extending the original derivation on K, such that $\delta(y_i) = t^{-1}y_i$ for all i. It is easily checked that the subring of constants of R is just F.

Set $a = t^2$, which is an invertible element of R. For each $i = 1, 2, \ldots$, we compute that $\delta^2(y_i) = -t^{-2}y_i + t^{-1}\delta(y_i) = 0$, $2a\delta(y_i) = 2ty_i = y_i\delta(a)$, from which it follows that $(a\theta^2)(y_i\theta) = (y_i\theta)(a\theta^2)$. Therefore all the $y_i\theta$ lie in the centralizer of $a\theta^2$. As the y_i are linearly independent over F, we infer that the $F[a\theta^2]$-module spanned by the $y_i\theta$ cannot be finitely generated. On the other hand, $F[a\theta^2]$ is a P.I.D., whence all submodules of finitely generated $F[a\theta^2]$-modules are finitely generated. Therefore $C_{R[\theta;\delta]}(a\theta^2)$ cannot be a finitely generated $F[a\theta^2]$-module.

While Theorem 1.2 covers all positive order differential operators in characteristic zero, it is incomplete in positive characteristic cases. We shall show that in positive characteristic an analog of Theorem 1.2 remains valid, except that the basis for the centralizer is not explicitly given, and the bound for the rank of the centralizer is not quite so restrictive. This result is a direct consequence of an old result of Ore [**20**, Satz 3, p. 236] that is independent of characteristic.

Before developing Ore's result, we note that the positive characteristic case of the situation in Theorem 1.2 is fairly restrictive, in that R must a field. Namely, suppose that R is a semiprime commutative differential ring such that the subring F of constants of R is a field of characteristic $p > 0$. Given any nonzero element $x \in R$, we have $x^p \neq 0$ because R is semiprime. Also, $\delta(x^p) = px^{p-1}\delta(x) = 0$ and so $x^p \in F$, whence x^p is invertible in F, and thus x is invertible in R. Therefore R is a field, as claimed.

Thus to fill in the missing positive order cases of Theorem 1.2, we need only deal with the situation in which R is a field. We need bounds for the dimensions of solution spaces for systems of homogeneous linear differential equations over R. This may be proved by a standard classical argument involving Wronskians, as noted, for example, by Ore in [**20**, p. 238]. A more direct proof is also possible, which avoids the use of determinants and so does not require commutativity. That is, R need only be a division ring. Note that in this case the subring F of constants of R is another division ring. Namely, for any nonzero $x \in F$, one need only differentiate the equation $xx^{-1} = 1$ to see that $x^{-1} \in F$.

PROPOSITION 1.7. (AMITSUR [**2**, Theorem 1]). *Let R be a differential division ring, and let F be the sub-division-ring of constants of R. Let A be a $t \times t$ matrix over R, and set $X = \{x \in R^t \mid \delta(x) = Ax\}$. Then X is a right vector space over F, of dimension at most t.*

PROOF. Obviously X is closed under sums and differences. Given $x \in X$ and $\alpha \in F$, we have $\delta(x\alpha) = \delta(x)\alpha = (Ax)\alpha = A(x\alpha)$, whence $x\alpha \in X$. Thus X is a right F-subspace of R^t. [Unless the elements of F all commute with the entries of A, the set X need not be a left F-subspace of R^t.]

Since R^t is a right R-vector-space of dimension t, it does not contain a set of $t + 1$ vectors that are right linearly independent over R. Thus it suffices to prove that any vectors $x_1, \ldots, x_n \in X$ that are right linearly independent over F are also right linearly independent over R. This is clear for $n = 1$.

Now let $n > 1$, let $x_1, \ldots, x_n \in X$ be right linearly independent over F, and assume that $x_2, \ldots, x_n$ are right linearly independent over R. If

$x_1, \ldots, x_n$ are not right linearly independent over R, then $x_1r_1 + \cdots + x_nr_n = 0$ for some $r_i \in R$, not all zero. As $x_2, \ldots, x_n$ are right linearly independent over R, we must have $r_1 \neq 0$. Thus, replacing each r_i by $r_ir_1^{-1}$, we may assume that $r_1 = 1$, that is, $x_1 + x_2r_2 + x_3r_3 + \cdots + x_nr_n = 0$. Consequently, we compute that

$$\begin{aligned} x_2\delta(r_2) &+ \cdots + x_n\delta(r_n) \\ &= A(x_1 + x_2r_2 + \cdots + x_nr_n) + x_2\delta(r_2) + \cdots + x_n\delta(r_n) \\ &= \delta(x_1) + \delta(x_2)r_2 + \cdots + \delta(x_n)r_n + x_2\delta(r_2) + \cdots + x_n\delta(r_n) \\ &= \delta(x_1 + x_2r_2 + \cdots + x_nr_n) = 0. \end{aligned}$$

Using the right linear independence of $x_2, \ldots, x_n$ over R again, we find that $\delta(r_2), \ldots, \delta(r_n)$ are all zero, so that $r_2, \ldots, r_n$ all lie in F. But then the equation $x_1 + x_2r_2 + \cdots + x_nr_n = 0$ contradicts the assumption that $x_1, \ldots, x_n$ are right linearly independent over F. Therefore $x_1, \ldots, x_n$ must be right linearly independent over R, which completes the induction step and thereby completes the proof of the proposition.

By introducing appropriate extra unknowns, Proposition 1.7 can be extended to cover a system of n-th order linear differential equations of the form

$$\delta^n(x) = A_0x + A_1\delta(x) + \cdots + A_{n-1}\delta^{n-1}(x),$$

where $x \in R^k$ and the A_i are $k \times k$ matrices over R. The result we are aiming toward (Theorem 1.9) was proved by Ore by means of a rather messy reduction to a system of n-th order equations in k unknowns of the form just mentioned. However, from a different perspective the problem easily reduces to a system of kn first-order linear differential equations, to which Proposition 1.7 applies directly.

This perspective requires viewing certain objects as homomorphism groups over the differential operator ring. Thus we look at left S-modules where $S = R[\theta; \delta]$ for a differential field R. As R is a subring of S, any left S-modules A and B may also be viewed as left vector spaces over R, simply by ignoring scalar multiplication by elements of $S - R$. Any S-module homomorphism (i.e., S-linear map) from A to B is automatically R-linear, hence the group $\mathrm{Hom}_S(A, B)$ of S-module homomorphisms from A to B is a subgroup of the group $\mathrm{Hom}_R(A, B)$ of R-linear maps from A to B. Provided A and B are both finite-dimensional over R, the maps in $\mathrm{Hom}_R(A, B)$ correspond to rectangular matrices over R, and it is not difficult to specify which matrices correspond to S-module homomorphisms. The criterion can be given as a system of first-order linear differential equations in the entries of the matrices, hence we can conclude that $\mathrm{Hom}_S(A, B)$ is finite-dimensional over the subfield of constants of R, as follows.

PROPOSITION 1.8. *Let R be a differential field, let F be the subfield of constants of R, and set $S = R[\theta; \delta]$. Let A and B be left S-modules such that the dimensions $n = \dim_R(A)$ and $k = \dim_R(B)$ are both finite. Then $\dim_F(\mathrm{Hom}_S(A, B)) \leqq kn$.*

PROOF. This is clear if either n or k is zero, so assume that $n > 0$ and $k > 0$.

Let $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_k\}$ be bases for A and B over R. For $i = 1, \ldots, n$ and $j = 1, \ldots, k$, we have $\theta a_i = \sum_{p=1}^n u_{ip}a_p$ and $\theta b_j = \sum_{q=1}^k v_{jq}b_q$ for some $u_{ip}, v_{jq} \in R$.

Let M denote the set of all $n \times k$ matrices over R. For each matrix $x = (x_{ij})$ in M, let f_x denote the corresponding R-linear map $A \to B$ with respect to the bases chosen above, so that $f_x(a_i) = \sum_{j=1}^k x_{ij}b_j$ for all $i = 1, \ldots, n$. Then the rule $x \mapsto f_x$ defines an F-linear isomorphism of M onto $\mathrm{Hom}_R(A, B)$. Set $X = \{x \in M \mid f_x \text{ is } S\text{-linear}\}$. Then X is mapped isomorphically (as a vector space over F) onto $\mathrm{Hom}_S(A, B)$, hence we need only show that $\dim_F(X) \leqq kn$.

For any $x \in M$, the map f_x is already R-linear, hence f_x is S-linear if and only if $f_x(\theta a) = \theta f_x(a)$ for all $a \in A$. This happens if and only if $f_x(\theta r a_i) = \theta f_x(r a_i)$ for all $r \in R$ and $i = 1, \ldots, n$. Since

$$f_x(\theta r a_i) = f_x((r\theta + \delta(r))a_i) = rf_x(\theta a_i) + \delta(r)f_x(a_i),$$

$$\theta f_x(ra_i) = \theta r f_x(a_i) = r\theta f_x(a_i) + \delta(r)f_x(a_i),$$

it suffices to check the case $r = 1$. Thus $X = \{x \in M \mid f_x(\theta a_i) = \theta f_x(a_i) \text{ for all } i = 1, \ldots, n\}$.

Given $x \in M$ and $i \in \{1, 2, \ldots, n\}$, we compute that

$$f_x(\theta a_i) = \sum_{p=1}^n u_{ip}f_x(a_p) = \sum_{p=1}^n u_{ip}(\sum_{j=1}^k x_{pj}b_j) = \sum_{j=1}^k (\sum_{p=1}^n u_{ip}x_{pj})b_j,$$

$$\begin{aligned}\theta f_x(a_i) &= \sum_{q=1}^k \theta x_{iq}b_q = \sum_{q=1}^k x_{iq}\theta b_q + \sum_{q=1}^k \delta(x_{iq})b_q \\ &= \sum_{q=1}^k x_{iq}(\sum_{j=1}^k v_{qj}b_j) + \sum_{j=1}^k \delta(x_{ij})b_j = \sum_{j=1}^k (\delta(x_{ij}) + \sum_{q=1}^k x_{iq}v_{qj})b_j.\end{aligned}$$

Consequently, X is exactly the set of those $x \in M$ satisfying

$$\delta(x_{ij}) = \sum_{p=1}^n u_{ip}x_{pj} - \sum_{q=1}^k v_{qj}x_{iq} \tag{1.1}$$

for all $i = 1, \ldots, n$ and $j = 1, \ldots, k$. [The transposition of $x_{iq}v_{qj}$ to $v_{qj}x_{iq}$ in the second summation is the only point in the proof where the commutativity of R is needed.]

Equation (1.1) is just a system of kn first-order linear differential equations over R in kn unknowns, hence we conclude from Proposition 1.7 that $\dim_F(X) \leqq kn$, as desired.

Recall that given an element b in a ring S, the expression Sb is used to denote the set $\{sb \mid s \in S\}$, which is a left ideal of S. Thus the quotient S/Sb is a left S-module. It is also cyclic, since it is generated by the single coset $1 + Sb$.

THEOREM 1.9. (ORE [**20**, Satz 3, p. 236]). *Let R be a differential field, let F be the subfield of constants of R, and set $S = R[\theta; \delta]$. Let $a, b \in S$ be nonzero operators of orders n and k, and set $I = \{c \in S \mid ac \in Sb\}$. Then* $\dim_F(I/Sb) \leqq kn$.

PROOF. We consider S-module homomorphisms between the cyclic S-modules S/Sa and S/Sb. It is a trivial exercise to show that the rule $f \mapsto f(1 + Sa)$ defines an F-linear isomorphism of $\mathrm{Hom}_S(S/Sa, S/Sb)$ onto I/Sb.

As a has order n and its leading coefficient is invertible in R, every element of S can be uniquely expressed in the form $p + qa$ with $p, q \in S$ and $\mathrm{ord}(p) \leqq n - 1$. Consequently, S/Sa is finite-dimensional as a vector space over R, with basis $\{1 + Sa, \theta + Sa, \ldots, \theta^{n-1} + Sa\}$. Thus $\dim_R(S/Sa) = n$. Likewise, $\dim_R(S/Sb) = k$.

Now $\dim_F(\mathrm{Hom}_S(S/Sa, S/Sb)) \leqq kn$ by Proposition 1.8, and therefore $\dim_F(I/Sb) \leqq kn$.

For later use, we also record the symmetric version of Theorem 1.9. It can be proved in the same manner as Theorem 1.9, provided one first proves the right-module version of Proposition 1.8. However, there is an easy way to obtain it directly from Theorem 1.9, as follows.

COROLLARY 1.10. *Let R be a differential field, let F be the subfield of constants of R, and set $S = R[\theta; \delta]$. Let $a, b \in S$ be nonzero operators of orders n and k, and set $J = \{d \in S \mid db \in aS\}$. Then* $\dim_F(J/aS) \leqq kn$.

PROOF. Define I as in Theorem 1.9, and set $K = \{(c, d) \in S^2 \mid ac = db\}$, $L = \{(sb, as) \mid s \in S\}$. The projection $(c, d) \mapsto c$ provides an F-linear map f of K onto I, and we observe that $f^{-1}(Sb) = L$. Thus $K/L \cong I/Sb$. Similarly, the projection $(c, d) \mapsto d$ induces an F-linear isomorphism of K/L onto J/aS, and so $J/aS \cong I/Sb$. Therefore $\dim_F(J/aS) \leqq kn$ by Theorem 1.9.

With the aid of Theorem 1.9, we can now complete the remaining cases of Theorem 1.2, as follows.

THEOREM 1.11. *Let R be a differential field, let F be the subfield of constants of R, and set $S = R[\theta; \delta]$. Let $a \in S$ be an operator of positive order* *n. Then $C_S(a)$ is a free $F[a]$-module of finite rank, and that rank is at most n^2.*

PROOF. Set $I = \{c \in S \mid ac \in Sa\}$ and $K = C_S(a)/[C_S(a) \cap Sa]$. Then $\dim_F(I/Sa) \leqq n^2$, by Theorem 1.9. Clearly $C_S(a) \subseteqq I$, hence K embeds in I/Sa, and so $\dim_F(K) \leqq n^2$.

Let $B_0 = \{b \in C_S(a) \mid \operatorname{ord}(b) \leqq 0\}$. As a has positive order, $B_0 \cap Sa = \{0\}$. Thus B_0 embeds in K and so is finite-dimensional over F. Choose a basis $\{b_1, \ldots, b_s\}$ for B_0 over F, and note that the images of $b_1, \ldots, b_s$ in K are linearly independent over F. Thus we may choose elements $b_{s+1}, \ldots, b_t$ in $C_S(a)$ such that the images of $b_1, \ldots, b_t$ form a basis for K over F. Note that $t \leqq n^2$.

We claim that $\{b_1, \ldots, b_t\}$ is a basis for $C_S(a)$ as a free $F[a]$-module, which will prove the theorem.

If the b_i are not linearly independent over $F[a]$, than $p_1b_1 + \cdots + p_tb_t = 0$ for some p_i in $F[a]$, not all zero. Choose such a dependence relation for which the number $m = \max\{\operatorname{ord}(p_1), \ldots, \operatorname{ord}(p_t)\}$ is minimal. Write each $p_i = \alpha_i + aq_i$ with $\alpha_i \in F$ and $q_i \in F[a]$. Then

$$\begin{aligned}\alpha_1b_1 + \cdots + \alpha_tb_t &= (p_1 - aq_1)b_1 + \cdots + (p_t - aq_t)b_t \\ &= -(q_1b_1 + \cdots + q_tb_t)a \in C_S(a) \cap Sa.\end{aligned}$$

Since the b_i map to F-linearly independent elements of K, we must have all $\alpha_i = 0$. Note that now not all q_i can be zero. We have $a(q_1b_1 + \cdots + q_tb_t) = p_1b_1 + \cdots + p_tb_t = 0$ and so $q_1b_1 + \cdots + q_tb_t = 0$. However, this is a dependence relation for which $\max\{\operatorname{ord}(q_1), \ldots, \operatorname{ord}(q_t)\} = m - n$, which contradicts the minimality of m. Therefore the b_i must be linearly independent over $F[a]$.

Let B be the $F[a]$-submodule of $C_S(a)$ spanned by the b_i. We prove by induction on order that any $c \in C_S(a)$ must lie in B. Obviously $B_0 \subseteqq B$, so this is clear when $\operatorname{ord}(c) \leqq 0$.

Now let $\operatorname{ord}(c) = k > 0$, and assume that all operators in $C_S(a)$ with order less than k lie in B. As the b_i span K over F, we must have $c = \alpha_1b_1 + \cdots + \alpha_tb_t + da$ for some $\alpha_i \in F$ and some $d \in S$ such that $da \in C_S(a)$. Clearly $d \in C_S(a)$ as well, and $\operatorname{ord}(d) < \operatorname{ord}(c)$, hence $d \in B$ by the induction hypothesis. Consequently $da \in B$, and thus $c \in B$, which completes the induction step.

Therefore the b_i span $C_S(a)$ over $F[a]$, whence $C_S(a)$ is indeed a free $F[a]$-module with basis $\{b_1, \ldots, b_t\}$, as desired.

The upper bound given for the rank of $C_S(a)$ in Theorem 1.11, higher than the upper bound for the cases covered in Theorem 1.2, can be attained, as the following example shows.

EXAMPLE 1.12. Given a prime number p, there exists a differential field

R of characteristic p, with subfield F of constants, such that $C_{R[\theta;\delta]}(\theta^p)$ is a free $F[\theta^p]$-module of rank p^2.

PROOF. Choose a field K of characteristic p, let R be the field $K(t)$ of rational functions over K in an indeterminate t, and let δ be the usual derivation on R, so that δ is K-linear and $\delta(t) = 1$. Letting F denote the subfield of constants of R, we have $K \subseteqq F$ automatically and $t^p \in F$ because of characteristic p, whence $K(t^p) \subseteqq F$. As the dimension of R over $K(t^p)$ is prime, there are no intermediate fields strictly between $K(t^p)$ and R. On the other hand, $F \neq R$, because $t \notin F$. Thus $F = K(t^p)$, so that R has dimension p over F, with basis $\{1, t, \ldots, t^{p-1}\}$.

Note that for $i = 1, 2, \ldots, p - 1$, the binomial coefficients $\binom{p}{i}$ are divisible by p. Consequently, it follows from Leibniz' Rule that $\delta^p(xy) = \delta^p(x)y + x\delta^p(y)$ for all $x, y \in R$, so that δ^p is a derivation on R. Since $\delta^p(t) = 0$, we infer from this that $\delta^p = 0$ on R. Now

$$\theta^p r = \sum_{i=0}^{p} \binom{p}{i} \delta^i(r)\theta^{p-i} = r\theta^p$$

for all $r \in R$, whence θ^p commutes with all operators in $R[\theta; \delta]$.

Therefore $C_{R[\theta;\delta]}(\theta^p) = R[\theta;\delta]$. To complete the proof, we observe that $R[\theta; \delta]$ is a free $F[\theta^p]$-module of rank p^2, with basis $\{t^i\theta^j \mid i, j = 0, 1, \ldots, p - 1\}$.

We conclude this section by using the finiteness of ranks of centralizers obtained in Theorems 1.2 and 1.11 to show that most commuting pairs of differential operators satisfy polynomials in two variables with constant coefficients, as follows.

THEOREM 1.13. *Let R be a semiprime commutative differential ring, such that the subring F of constants of R is a field, and set $S = R[\theta; \delta]$. Let $a \in S$ be an operator of positive order whose leading coefficient is invertible in R. Given any operator $b \in S$ which commutes with a, there exist polynomials $p_0(a), p_1(a), \ldots, p_{t-1}(a)$ in $F[a]$ such that*

$$p_0(a) + p_1(a)b + \cdots + p_{t-1}(a)b^{t-1} + b^t = 0.$$

In particular, there is a non-constant polynomial $q(x, y)$ over F in two commuting indeterminates x, y such that $q(a, b) = 0$.

PROOF. If the order n of a is invertible in F, then Theorem 1.2 shows that $C_S(a)$ is a free $F[a]$-module of rank at most n. If n is not invertible in F, then F has positive characteristic. In this case, R must be a field (recall the discussion prior to Proposition 1.7), whence Theorem 1.11 shows that $C_S(a)$ is a free $F[a]$-module of rank at most n^2.

Thus in any case $C_S(a)$ is a finitely generated $F[a]$-module. As $F[a]$ is a commutative ring, the existence of the relation $p_0(a) + p_1(a)b + \cdots +$

$p_{t-1}(a)b^{t-1} + b^t = 0$ is a well-known ring-theoretic consequence. It is proved, for instance, in [**4**, Proposition 5.1], and in [**12**, Theorem 12].

II. Pseudo-differential operator rings, Part 1. Centralizers in differential operator rings, in addition to having finite rank over polynomial subrings, often turn out to be commutative rings. To obtain such results, we adopt a method introduced by Schur in a 1904 paper [**24**], apparently forgotten and later rediscovered by later authors. The key to this method is to embed the ring of differential operators in a larger ring, which, following recent papers such as [**1**, p. 229], [**10**, p. 45], [**15**, p. 44], we refer to as the ring of formal linear pseudo-differential operators.

In the present section, we discuss the construction of this pseudo-differential operator ring, and we investigate a few basic properties, such as the existence of inverses and roots. Applications of this ring to commutativity results are given in §III and §IV.

In the case of a noetherian coefficient ring, a natural localization and completion process may be used to construct this pseudo-differential operator ring, and a glance at this construction process helps to indicate why the general construction takes the form it does. Hence, let R be a differential ring, and let X be the set of monic operators (i.e., nonzero operators with leading coefficient 1) in the differential operator ring $S = R[\theta; \delta]$. Then X is a multiplicatively closed set of non-zero-divisors in S. If the coefficient ring R is right noetherian, then X is a right denominator set in S, by [**22**, Proposition 2.2]. (Unfortunately, in the non-noetherian case X need not be a right denominator set in S, as noted in [**22**, Section 2].) This allows us to form the localization of S with respect to X, which is a ring S_X containing S such that all elements of X are invertible in S_X, and all elements of S_X have the form sx^{-1} for $s \in S$ and $x \in X$. We may define the order of any element sx^{-1} in S_X according to the rule $\operatorname{ord}(sx^{-1}) = \operatorname{ord}(s) - \operatorname{ord}(x)$. Then, S_X may be made into a topological ring in which the neighborhoods of 0 are the sets $\{a \in S_X | \operatorname{ord}(a) \leqq k\}$ for all integers k, and the completion of S_X is the pseudo-differential operator ring we wish to study. Any infinite series involving coefficients from R and descending powers of θ converges in this completion. Consequently, the elements of this completion may be represented as formal Laurent series in the indeterminate θ^{-1}. To construct the pseudo-differential operator ring in general, we start with the collection of all formal Laurent series in θ^{-1} with coefficients from R, and build a ring structure there, as follows.

DEFINITION. Let R be a differential ring, and let θ be an indeterminate. We denote by $R((\theta^{-1}; \delta))$ the set of all formal expressions $a = \sum_{i=-\infty}^{n} a_i\theta^i$ where $n \in \mathbf{Z}$ and the $a_i \in R$. [The notation $R((\theta^{-1}; \delta))$ is used because these expressions look like Laurent series in θ^{-1}, and because the multiplication rule we will use for these expressions is a version of Laurent series multiplication that has been twisted by means of δ.] If $a_n \neq 0$, then n is called the *order* of a, denoted ord(a), and a_n is called the *leading coefficient* of a. In case all the $a_i = 0$, the order of a is defined to be $-\infty$, and the leading coefficient of a is defined to be 0. We regard a as unchanged if terms with zero coefficients are appended to or deleted from a. Thus an expression $b = \sum_{j=-\infty}^{m} b_j\theta^j$ in $R((\theta^{-1}; \delta))$ is identified with a if and only if $a_i = b_i$ for all $i \leqq \min\{n, m\}$ while $a_i = 0$ for all $i > \min\{n, m\}$ and $b_j = 0$ for all $j > \min\{n, m\}$.

An addition operation is defined in $R((\theta^{-1}; \delta))$ to be performed by adding coefficients of like powers of θ. Thus for expressions a, b as above, and, say, $n \geqq m$, then

$$a + b = \sum_{i=-\infty}^{m} (a_i + b_i)\theta^i + \sum_{i=m+1}^{n} a_i\theta^i.$$

It is clear that under this operation, $R((\theta^{-1}; \delta))$ becomes an abelian group.

There is a natural topology on $R((\theta^{-1}; \delta))$ making it a topological group, where the neighborhoods of 0 are the sets $\{a \in R((\theta^{-1}; \delta)) | \text{ord}(a) \leqq k\}$, for all $k \in \mathbf{Z}$. We note that $R((\theta^{-1}; \delta))$ is complete in this topology.

Our objective is to make $R((\theta^{-1}; \delta))$ into a ring, in such a way that the natural copy of $R[\theta; \delta]$ inside it is a subring. Assuming we also wish the multiplication to be uniformly continuous with respect to the topology defined above, we may work out what the appropriate definition ought to be as follows.

Given $r \in R$, we have $r\theta = \theta r - \delta(r)$ in $R[\theta; \delta]$. Assuming a suitable ring structure in $R((\theta^{-1}; \delta))$, it follows on multiplying both sides of each term in this equation by θ^{-1} that

$$\theta^{-1}r = r\theta^{-1} - \theta^{-1}\delta(r)\theta^{-1}. \tag{2.1}$$

Replacing r by $\delta^i(r)$ in (2.1), we obtain

$$\theta^{-1}\delta^i(r) = \delta^i(r)\theta^{-1} - \theta^{-1}\delta^{i+1}(r)\theta^{-1} \tag{2.2}$$

for $i = 1, 2, \ldots$. Successive substitutions of (2.2) in (2.1) then yield

$$\begin{aligned} \theta^{-1}r &= r\theta^{-1} - \theta^{-1}\delta(r)\theta^{-1} = r\theta^{-1} - \delta(r)\theta^{-2} + \theta^{-1}\delta^2(r)\theta^{-2} = \cdots \\ &= r\theta^{-1} - \delta(r)\theta^{-2} + \cdots + (-1)^{i-1}\delta^{i-1}(r)\theta^{-i} + (-1)^i\theta^{-1}\delta^i(r)\theta^{-i}. \end{aligned} \tag{2.3}$$

Taking the limit in (2.3), we conclude that $\theta^{-1}r$ should be given by

$$\theta^{-1}r = \sum_{i=1}^{\infty} (-1)^{i-1}\delta^{i-1}(r)\theta^{-i}. \tag{2.4}$$

We may then multiply each term of (2.4) on the left by θ^{-1} and use (2.4) to arrange each of the expressions $\theta^{-1}\delta^{i-1}(r)\theta^{-i}$ as a series with coefficients on the left. Collecting terms, we obtain an expression for $\theta^{-2}r$. Continuing by induction, we find that we should have

$$(2.5)\qquad \theta^{-n}r = \sum_{i=n}^{\infty}(-1)^{i-n}\binom{i-1}{n-1}\delta^{i-n}(r)\theta^{-i}$$

for all $n \in \mathbf{N}$ and $r \in R$.

Unfortunately, equation (2.5) has a somewhat different form than the corresponding equation for positive powers of θ, namely

$$(2.6)\qquad \theta^{n}r = \sum_{i=0}^{n}\binom{n}{i}\delta^{i}(r)\theta^{n-i}$$

for all $n \in \mathbf{N}$ and $r \in R$. In order to put equations (2.5) and (2.6) into a common form, we need generalized bionomial coefficients.

DEFINITION. Recall that for any integer m, the expressions $\binom{m}{0}$, $\binom{m}{1}$, $\binom{m}{2}$, ... are defined by setting $\binom{m}{0} = 1$ and $\binom{m}{i} = m(m-1)\cdots(m-i+1)/i!$ for $i = 1, 2, \ldots$. This latter expression is always an integer, as is well-known for $m \geqq 0$. In case $m = -n$ for some $n \in \mathbf{N}$, we may verify the integrality of $\binom{m}{i}$ by checking that

$$\begin{aligned}\binom{-n}{i} &= (-n)(-n-1)\cdots(-n-i+1)/i! = (-1)^{i}n(n+1)\cdots(n+i-1)/i!\\ &= (-1)^{i}(n+i-1)!/i!(n-1)! = (-1)^{i}\binom{n+i-1}{n-1}\end{aligned}$$

for all $i \in \mathbf{N}$.

In particular, for integers $i > n > 0$, we obtain $\binom{-n}{i-n} = (-1)^{i-n}\binom{i-1}{n-1}$, so that equation (2.5) becomes

$$(2.7)\qquad \theta^{-n}r = \sum_{i=n}^{\infty}\binom{-n}{i-n}\delta^{i-n}(r)\theta^{-i} = \sum_{j=0}^{\infty}\binom{-n}{j}\delta^{j}(r)\theta^{-n-j}.$$

Note that (2.6) may be written in the corresponding form

$$(2.8)\qquad \theta^{n}r = \sum_{i=0}^{\infty}\binom{n}{i}\delta^{i}(r)\theta^{n-i}$$

because $\binom{n}{i} = 0$ for all $i > n$. Thus (2.7) and (2.8) provide a common formula with which to describe multiplication of monomials in $R((\theta^{-1}; \delta))$, namely

$$(2.9)\qquad (r\theta^{n})(s\theta^{m}) = \sum_{i=0}^{\infty}\binom{n}{i}r\delta^{i}(s)\theta^{n+m-i}$$

for all $r, s \in R$ and all $n, m \in \mathbf{Z}$.

The form of (2.9) indicates that it will be convenient to write expressions in $R((\theta^{-1}; \delta))$ in the form $\sum_{i=0}^{\infty} a_i\theta^{n-i}$, with $n \in \mathbf{Z}$ and coefficients $a_i \in R$.

Adopting this mode, we are now ready to give a complete definition of multiplication in $R((\theta^{-1}; \delta))$. [Remember that equations (2.1)–(2.5), (2.7), (2.9) are at present only the anticipated behavior of an operation yet to be defined.]

DEFINITION. Given any expressions $a = \sum_{i=0}^{\infty} a_i\theta^{n-i}$ and $b = \sum_{j=0}^{\infty} b_j\theta^{m-j}$ in $R((\theta^{-1}; \delta))$, with $n, m \in \mathbf{Z}$ and all $a_i, b_j \in R$, we define

$$ab = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (a_i\theta^{n-i})(b_j\theta^{m-j}),$$

where the products of the monomials are defined as

$$(a_i\theta^{n-i})(b_j\theta^{m-j}) = \sum_{k=0}^{\infty} \binom{n-i}{k} a_i\delta^k(b_j)\theta^{n+m-i-j-k}$$

[as in (2.9)]. When the terms are collected according to the powers of θ that appear, this definition for the product ab may be rewritten as

$$ab = \sum_{k=0}^{\infty} c_k\theta^{n+m-k}$$

where each

$$c_k = \sum_{i=0}^{k} \sum_{j=0}^{k-i} \binom{n-i}{k-i-j} a_i\delta^{k-i-j}(b_j).$$

While this formula is rather obnoxious, the first few coefficients are relatively easy to work out. Thus

$$\begin{aligned} ab &= a_0b_0\theta^{n+m} + [na_0\delta(b_0) + a_0b_1 + a_1b_0]\theta^{n+m-1} \\ &+ \left[\binom{n}{2}a_0\delta^2(b_0) + na_0\delta(b_1) + a_0b_2 + (n-1)a_1\delta(b_0) + a_1b_1 + a_2b_0\right]\theta^{n+m-2} \\ &+ [\text{lower terms}]. \end{aligned}$$

Since expressions in $R((\theta^{-1}; \delta))$ may be modified by insertion or deletion of zero terms, it must be checked that the multiplication rule given above is well-defined. This is a trivial exercise, as is the checking of the two distributive laws, and we leave that to the reader.

Associativity, however, is more stubborn. Some authors, beginning with Schur [**24**], simply ignore the question, while some — e.g., [**28**, pp. 133, 134]—say that it is straightforward to prove associativity. Several other authors—e.g., [**1**, p. 229], [**9**, p. 266], [**15**, p. 44]—prove associativity by means of the following trick. First define an alternate multiplication $\circ$ on $R((\theta^{-1}; \delta))$, under which $R((\theta^{-1}; \delta))$ is the ring of ordinary Laurent series in θ^{-1} over R. Then $\circ$, at least, is associative. Next extend δ to all of $R((\theta^{-1}; \delta))$, and define a map ∂ on $R((\theta^{-1}; \delta))$, using the rules

$$\delta\Big(\sum_{i=0}^{\infty} a_i\theta^{n-i}\Big) = \sum_{i=0}^{\infty} \delta(a_i)\theta^{n-i}$$

$$\partial\Big(\sum_{i=0}^{\infty} a_i\theta^{n-i}\Big) = \sum_{i=0}^{\infty} (n-i)a_i\theta^{n-i-1}.$$

It may be checked that on the ring $(R((\theta^{-1};\delta)), \circ)$, the maps δ and ∂ are commuting derivations. The original multiplication on $R((\theta^{-1};\delta))$ is expressible in terms of $\circ$, δ, ∂ as

$$ab = \sum_{k=0}^{\infty} (1/k!)\partial^k(a) \circ \delta^k(b).$$

[If R is not an algebra over $\mathbf{Q}$, care must be taken to ensure that $\partial^k(a)\circ\delta^k(b)$ is divisible by $k!$. The infinite sum is no problem because it converges in the topology mentioned earlier.] Finally, with the aid of this formula a short direct computation shows that our multiplication is associative.

To set up this trick and carry out the details is actually a longer process than a direct proof, which we prefer. Here the only difficulty appears in proving a rather involved identity concerning sums of products of binomial coefficients. We start with a classical identity sometimes known as Vandermonde's Theorem.

LEMMA 2.1. *For all $p, q, k \in \mathbf{Z}$ with $k \geqq 0$,*

$$\sum_{j=0}^{k}\binom{p}{k-j}\binom{q}{j} = \binom{p+q}{k}.$$

PROOF. [7, Theorem 38].

LEMMA 2.2 *For all $p, q, i, k \in \mathbf{Z}$ with $i, k \geqq 0$,*

$$\sum_{j=0}^{k}\binom{p}{k+i-j}\binom{k+i-j}{i}\binom{q}{j} = \binom{p}{i}\binom{p+q-i}{k}.$$

PROOF. We first claim that

$$\binom{p}{k+i-j}\binom{k+i-j}{i} = \binom{p}{i}\binom{p-i}{k-j} \tag{2.10}$$

for all $j = 0, \ldots, k$. If either $i = 0$ or $j = k$, this is trivial. In case $i > 0$ and $j < k$, we check that

$$\begin{aligned}
&\binom{p}{i}\binom{p-i}{k-j}\\
&=[p(p-1)\cdots(p-i+1)/i!][(p-i)(p-i-1)\cdots(p-i-k+j+1)/(k-j)!]\\
&=[p(p-1)\cdots(p-i-k+j+1)/(k+i-j)!][(k+i-j)!/i!(k-j)!]\\
&=\binom{p}{k+i-j}\binom{k+i-j}{i},
\end{aligned}$$

as claimed.

Using (2.10) in conjunction with Lemma 2.1, we conclude that

$$\sum_{j=0}^{k}\binom{p}{k+i-j}\binom{k+i-j}{i}\binom{q}{j}=\sum_{j=0}^{k}\binom{p}{i}\binom{p-i}{k-j}\binom{q}{j}=\binom{p}{i}\binom{p+q-i}{k},$$

as desired.

With the aid of Lemma 2.2, we can now check associativity of multiplication in $R((\theta^{-1};\delta))$. First consider monomials $a\theta^p$, $b\theta^q$, $c\theta^r$, for some $a, b, c \in R$ and some $p, q, r \in \mathbf{Z}$. On one hand,

$$\begin{aligned}[(a\theta^p)(b\theta^q)](c\theta^r) &= \left(\sum_{i=0}^{\infty}\binom{p}{i}a\delta^i(b)\theta^{p+q-i}\right)(c\theta^r)\\ &= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k}\binom{p}{i}\binom{p+q-i}{k-i}a\delta^i(b)\delta^{k-i}(c)\right)\theta^{p+q+r-k}.\end{aligned}$$

On the other hand,

$$\begin{aligned}(a\theta^p)[(b\theta^q)(c\theta^r)] &= (a\theta^p)\left(\sum_{j=0}^{\infty}\binom{q}{j}b\delta^j(c)\theta^{q+r-j}\right)\\ &= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\binom{p}{k-j}\binom{q}{j}a\delta^{k-j}(b\delta^j(c))\right)\theta^{p+q+r-k}\\ &= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\sum_{i=0}^{k-j}\binom{p}{k-j}\binom{q}{j}\binom{k-j}{i}a\delta^i(b)\delta^{k-j-i}(\delta^j(c))\right)\theta^{p+q+r-k}\\ &= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k}\left(\sum_{j=0}^{k-i}\binom{p}{k-j}\binom{q}{j}\binom{k-j}{i}\right)a\delta^i(b)\delta^{k-i}(c)\right)\theta^{p+q+r-k}.\end{aligned}$$

For $k = 0, 1, 2, \ldots$ and $i = 0, 1, \ldots, k$ and $j = 0, 1, \ldots, k-i$, we have

$$\sum_{j=0}^{k-i}\binom{p}{k-j}\binom{q}{j}\binom{k-j}{i}=\binom{p}{i}\binom{p+q-i}{k-i}$$

by Lemma 2.2. Therefore $[(a\theta^p)(b\theta^q)](c\theta^r) = (a\theta^p)[(b\theta^q)(c\theta^r)]$.

Thus multiplication of monomials in $R((\theta^{-1};\delta))$ is associative. Invoking the distributive laws, it follows that multiplication of finite sums of monomials is associative. Inasmuch as the set of finite sums of monomials is dense in $R((\theta^{-1};\delta))$ in a topology with respect to which multiplication is uniformly continuous, we conclude that multiplication in $R((\theta^{-1};\delta))$ is associative.

Thus $R((\theta^{-1};\delta))$ is finally a ring. We still need to check that the natural copy of $R[\theta;\delta]$ inside $R((\theta^{-1};\delta))$ is a subring. It suffices to check that any product of monomials $a\theta^n$ and $b\theta^m$, where $a, b \in R$ and $n, m \in \mathbf{Z}^+$, is the same whether computed in $R[\theta;\delta]$ or in $R((\theta^{-1};\delta))$. In $R[\theta;\delta]$,

$$(a\theta^n)(b\theta^m) = \sum_{k=0}^{n}\binom{n}{k}a\delta^k(b)\theta^{n-k+m}.$$

On the other hand, in $R((\theta^{-1};\delta))$ we have

$$(a\theta^n)(b\theta^m) = \sum_{k=0}^{\infty}\binom{n}{k}a\delta^k(b)\theta^{n+m-k} = \sum_{k=0}^{n}\binom{n}{k}a\delta^k(b)\theta^{n+m-k}$$

because $\binom{n}{k} = 0$ for all $k > n$. Thus the product $(a\theta^n)(b\theta^m)$ is the same in each ring, as desired.

To mark the successful conclusion of checking the ring axioms in $R((\theta^{-1};\delta))$, we record the following proposition.

PROPOSITION 2.3. *If R is any differential ring, then $R((\theta^{-1};\delta))$ is a ring, containing $R[\theta;\delta]$ as a unital subring.*

DEFINITION. We shall refer to $R((\theta^{-1};\delta))$ as the *ring of formal linear pseudo-differential operators* over R. In those contexts where $R[\theta;\delta]$ is known as the Ore extension of R, we propose calling $R((\theta^{-1};\delta))$ the *Schur extension* of R.

The usefulness of $R((\theta^{-1};\delta))$ lies in the existence of suitable inverses and roots therein, as follows.

PROPOSITION 2.4. *Let R be a differential ring, and let a be a nonzero operator in $R((\theta^{-1};\delta))$ with order n and leading coefficient a_0. If a_0 is invertible in R, then a is invertible in $R((\theta^{-1};\delta))$; moreover, a^{-1} has order $-n$ and leading coefficient a_0^{-1}.*

PROOF. Write out $a = \sum_{i=0}^{\infty} a_i\theta^{n-i}$ with the $a_i \in R$. We seek an operator $b = \sum_{j=0}^{\infty} b_j\theta^{-n-j}$ in $R((\theta^{-1};\delta))$ such that $ab = 1$. Expanding the product ab and comparing coefficients in the equation $ab = 1$, we find that $a_0b_0 = 1$ and

$$\sum_{i=0}^{k}\sum_{j=0}^{k-i}\binom{n-i}{k-i-j}a_i\delta^{k-i-j}(b_j) = 0 \tag{2.11}$$

for all $k = 1, 2, \ldots$. We satisfy the equation $a_0b_0 = 1$ with $b_0 = a_0^{-1}$, which exists by hypothesis. The case $k = 1$ of (2.11) reads $na_0\delta(b_0) + a_0b_1 + a_1b_0 = 0$, which is solvable also, namely $b_1 = -a_0^{-1}a_1b_0 - n\delta(b_0)$.

Now let $k > 1$, and assume we have found elements $b_0, b_1, \ldots, b_{k-1}$ in R satisfying the first $k-1$ cases of (2.11). The next case of (2.11) reads

$$a_0b_k + \sum_{j=0}^{k-1}\binom{n}{k-j}a_0\delta^{k-j}(b_j) + \sum_{i=1}^{k}\sum_{j=0}^{k-i}\binom{n-i}{k-i-j}a_i\delta^{k-i-j}(b_j) = 0.$$

Since the term a_0b_k is the only one containing b_k, we may solve this equation for b_k.

Therefore by induction we obtain elements $b_0, b_1, \ldots$ in R such that $a_0b_0 = 1$ and all cases of (2.11) are satisfied. As a result, the operator $b = \sum_{j=0}^{\infty} b_j\theta^{-n-j}$ in $R((\theta^{-1};\delta))$ satisfies the equation $ab = 1$.

The leading coefficient of b is a_0^{-1}, which is invertible in R. Repeating the process above, we obtain an operator c in $R((\theta^{-1};\delta))$ such that $bc = 1$. Now $a = abc = c$, whence $ba = 1$. Therefore b is the inverse of a in $R((\theta^{-1};\delta))$.

COROLLARY 2.5. *Let R be a differential ring. If R is a division ring, then so is $R((\theta^{-1};\delta))$.*

LEMMA 2.6. *Let R be a commutative differential ring, let $b \in R((\theta^{-1};\delta))$, and let $n \in \mathbf{N}$. Write out $b = \sum_{i=0}^{\infty} b_i\theta^{m-i}$ and $b^n = \sum_{j=0}^{\infty} c_j\theta^{mn-j}$ with $m \in \mathbf{Z}$ and each $b_i, c_j \in R$. Then for $j = 1, 2, \ldots, c_j = nb_0^{n-1}b_j + d_j$, where d_j lies in the differential subring of R generated by $b_0, \ldots, b_{j-1}$.*

PROOF. This is clear for $n = 1$. Now assume the lemma holds for some n, and write out $b^{n+1} = \sum_{k=0}^{\infty} e_k\theta^{mn+m-k}$ with all $e_k \in R$. Comparing coefficients of powers of θ in the equation $b^{n+1} = b^n b$, we obtain

$$e_k = \sum_{i=0}^{k}\sum_{j=0}^{k-i}\binom{mn-i}{k-i-j}c_i\delta^{k-i-j}(b_j)$$

for all $k = 0, 1, 2, \ldots$. In case $k > 0$, there are exactly two terms involving either c_k or b_k, namely c_0b_k and c_kb_0, and

$$c_0b_k + c_kb_0 = b_0^nb_k + (nb_0^{n-1}b_k + d_k)b_0 = (n+1)b_0^nb_k + d_kb_0.$$

All other terms just involve $c_0, \ldots, c_{k-1}, b_0, \ldots, b_{k-1}$ and their derivatives. In addition, each of $c_0, \ldots, c_{k-1}$ involves only $b_0, \ldots, b_{k-1}$ and their derivatives. Thus $e_k = (n+1)b_0^nb_k + f_k$, where f_k lies in the differential subring of R generated by $b_0, \ldots, b_{k-1}$, completing the induction step.

PROPOSITION 2.7. *Let R be a commutative differential ring, and let a be a nonzero operator in $R((\theta^{-1};\delta))$ with order s and leading coefficient a_0, such that a_0 is invertible in R. Let n be a positive integer which is a divisor of s and which is invertible in R. If there exists $b_0 \in R$ such that $b_0^n = a_0$, then there exists an operator b in $R((\theta^{-1};\delta))$ such that $b^n = a$, while b has order s/n and leading coefficient b_0.*

PROOF. Write out $a = \sum_{i=0}^{\infty}a_i\theta^{s-i}$ with all $a_i \in R$. Set $m = s/n$. We seek an operator $b = \sum_{i=0}^{\infty}b_i\theta^{m-i}$ in $R((\theta^{-1};\delta))$ such that $b^n = a$. Expanding b^n with the help of Lemma 2.6, and comparing coefficients of powers of θ in the equation $b^n = a$, we find that $b_0^n = a_0$ and

$$nb_0^{n-1}b_j + d_j = a_j \tag{2.12}$$

for each $j = 1, 2, \ldots$, where d_j lies in the differential subring of R generated by $b_0, \ldots, b_{j-1}$.

We are given a particular solution to the equation $b_0^n = a_0$. Note that since a_0 is invertible in R, so is b_0. As n is also invertible in R, we can successively solve the equations (2.12) for $b_1, b_2, \ldots$, providing us with the coefficients for an operator b in $R((\theta^{-1}; \delta))$ such that $b^n = a$.

III. **Pseudo-differential operator rings, Part 2.** We investigate in this section centralizers in rigns of pseudo-differential operators, showing that they are often commutative. In some cases, we can also precisely identify the centralizer.

As before, we work over a commutative differential ring R. We consider a nonzero operator a in the ring T of pseudo-differential operators over R, and we study the centralizer $C_T(a)$. Provided the order of a and the leading coefficient of a are both non-zero-divisors in R, we show that $C_T(a)$ is commutative. In case R has no $\mathbf{Z}$-torsion, while a has order 0 and the derivative of its leading coefficient is a non-zero-divisor in R, we again show that $C_T(a)$ is commutative. A more general result combining the $\mathbf{Z}$-torsion-free cases of both results in also proved. The size of $C_T(a)$ can be specified in case R is semiprime, the subring F of constants of R is a field, the order of a is positive and invertible in F, and the leading coefficient of a is invertible in R. In this case, $C_T(a)$ contains a natural copy of the Laurent series field $F((a^{-1}))$, and the dimension of $C_T(a)$ over $F((a^{-1}))$ is a divisor of the order of a.

These results are all based on techniques developed by Schur in **[24]** for an unspecified case, probably the ring of complex-valued C^∞ functions on the real line. Schur dealt with an operator $a \in T$ of positive order n, constructing an n-th root b for a as in Proposition 2.7, and proving that $C_T(a)$ is just the natural copy of the Laurent series field $F((b^{-1}))$ inside T. It is immediate from this description that $C_T(a)$ is commutative. The case in which R is the ring of complex power series in a single indeterminate was worked out by Mumford, using the same procedure, in **[17**, Lemma, p. 140]. The commutativity of $C_T(a)$, when R is a differential field of characteristic zero, was proved without the use of n-th roots by Van Deuren [**26**, Théorème 1; **27**, Théorème III.2.1]. In this case, Van Deuren also proved that $C_T(a)$ is finite-dimensional over the subfield $F((a^{-1}))$ [**26**, Proposition 3; **27**, Proposition IV.3.1.2].

The completeness of a pseudo-differential operator ring in its natural topology means that it contains natural copies of various Laurent series rings. For instance, let R be a commutative differential ring, let F be the subring of constants of R, and let a be an invertible operator in $R((\theta^{-1}; \delta))$ with positive order n. Then a^k has order kn for any $k \in \mathbf{Z}$, hence we see that any series $\sum_{k=-\infty}^{m} r_k a^k$, with $m \in \mathbf{Z}$ and all $r_k \in R$, must converge in $R((\theta^{-1}; \delta))$. In particular, the set L of such series for which all the r_k lie in F is a subring of $R((\theta^{-1}; \delta))$, isomorphic to the Laurent series ring $F((t))$ over F in an indeterminate t, via the map

$$\sum_{k=-m}^{\infty} \alpha_k t^k \mapsto \sum_{k=-\infty}^{m} \alpha_{-k} a^k.$$

Thus we shall use the notation $F((a^{-1}))$ for this subring L. Note that since the constants in F commute with all operators in $R((\theta^{-1}; \delta))$, the subring $F((a^{-1}))$ is always contained in the centralizer of a in $R((\theta^{-1}; \delta))$, and is also contained in the centralizer of any operator that commutes with a. For an invertible operator b in $R((\theta^{-1}; \delta))$ with negative order, $R((\theta^{-1}; \delta))$ contains a corresponding Laurent series subring $F((b))$, which is contained in the centralizer of any operator commuting with b.

THEOREM 3.1. *Let R be a commutative differential ring, let F be the subring of constants of R, and set $T = R((\theta^{-1}; \delta))$. Let $a \in T$ be an operator with positive order n and leading coefficient a_0. Assume that n is invertible in F and a_0 is invertible in R, and that a_0 has an n-th root in R. Then a has an n-th root b in T, and $C_T(a) = F((b^{-1}))$. Thus $C_T(a)$ is commutative, and is a free $F((a^{-1}))$-module of rank n.*

PROOF. The existence of b is given by Proposition 2.7. Note that b has order 1, and write out $a = \sum_{i=0}^{\infty} a_i\theta^{n-i}$ and $b = \sum_{j=0}^{\infty} b_j\theta^{1-j}$ with all $a_i, b_j \in R$ and $b_0^n = a_0$. Since a_0 is invertible in R, so is b_0.

We claim that given $c \in C_T(a)$ and $t \in \mathbf{Z}$ with $\operatorname{ord}(c) \leqq t$, there exists $\alpha \in F$ such that $c - \alpha b^t$ has order less than t. We may write $c = \sum_{j=0}^{\infty} c_j\theta^{t-j}$ with the $c_j \in R$.

Comparing coefficients of θ^{n+t-1} in the equation $ac = ca$, we find that

$$na_0\delta(c_0) + a_0c_1 + a_1c_0 = tc_0\delta(a_0) + c_0a_1 + c_1a_0,$$

whence $na_0\delta(c_0) = tc_0\delta(a_0)$. As $a_0 = b_0^n$, it follows that $nb_0^n\delta(c_0) = ntc_0b_0^{n-1}\delta(b_0)$, and consequently $\delta(c_0) = tc_0b_0^{-1}\delta(b_0)$, because n and b_0 are invertible in R. Setting $\alpha = c_0b_0^{-t}$, we compute that

$$\delta(\alpha) = \delta(c_0)b_0^{-t} - tc_0b_0^{-t-1}\delta(b_0) = 0,$$

hence $\alpha \in F$ and $c_0 = \alpha b_0^t$. Since b^t has order t and leading coefficient b_0^t, we conclude that $c - \alpha b^t$ has order at most $t - 1$, as claimed.

Note that $b \in C_T(a)$ (because $b^n = a$), and so $F((b^{-1})) \subseteqq C_T(a)$. Given any nonzero $c \in C_T(a)$ with order s, we may inductively apply the claim above to obtain constants $\alpha_0, \alpha_1, \ldots$ in F such that for all $k = 0, 1, 2, \ldots,$ the operator

$$c - \alpha_0 b^s - \alpha_1 b^{s-1} - \cdots - \alpha_k b^{s-k}$$

has order less than $s - k$. As a result, $c = \sum_{k=0}^{\infty} \alpha_k b^{s-k}$, whence $c \in F((b^{-1}))$.

Therefore $C_T(a) = F((b^{-1}))$. In particular, it follows that $C_T(a)$ is commutative. Also, we note that $C_T(a)$ is a free $F((a^{-1}))$-module with basis $\{1, b, \ldots, b^{n-1}\}$.

The analogous version of Theorem 3.1 for operators of negative order holds as well, but we shall not write out the detailed statement.

COROLLARY 3.2. *Let R be a commutative differential ring, and set $T = R((\theta^{-1};\ \delta))$. Let $a \in T$ be a nonzero operator with order n and leading coefficient a_0. If n and a_0 are non-zero-divisors in R, then $C_T(a)$ is commutative.*

PROOF. There is no loss of generality in enlarging R. Namely, if R is embedded in a differential ring R'', then $R((\theta^{-1};\ \delta))$ embeds in $R''(\theta^{-1};\ \delta))$, hence it suffices to show that the centralizer of a in $R''((\theta^{-1};\ \delta))$ is commutative. We construct such an embedding so that n and a_0 are invertible in R'', and a_0 has an n-th root in R''. If $n > 0$, then Theorem 3.1 immediately shows that the centralizer of a in $R''((\theta^{-1};\ \delta))$ is commutative. If $n < 0$, then we work with a^{-1}, which exists in $R''((\theta^{-1};\ \delta))$ by Proposition 2.4. As a^{-1} has order $-n$ and leading coefficient a_0^{-1}, both invertible in R'', and a_0^{-1} has an n-th root in R'', Theorem 3.1 shows that the centralizer of a^{-1} in $R''((\theta^{-1};\ \delta))$ is commutative. This suffices because a and a^{-1} have the same centralizer in $R''((\theta^{-1};\ \delta))$.

Therefore it only remains to embed R in a commutative differential ring R'' is such a way that n and a_0 are invertible in R'', and a_0 has an n-th root in R''.

First let R' be the commutative ring obtained from R by formally inverting n and a_0. (See, e.g., [**4**, pp. 36, 37].) Since n and a_0 are non-zero-divisors in R, the natural map $R \to R'$ is an embedding, and we may identify R with its image in R'. Extend δ to R' via the quotient rule, so that R' becomes a differential ring containing R as a differential subring.

Now consider the polynomial ring $R'[x]$ over R' in an indeterminate x, and extend δ from R' to $R'[x]$ so that $\delta(x) = n^{-1}a_0^{-1}\delta(a_0)x$. We check that

$$\delta(x^n - a_0) = nx^{n-1}n^{-1}a_0^{-1}\delta(a_0)x - \delta(a_0) = a_0^{-1}\delta(a_0)(x^n - a_0),$$

whence $\delta(x^n - a_0)$ is a multiple of $x^n - a_0$. Thus if I is the ideal of $R'[x]$ consisting of all multiples of $x^n - a_0$, then $\delta(I) \subseteq I$. Consequently, δ induces a derivation on the ring $R'' = R'[x]/I$. As $x^n - a_0$ is a monic polynomial, the natural map $R' \to R'[x] \to R''$ is an embedding.

Therefore we have embedded R as a differential subring of the commutative differential ring R'', such that n and a_0 are invertible in R'', and a_0 has an n-th root in R''.

For pseudo-differential operators of order 0, there are also cases in which the centralizer must be commutative, but completely different methods of proof are needed, as in the following theorem and corollary.

THEOREM 3.3. *Let R be a commutative differential* **Q**-*algebra, and set*

$T = R((\theta^{-1}; \delta))$. Let $a \in T$ be a nonzero operator with order 0 *and leading coefficient a_0. If $\delta(a_0)$ is invertible in R, then $C_T(a)$ is isomorphic (as a ring) to R. In particular, $C_T(a)$ is commutative.*

PROOF. Write out $a = \sum_{i=0}^{\infty} a_i\theta^{-i}$ with the $a_i \in R$. We claim that any nonzero operator b in $C_T(a)$ must also have order 0. Write $b = \sum_{j=0}^{\infty} b_j\theta^{m-j}$ with $m \in \mathbf{Z}$, the $b_j \in R$, and $b_0 \neq 0$. Comparing coefficients of θ^{m-1} in the equation $ab = ba$, we find that $a_0b_1 + a_1b_0 = mb_0\delta(a_0) + b_0a_1 + b_1a_0$, whence $mb_0\delta(a_0) = 0$. Since $\delta(a_0)$ is invertible in R, it follows that $mb_0 = 0$. As $b_0 \neq 0$ and R is a $\mathbf{Q}$-algebra, this forces $m = 0$. Thus b does have order 0, as claimed.

For all nonzero operators $b \in C_T(a)$, set $\varphi(a)$ equal to the leading coefficient of b. Also set $\varphi(0) = 0$. Thus we obtain a map $\varphi\colon C_T(a) \to R$. Inasmuch as all nonzero operators in $C_T(a)$ have order 0, we see that φ is a unital ring homomorphism, and that φ is injective.

It remains to show that φ is surjective. Given $c_0 \in R$, we seek an operator $c = \sum_{j=0}^{\infty} c_j\theta^{-j}$ in T such that $ac = ca$. Comparing coefficients of powers of θ in this equation, we see that we need

$$\sum_{i=0}^{k}\sum_{j=0}^{k-i}\binom{-i}{k-i-j}a_i\delta^{k-i-j}(c_j) = \sum_{j=0}^{k}\sum_{i=0}^{k-j}\binom{-j}{k-i-j}c_j\delta^{k-i-j}(a_i) \tag{3.1}$$

for all $k = 0, 1, 2, \ldots$.

The cases $k = 0$ and $k = 1$ of (3.1) just say $a_0c_0 = c_0a_0$ and $a_0c_1 + a_1c_0 = c_0a_1 + c_1a_0$ and so provide no restrictions. The case $k = 2$ reads

$$a_0c_2 - a_1\delta(c_0) + a_1c_1 + a_2c_0 = c_0a_2 - c_1\delta(a_0) + c_1a_1 + c_2a_0,$$

whence $c_1\delta(a_0) = a_1\delta(c_0)$. Since $\delta(a_0)$ is invertible in R, we can solve this equation for c_1.

Now assume, for some $k \geqq 2$, that we have found $c_0, \ldots, c_{k-1}$ in R satisfying the cases $0, 1, \ldots, k$ of (3.1). In the case $k + 1$ of (3.1), the unknown c_{k+1} appears in only two terms, namely the term a_0c_{k+1} on the left-hand side and the term $c_{k+1}a_0$ on the right-hand side. Thus this equation provides no restriction on c_{k+1}. The unknown c_k appears three times, in the term a_1c_k on the left-hand side, and in the terms $-kc_k\delta(a_0)$ and c_ka_1 on the right-hand side. Thus the equation may be rewritten to express $kc_k\delta(a_0)$ as an R-linear combination of $c_0, c_1, \ldots, c_{k-1}$ and their derivatives. As k and $\delta(a_0)$ are invertible in R, this equation is solvable for c_k.

Therefore we can inductively find $c_1, c_2, \ldots$ in R which, together with the given c_0, satisfy all cases of (3.1). Setting $c = \sum_{j=0}^{\infty} c_j\theta^{-j}$, we obtain an operator $c \in C_T(a)$ such that $\varphi(c) = c_0$, proving that φ is surjective.

Therefore φ is a ring isomorphism of $C_T(a)$ onto R.

DEFINITION. We say that a ring R *has no* $\mathbf{Z}$*-torsion* if R is torsion-free when considered as an abelian group, that is, $nr \neq 0$ for all nonzero $n \in \mathbf{Z}$ and all nonzero $r \in R$.

COROLLARY 3.4. *Let R be a commutative differential ring with no $\mathbf{Z}$-torsion, and set $T = R((\theta^{-1}; \delta))$. Let $a \in T$ be a nonzero operator with order* 0 *and leading coefficient a_0. If $\delta(a_0)$ is a non-zero-divisor in R, then $C_T(a)$ is commutative.*

PROOF. Set $R' = R \otimes_{\mathbf{Z}} \mathbf{Q}$, and note that the natural map $R \to R'$ is an embedding, because R has no $\mathbf{Z}$-torsion. Identifying R with its image in R', we may write all elements of R' in the form r/n, for $r \in R$ and $n \in \mathbf{N}$. The derivation δ on R may be extended to R' by setting $\delta(r/n) = \delta(r)/n$ for all $r \in R$ and $n \in \mathbf{N}$. Then R' is a commutative differential $\mathbf{Q}$-algebra containing R as a differential subring.

Now let R'' be the commutative ring obtained from R' by formally inverting $\delta(a_0)$. As $\delta(a_0)$ is a non-zero-divisor in R, it is also a non-zero-divisor in R', hence the natural map $R' \to R''$ is an embedding. Extend δ from R' to R'' via the quotient rule. Thus R'' becomes a commutative differential $\mathbf{Q}$-algebra, containing R as a differential subring, such that $\delta(a_0)$ is invertible in R''.

According to Theorem 3.3, the centralizer of a in $R''((\theta^{-1}; \delta))$ is commutative. As T is embedded in $R''((\theta^{-1}; \delta))$; we conclude that $C_T(a)$ is commutative.

Corollaries 3.2 and 3.4 cover all non-central operators in case R is a field of characteristic zero, as follows.

THEOREM 3.5. *Let R be a field of characteristic zero with a nonzero derivation, let F be the subfield of constants of R, and set $T = R((\theta^{-1}; \delta))$. Then the center of T is F. For any non-central operator $a \in T$, the centralizer $C_T(a)$ is commutative.*

PROOF. Clearly F is contained in the center of T. Elements of $R - F$ fail to commute with θ and so are not central. There is some $r \in R$ satisfying $\delta(r) \neq 0$, and because of characteristic zero we see that operators in $T - R$ fail to commute with r. Thus the center of T is exactly F.

Now consider an operator $a \in T - F$. Because R is a field, the leading coefficient a_0 of a is invertible in R. If a has nonzero order n, then n is invertible in F, and Corollary 3.2 shows that $C_T(a)$ is commutative. Now suppose that a has order 0. If $\delta(a_0) \neq 0$, then $\delta(a_0)$ is invertible in R, and Corollary 3.4 shows that $C_T(a)$ is commutative. If $\delta(a_0) = 0$, then $a_0 \in F$, whence $a \neq a_0$. Since a_0 is central in T, we have $C_T(a) = C_T(a - a_0)$. As $a - a_0$ is a nonzero operator with negative order, the first case covered above shows that $C_T(a - a_0)$ is commutative. Thus $C_T(a)$ is commutative in this case also.

We may also combine Corollaries 3.2 and 3.4, when R has no $\mathbf{Z}$-torsion, to show that a pseudo-differential operator $a = \sum_{i=0}^{\infty} a_i\theta^{n-i}$, for which the set $\{\delta(a_n)\} \cup \{a_i | i \neq n\}$ acts like a non-zero-divisor, has a commutative centralizer. This may be illustrated most easily for the case of the ring R of complex-valued C^∞ functions on the real line. Suppose, for example, that we are given an ordinary differential operator $a = a_0 + a_1\theta + a_2\theta^2$. If I_2 is an open interval on which a_2 does not vanish, then by applying Corollary 3.2 to the ring R_2 of complex-valued C^∞ functions on I_2, we see that the centralizer of a is "commutative on I_2", that is, restricting the coefficients of a to functions on I_2, we obtain an operator in $R_2((\theta^{-1}; \delta))$ whose centralizer is commutative. Next, if I_1 is an open interval on which a_2 vanishes but a_1 does not vanish, a second application of Corollary 3.2 shows that the centralizer of a is "commutative on I_1". Finally, if I_0 is an open interval on which a_1 and a_2 vanish but $\delta(a_0)$ does not vanish, then Corollary 3.4 shows that the centralizer of a is "commutative on I_0". Provided $I_0 \cup I_1 \cup I_2$ is dense in $\mathbf{R}$, the information above may be patched together to show that the centralizer of a in $R((\theta^{-1}; \delta))$ is commutative.

In order to recast this method in an algebraic form, we must enlarge our coefficient ring to its "maximal quotient ring". For the benefit of non-ring-theoretic readers, we sketch the concepts involved, sticking to the commutative case. We shall follow the development in [**14**], with a few modifications.

DEFINITION. Let R be a commutative ring. An ideal I of R is said to be *dense* (or *rational in R*) provided the annihilator of I is zero, that is, the only element $r \in R$ satisfying $rI = \{0\}$ is $r = 0$. Several properties of dense ideals should be noted [**14**, p. 37]:

(a) R is dense.

(b) If I is a dense ideal of R, then any ideal containing I is dense.

(c) If I and I' are dense ideals of R, then so are II' and $I \cap I'$. [Recall that the ideal product II' is defined as the set of all sums of products rr' with $r \in I$ and $r' \in I'$.]

DEFINITION. A *quotient ring* (or *ring of quotients*) of R is a commutative ring Q which contains R as a unital subring in such a way that for any $x \in Q$, the ideal $\{r \in R \mid xr \in R\}$ is dense in R. Thus for any $x \in Q$, there is a dense ideal I in R such that $xI \subseteq R$ and $xI \neq \{0\}$ [**14**, Proposition 6, p. 40]. Also, if $y \in Q$ and J is any dense ideal of R, then $yJ \neq \{0\}$ unless $y = 0$ [**14**, Corollary, p. 41].

There is always a "largest" quotient ring of R, which may be constructed as a direct limit of the homomorphism groups $\mathrm{Hom}_R(I, R)$, where I ranges over the dense ideals of R [**14**, pp. 37, 38]. The quotient ring Q obtained this way has the property that all quotient rings of R embed in Q [**14**, Proposition 6, p. 40]. For this reason, Q is called the *maximal quotient ring* (or *maximal ring of quotients*, or *complete ring of quotients*) of R. It follows easily from the construction of Q that given any dense ideal I in R and any R-linear map $f\colon I \to Q$, there is a unique $y \in Q$ such that $f(r) = yr$ for all $r \in I$. [Namely, $f^{-1}(R)$ is a dense ideal of R, so the restriction of f to $f^{-1}(R)$ is one of the "fractions" used in constructing Q.]

PROPOSITION 3.6. [**25**, Corollary 2 to Theorem 4.1]. *Let R be a commutative ring, and let Q be its maximal quotient ring. Then any derivation on R extends uniquely to a derivation on Q.*

PROOF. As usual, the derivation on R is denoted δ.

Given any $x \in Q$, we claim there exists a unique element $y \in Q$ such that for some dense ideal I of R, we have $xI \subseteqq R$ and $yr = \delta(xr) - x\delta(r)$ for all $r \in I$.

There is at least one dense ideal I of R such that $xI \subseteqq R$. Define a map $f\colon I \to Q$ by setting $f(r) = \delta(xr) - x\delta(r)$ for all $r \in I$, and note that f is additive. Moreover, for all $r \in I$ and $s \in R$,

$$f(rs) = \delta(xrs) - x\delta(rs) = \delta(xr)s + xr\delta(s) - x\delta(r)s - xr\delta(s) = f(r)s,$$

hence f is an R-linear map. Consequently, there exists $y \in Q$ such that $yr = f(r)$ for all $r \in I$.

Suppose we also have an element $y' \in Q$ and a dense ideal I' of R such that $xI' \subseteqq R$ and $y'r = \delta(xr) - x\delta(r)$ for all $r \in I'$. Then $I \cap I'$ is a dense ideal of R, and $(y - y')r = 0$ for all $r \in I \cap I'$, whence $y = y'$. Therefore y is unique, and the claim is proved.

For any $x \in Q$, we now define $\delta'(x)$ to be the unique element $y \in Q$ given in the claim. Note that if $x \in R$, then since $\delta(x)r = \delta(xr) - x\delta(r)$ for all $r \in R$, we must have $\delta'(x) = \delta(x)$. Thus $\delta'\colon Q \to Q$ is a map extending δ.

Given $x_1, x_2 \in Q$, choose dense ideals I_1, I_2 in R such that each $x_jI_j \subseteqq R$ and $\delta'(x_j)r = \delta(x_jr) - x_j\delta(r)$ for all $r \in I_j$. Then $I_1 \cap I_2$ is a dense ideal of R, and $(x_1 + x_2)(I_1 \cap I_2) \subseteqq R$. Since

$$\begin{aligned}(\delta'(x_1) + \delta'(x_2))r &= \delta(x_1r) - x_1\delta(r) + \delta(x_2r) - x_2\delta(r)\\ &= \delta((x_1 + x_2)r) - (x_1 + x_2)\delta(r)\end{aligned}$$

for all $r \in I_1 \cap I_2$, we see that $\delta'(x_1 + x_2) = \delta'(x_1) + \delta'(x_2)$.

Also, I_1I_2 is a dense ideal of R, and $(x_1x_2)(I_1I_2) = (x_1I_1)(x_2I_2) \subseteqq R$. For any $r_1 \in I_1$ and $r_2 \in I_2$, we obtain

$$\begin{aligned}(\delta'(x_1)x_2 + x_1\delta'(x_2))(r_1r_2) &= \delta'(x_1)r_1x_2r_2 + x_1r_1\delta'(x_2)r_2\\ &= (\delta(x_1r_1) - x_1\delta(r_1))x_2r_2 + x_1r_1(\delta(x_2r_2) - x_2\delta(r_2))\\ &= \delta(x_1x_2r_1r_2) - x_1x_2\delta(r_1r_2).\end{aligned}$$

Taking sums, it follows that $(\delta'(x_1)x_2 + x_1\delta'(x_2))r = \delta(x_1x_2r) - x_1x_2\delta(r)$ for all $r \in I_1I_2$, whence $\delta'(x_1x_2) = \delta'(x_1)x_2 + x_1\delta'(x_2)$.

Therefore δ' is a derivation on Q, extending δ. We leave the uniqueness of δ' as an exercise for the reader.

DEFINITION. A ring Q is (*von Neumann*) *regular* provided that given any $x \in Q$ there is some $y \in Q$ satisfying $xyx = x$. Note that the product xy is *idempotent*, that is, $(xy)^2 = xy$. Idempotents are common in regular rings, since every finitely generated one-sided ideal in a regular ring can be generated by an idempotent [**14**, Lemma, p. 68]. It is clear from this abundance of idempotents that every regular ring is semiprime.

PROPOSITION 3.7. *If R is a semiprime commutative ring, then its maximal quotient ring is a regular ring.*

PROOF. [**14**, Proposition 1, p. 42].

THEOREM 3.8. *Let R be a semiprime commutative differential ring with no $\mathbf{Z}$-torsion, and set $T = R((\theta^{-1}; \delta))$. Let $a = \sum_{i=0}^{\infty} a_i\theta^{n-i}$ be an operator in T, with $n \in \mathbf{Z}$ and all $a_i \in R$. Assume that the set*

$$A = \{\delta(a_n)\} \cup \{a_i \mid i = 0, 1, 2, \ldots; i \neq n\}$$

has zero annihilator in R, that is, $rA \neq \{0\}$ for all nonzero $r \in R$. Then $C_T(a)$ is commutative.

PROOF. Let Q be the maximal quotient ring of R. Extend the derivation from R to Q by Proposition 3.6, and note from Proposition 3.7 that Q is a regular ring.

Suppose $x \in Q$ and $xA = \{0\}$. There is a dense ideal I in R such that $xI \subseteq R$. For any $r \in I$, we have $xr \in R$ and $(xr)A = \{0\}$, whence $xr = 0$ by our hypothesis on A. Then $xI = \{0\}$, hence $x = 0$. Therefore A has zero annihilator in Q. In the same manner, since R has no $\mathbf{Z}$-torsion it follows that Q has no $\mathbf{Z}$-torsion.

Inasmuch as it suffices to show that the centralizer of a in $Q((\theta^{-1}; \delta))$ is commutative, we may replace R by Q. Thus we may assume, without loss of generality, that R is a regular ring.

Define $b_0, b_1, \ldots \in R$ so that $b_j = a_j$ for all $j \neq n$, but $b_n = \delta(a_n)$. We define ideals $I_0, I_1, \ldots$ in R by setting $I_j = b_0R + b_1R + \cdots + b_jR$, that is, I_j is the collection of all R-linear combinations of $b_0, \ldots, b_j$. Then each I_j is a finitely generated ideal of R, and $I_0 \subseteq I_1 \subseteq \ldots$.

As R is regular, there exist idempotents $f_0, f_1, f_2, \ldots$ in R such that $f_jR = I_j$ for all j [**14**, Lemma, p. 68]. Set $e_0 = f_0$, and set $e_j = f_j - f_{j-1}$ for all $j = 1, 2, \ldots$. These e_j are idempotents in R, they are pairwise orthogonal (that is, $e_je_k = 0$ whenever $j \neq k$), and $I_j = (e_0 + e_1 + \cdots + e_j)R$ for all j. Since each $b_j \in I_j$, each element of A is an R-linear combination of the e_j, hence the set $\{e_0, e_1, \dots\}$ has zero annihilator in R. Consequently, we see that the only operator $t \in T$ for which $e_j t = 0$ for all j is $t = 0$.

We claim that each e_j is a constant. Differentiating the equation $e_j^2 = e_j$, we obtain $2e_j\delta(e_j) = \delta(e_j)$. Multiplying this equation by e_j yields $2e_j\delta(e_j) = e_j\delta(e_j)$, whence $e_j\delta(e_j) = 0$. Returning to the equation $2e_j\delta(e_j) = \delta(e_j)$, we obtain $\delta(e_j) = 0$, as claimed.

As a result, the ring e_jR is a differential ring (with identity element e_j), and $(e_jR)((\theta^{-1}; \delta)) = e_jT$. Also, e_j commutes with everything in T, so $C_T(a) \subseteq C_T(e_ja)$. We shall show that each of the centralizers

$$C_j(a) = C_{e_jT}(e_ja)$$

is commutative.

First suppose that $j < n$. For $i = 0, 1, \dots, j-1$, we have $e_je_i = 0$. Since each of $a_0, \dots, a_{j-1}$ is an R-linear combination of $e_0, \dots, e_{j-1}$, we see that $e_ja_i = 0$ for $i = 0, \dots, j-1$. On the other hand, $e_j = a_0r_0 + \cdots + a_jr_j$ for some $r_0, \dots, r_j \in R$. Multiplying this equation by e_j^2, we obtain $e_j = (e_ja_j)(e_jr_j)$, showing that e_ja_j is an invertible element of the ring e_jR. Thus e_ja is an operator in e_jT with order $n-j$ and leading coefficient e_ja_j invertible in e_jR. Also, $n-j$ is a non-zero-divisor in e_jR, because R has no $\mathbf{Z}$-torsion. Consequently, Corollary 3.2 shows that $C_j(a)$ is commutative.

Next consider the case $j = n$. Proceeding as above, we see that e_na is an operator in e_nT with order 0 and leading coefficient e_na_n, and $\delta(e_na_n)$ is invertible in e_nR. In this case, Corollary 3.4 shows that $C_n(a)$ is commutative.

Finally, suppose that $j > n$. Here we find that $e_ja = \alpha + b$ for some constant α in e_jR and some operator $b \in e_jT$ with order $n-j$ and leading coefficient e_ja_j which is invertible in e_jR. The centralizers of e_ja and b in e_jT coincide because α is central, hence another application of Corollary 3.2 shows that $C_j(a)$ is commutative in this case also.

Therefore all the $C_j(a)$ are commutative. Given any $b, c \in C_T(a)$, we have e_jb and e_jc in $C_j(a)$ for any j, whence $(e_jb)(e_jc) = (e_jc)(e_jb)$. Thus $e_j(bc - cb) = 0$ for all j, and consequently $bc - cb = 0$. Therefore $C_T(a)$ is commutative.

We now turn to the structure of centralizers in pseudo-differential operator rings, seeking analogs of results such as Theorem 1.2. Here we have a Laurent series field $F((a^{-1}))$ automatically contained in the centralizer of an operator a of positive order, and we seek to bound the dimension of the centralizer of a over this field.

THEOREM 3.9. *Let R be a semiprime commutative differential ring, such that the subring F of constants of R is a field, and set $T = R((\theta^{-1}; \delta))$.*

Let $a \in T$ be an operator with positive order n and leading coefficient a_0, such that n is invertible in F and a_0 is invertible in R. Let t be the smallest positive integer such that αa_0^t has an n-th root in R for some nonzero constant $\alpha \in F$. Then n is divisible by t, there exists an operator $b \in T$ such that $b^n = \alpha a^t$, and $C_T(a) = F((b^{-1}))$. In particular, $C_T(a)$ is an extension field of $F((a^{-1}))$ of dimension n/t.

PROOF. Since a_0^n has an n-th root in R, we see that t exists, and $t \leqq n$. Now αa^t is an operator with order nt and leading coefficient αa_0^t, and αa_0^t is invertible in R. Since n is invertible in R as well, Proposition 2.7 shows that there exists an operator $b \in T$ of order t satisfying $b^n = \alpha a^t$. Write out $a = \sum_{i=0}^{\infty} a_i\theta^{n-i}$, $b = \sum_{j=0}^{\infty} b_j\theta^{t-j}$ with the a_i, $b_j \in R$ and $b_0^n = \alpha a_0^t$. Since α and a_0 are invertible in R, so is b_0. Applying δ to the equation $b_0^n = \alpha a_0^t$, we obtain $nb_0^{n-1}\delta(b_0) = \alpha t a_0^{t-1}\delta(a_0)$, whence $\delta(b_0) = (t/n)(\alpha a_0^t)a_0^{-1}\delta(a_0)b_0^{-n}b_0 = (t/n)a_0^{-1}\delta(a_0)b_0$.

We claim that if $s \in \mathbf{Z}$ and $\beta \in F$ such that $\beta \neq 0$ and βa_0^s has an n-th root in R, then s must be divisible by t.

Write $s = qt + r$ with q, $r \in \mathbf{Z}$ and $0 \leqq r \leqq t - 1$. There is some $u \in R$ such that

$$u^n = \beta a_0^s = \beta(a_0^t)^q a_0^r = \beta\alpha^{-q}b_0^{nq}a_0^r,$$

whence $(ub_0^{-q})^n = (\beta\alpha^{-q})a_0^r$. As $\beta\alpha^{-q}$ is a nonzero element of F, the minimality of t forces $r = 0$, and so t divides s, as claimed.

In particular, since a_0^n has an n-th root in R, it follows that n is divisible by t. This implies in turn that t is invertible in F.

As $\alpha \in F$ and $a^t = \alpha^{-1}b^n$, we see that b commutes with a^t. Thus a and b both lie in $C_T(a^t)$. The operator a^t has order nt and leading coefficient a_0^t, both of which are invertible in R. Consequently, $C_T(a^t)$ is commutative, by Corollary 3.2. Thus a and b must commute, and so $F((b^{-1})) \subseteqq C_T(a)$.

We now claim that if c is any nonzero operator in $C_T(a)$ of order s, then s is divisible by t, and there exists a constant $\gamma \in F$ such that $c - \gamma b^{s/t}$ has order less than s. Write $c = \sum_{i=0}^{\infty} c_i\theta^{s-i}$ with the $c_i \in R$ and $c_0 \neq 0$.

Comparing coefficients of θ^{n+s-1} in the equation $ac = ca$, we find that

$$na_0\delta(c_0) + a_0c_1 + a_1c_0 = sc_0\delta(a_0) + c_0a_1 + c_1a_0$$

whence $\delta(c_0) = (s/n)a_0^{-1}\delta(a_0)c_0$. Setting $\beta = c_0^n a_0^{-s}$, we compute that

$$\delta(\beta) = nc_0^{n-1}\delta(c_0)a_0^{-s} - sc_0^n a_0^{-s-1}\delta(a_0) = 0,$$

hence $\beta \in F$ and $c_0^n = \beta a_0^s$. Since $c_0 \neq 0$ and R is semiprime, we must have $c_0^n \neq 0$, and so $\beta \neq 0$. As βa_0^s has an n-th root in R, the claim proved above shows that $s = mt$ for some $m \in \mathbf{Z}$. Setting $\gamma = c_0 b_0^{-m}$, we compute that

$$\begin{aligned}\delta(\gamma) &= \delta(c_0)b_0^{-m} - mc_0b_0^{-m-1}\delta(b_0)\\ &= (s/n)a_0^{-1}\delta(a_0)c_0b_0^{-m} - m(t/n)c_0b_0^{-m-1}a_0^{-1}\delta(a_0)b_0 = 0,\end{aligned}$$

hence $\gamma \in F$ and $c_0 = \gamma b_0^m$. Thus $c - \gamma b^m$ has order at most $s - 1$, as claimed.

We apply this claim inductively, as follows. Given a nonzero operator c in $C_T(a)$ of order s, we have $s = mt$ for some $m \in \mathbf{Z}$. Then we obtain constants $\gamma_0, \gamma_1, \ldots$ in F such that for all $k = 0, 1, 2, \ldots$, the operator $c - \gamma_0 b^m - \gamma_1 b^{m-1} - \cdots - \gamma_k b^{m-k}$ has order less than $(m - k)t$. As a result, $c = \sum_{k=0}^{\infty}\gamma_k b^{m-k}$, whence $c \in F((b^{-1}))$.

Therefore $C_T(a) = F((b^{-1}))$. In particular, $C_T(a)$ is an extension field of $F((a^{-1}))$. As $b^n = \alpha a^t$, we have $F((b^{-n})) = F((a^{-t}))$. Now $C_T(a)$ is an extension field of $F((b^{-n}))$ of dimension n, and $F((a^{-1}))$ is an extension field of $F((a^{-t}))$ of dimension t. Since

$$\dim_{F((a^{-t}))}(C_T(a)) = \dim_{F((a^{-1}))}(C_T(a))\cdot\dim_{F((a^{-t}))}(F((a^{-1}))),$$

we conclude that $C_T(a)$ has dimension n/t over $F((a^{-1}))$.

We conjecture that an analog of Theorem 1.11 should hold as well. Namely, given a differential field R with subfield F of constants, and an operator $a \in R((\theta^{-1}; \delta))$ of positive order n, the centralizer of a in $R((\theta^{-1}; \delta))$ should have dimension at most n^2 over the field $F((a^{-1}))$.

IV. Differential operator rings, Part 2. We return in this section to centralizers in differential operator rings, using the commutativity results proved in the previous section to obtain commutativity results here also. In particular, for a differential operator a over a commutative differential ring R, such that the order of a and the leading coefficient of a are non-zero-divisors in R, the centralizer of a in $R[\theta; \delta]$ is commutative. In case R is semiprime and has no $\mathbf{Z}$-torsion, and the subset of R consisting of the derivative of the zero-th order coefficient of a together with the remaining coefficients of a has zero annihilator in R, then the centralizer of a is again commutative. When R is the ring of complex-valued C^∞ functions on the real line, we find necessary and sufficient conditions for the centralizer of a given differential operator to be commutative.

Such commutativity results were first proved by Schur in **[24]** for an unspecified case (probably complex-valued C^∞ functions on the real line) by working within the ring of formal pseudo-differential operators, as we have done. The C^∞ function case also appeared in Flanders [**8**, Theorem 10.1], and was rediscovered later by Krichever [**13**, Corollary 1 to Theorem 1.2]. In the meantime, Amitsur had proved the commutativity result for the case when R is a differential field of characteristic zero [**3**, Theorem 1]. For R a commutative differential $\mathbf{Q}$-algebra containing no nonzero nilpotent solutions to equations of the form $\delta(x) = ax$, the commutativity of centralizers was proved by Carlson and the author [**6**, Theorem 1.4]. These results are all subsumed in the following theorem.

THEOREM 4.1. *Let R be a commutative differential ring, and set $S = R[\theta; \delta]$. Let $a \in S$ be an operator with positive order n and leading coefficient a_n. If n and a_n are non-zero-divisors in R, then $C_S(a)$ is commutative.*

PROOF. Set $T = R((\theta^{-1}; \delta))$. Then S is a subring of T, and $C_S(a)$ is a subring of $C_T(a)$. Inasmuch as $C_T(a)$ is commutative by Corollary 3.2, so is $C_S(a)$.

COROLLARY 4.2. *Let R be a semiprime commutative differential ring, such that the subring F of constants of R is a field, and set $S = R[\theta; \delta]$. Let $a \in S$ be an operator with positive order n and leading coefficient a_n. If n is invertible in F and a_n is invertible in R, then $C_S(a)$ is a commutative integral domain.*

PROOF. We obtain commutativity from Theorem 4.1, while Lemma 1.1 shows that there are no zero-divisors in the ring $C_S(a)$.

THEOREM 4.3. *Let R be a semiprime commutative differential ring with no $\mathbf{Z}$-torsion, and set $S = R[\theta; \delta]$.* Let $a = a_0 + a_1\theta + a_2\theta^2 + \cdots + a_n\theta^n$ *be a nonzero operator in S, with all $a_i \in R$. If the set $\{\delta(a_0), a_1, a_2, \ldots, a_n\}$ has zero annihilator in R, then $C_S(a)$ is commutative.*

PROOF. If $T = R((\theta^{-1}; \delta))$, then $C_T(a)$ is commutative by Theorem 3.8, whence $C_S(a)$ is commutative.

COROLLARY 4.4. *Let R be the ring of complex-valued C^∞ functions on the real line, and set $S = R[\theta; \delta]$. Let $a = a_0 + a_1\theta + a_2\theta^2 + \cdots + a_n\theta^n$ be a nonzero operator in S, with all $a_i \in R$. Then $C_S(a)$ is commutative if and only if there is no nonempty open interval on the line on which the functions $\delta(a_0), a_1, a_2, \ldots, a_n$ all vanish.*

PROOF. If $C_S(a)$ is not commutative, then by Theorem 4.3 there must exist a nonzero function $b \in R$ such that $b\delta(a_0) = ba_1 = ba_2 = \cdots = ba_n = 0$. There is a nonempty open interval I on which b does not vanish, hence $\delta(a_0), a_1, a_2, \ldots, a_n$ must all vanish on I.

Conversely, assume there is a nonempty open interval I on which $\delta(a_0), a_1, a_2, \ldots, a_n$ all vanish. Choose a function $c \in R$ which is nonzero and non-constant on I, but which vanishes outside I. The derivatives of c must also vanish outside I, hence all coefficients of the operator $ac - ca$ vanish outside I. On the other hand, inside I we have $a = a_0$ with a_0 constant, so the coefficients of $ac - ca$ must also vanish inside I. Thus $ac - ca = 0$, and $c \in C_S(a)$. Similarly, $c\theta \in C_S(a)$. However, as c is nonzero and non-constant on I, we have $c\delta(c) \neq 0$, hence c and $c\theta$ do not commute. Therefore $C_S(a)$ is not commutative.

V. Fractional differential operator rings. In this final section, we investigate centralizers in division rings of fractions of differential operators. Specifically, we start with a differential field R, form the differential operator ring $S = R[\theta; \delta]$, which is a principal right and left ideal domain, and then form the Ore quotient ring Q of S, a division ring containing S as a subring such that every element of Q can be written as a fraction with numerator and denominator coming from S. Our study of centralizers in Q is aided by the fact that Q may be naturally embedded in the pseudo-differential operator ring $R((\theta^{-1}; \delta))$, enabling us to apply some of the results of previous sections.

Commutativity results are of course the most direct applications of previous results. Specifically, we prove that if a is any nonzero element of Q whose order is invertible in R, then $C_Q(a)$ is commutative. Also, if R has characteristic zero and a is any element of Q not in the subfield of constants of R, then $C_Q(a)$ is commutative. This result was proved by Van Deuren in [**26**, Corollaire (Théorème 1); **27**, Théorème III.2.1].

Given a nonzero $a \in Q$, the centralizer $C_Q(a)$ clearly contains the field $F(a)$, where F is the subfield of constants of R, and in several cases we prove that $C_Q(a)$ must be finite-dimensional over $F(a)$. For example, if a is an operator in $R[\theta; \delta]$ with positive order n, then $C_Q(a)$ is a finite-dimensional division algebra over $F(a)$, of dimension at most n^2; if also n is invertible in F, then $C_Q(a)$ is a field extension of $F(a)$ of dimension a divisor of n. If $a = bc^{-1}$ for some commuting operators $b, c \in R[\theta; \delta]$ of orders n, k with $n - k$ invertible in F, then $C_Q(a)$ is a finite-dimensional extension field of $F(a)$ of dimension at most $n^2 + k^2$ (actually we obtain a somewhat lower, but more involved, bound on this dimension). In general, for $a \in Q$ of nonzero order, we know of no examples where $C_Q(a)$ is infinite-dimensional over $F(a)$, and we conjecture that it must always be finite-dimensional.

Proposition 5.1. *Let R be a differential field, and set $S = R[\theta; \delta]$. Then S is a principal right and left ideal domain.*

Proof. Clearly S has no zero-divisors and so is an integral domain. Consider an arbitrary right ideal I of S. As the zero ideal is certainly principal, assume that $I \neq \{0\}$. Choose a nonzero operator $a \in I$ with minimal order. Then $aS \subseteq I$, and we claim that $aS = I$.

Given $b \in I$, we may, because the leading coefficient of a is invertible in R, divide a into b, obtaining $b = aq + r$ for some $q, r \in S$ with $\operatorname{ord}(r) < \operatorname{ord}(a)$. Since $a, b \in I$ and $r = b - aq$, we have $r \in I$, hence the minimality of $\operatorname{ord}(a)$ forces $r = 0$. Now $b = aq$ and so $b \in aS$.

Thus $I = aS$ as claimed, whence I is principal. Similarly, all left ideals of S are principal.

COROLLARY 5.2. (Ore) [19, pp. 226, 227]. *Let R be a differential field, and set $S = R[\theta; \delta]$. Given any nonzero operators a, $b \in S$, there exist nonzero operators x, y, z, $w \in S$ such that $ax = by$ and $za = wb$.*

PROOF. To obtain x and y, the intersection $aS \cap bS$ must be nonzero. If not, then the right ideal $aS + bS$ is actually the direct sum of aS and bS. As S is a principal right ideal domain, $aS + bS = cS$ for some nonzero $c \in S$. Then the map $s \mapsto cs$ defines an isomorphism of S onto $aS \oplus bS$ (as right S-modules), whence $S = I \oplus J$ for some proper right ideals I and J. Write $1 = e + f$ with $e \in I$ and $f \in J$. On one hand, $e \in I$ implies $ef \in I$, while on the other hand, $ef = f - f^2$ and so $ef \in J$. Consequently, $ef = 0$, hence either $e = 0$ or $f = 0$. Since f lies in the proper right ideal J, we cannot have $f = 1$, whence $e \neq 0$. But then $f = 0$ and $e = 1$, which contradicts the fact that I is proper.

Therefore $aS \cap bS \neq \{0\}$, from which the existence of x and y follows. The existence of z and w is obtained by a symmetric argument.

In [18, 19], Ore used the common multiple properties obtained in Corollary 5.2 to construct a division ring **Q** whose elements are formal quotients of elements of S. However, since we already have available a division ring containing S as a subring, namely $R((\theta^{-1}; \delta))$ [recall Corollary 2.5], we may simply construct **Q** as a sub-division-ring of $R((\theta^{-1}; \delta))$, as follows.

PROPOSITION 5.3. *Let R be a differential field, and set $S = R[\theta; \delta]$. Set $T = R((\theta^{-1}; \delta))$, and let $Q = \{ab^{-1} \mid a, b \in S \text{ and } b \neq 0\}$. Then Q is a sub-division-ring of T, and also $Q = \{c^{-1}d \mid c, d \in S \text{ and } c \neq 0\}$.*

PROOF. Consider nonzero elements $a_1b_1^{-1}$ and $a_2b_2^{-1}$ in Q, where a_1, b_1, a_2, b_2 are nonzero operators in S. By Corollary 5.2, there exist nonzero operators x, $y \in S$ such that $b_1x = b_2y$, whence

$$(a_1b_1^{-1}) \pm (a_2b_2^{-1}) = (a_1xx^{-1}b_1^{-1}) \pm (a_2yy^{-1}b_2^{-1}) = (a_1x \pm a_2y)(b_1x)^{-1}.$$

Thus Q is closed under addition and subtraction. Using Corollary 5.2 again, we obtain nonzero operators u, $v \in S$ such that $a_2u = b_1v$, hence $b_1^{-1}a_2 = vu^{-1}$. Thus $(a_1b_1^{-1})(a_2b_2^{-1}) = a_1vu^{-1}b_2^{-1} = (a_1v)(b_2u)^{-1}$, proving that Q is closed under multiplication. Therefore Q is a subring of T. Lastly, we have $(a_1b_1^{-1})^{-1} = b_1a_1^{-1}$, which of course lies in Q. Therefore Q is a division ring.

Given a nonzero element $ab^{-1} \in Q$, where a, b are nonzero operators in S, we use Corollary 5.2 to obtain nonzero operators c, $d \in S$ such that $ca = db$, whence $ab^{-1} = c^{-1}d$. Conversely, given $c^{-1}d$, where c, d are nonzero operators in S, Corollary 5.2 provides nonzero operators a, $b \in S$ such that $ca = db$, hence $c^{-1}d = ab^{-1}$. Therefore $Q = \{c^{-1}d \mid c, d \in S \text{ and } c \neq 0\}$.

Definition. In the situation of Proposition 5.3, the division ring Q is called the *division ring of quotients of S* (or the *Ore quotient division ring of S*), and we sometime refer to the elements of Q as *fractional linear differential operators*. For notation, we write $Q = R(\theta;\, \delta)$, as the form of Q is analogous to that of a rational function field. In fact, if δ is the zero derivation on the field R, then Q is exactly the rational function field $R(\theta)$.

Since we have constructed $R(\theta;\, \delta)$ as a subring of $R((\theta^{-1};\, \delta))$, every element of $R(\theta;\, \delta)$ comes equipped with an *order* and a *leading coefficient*. If an element of $R(\theta;\, \delta)$ is expressed in the form ab^{-1} with $a, b \in R[\theta;\, \delta]$ and $b \neq 0$, then the order and the leading coefficient of ab^{-1} are easily determined from a and b, namely

$$\begin{aligned} \operatorname{ord}(ab^{-1}) &= \operatorname{ord}(a) - \operatorname{ord}(b), \\ \text{l.coeff.}(ab^{-1}) &= [\text{l.coeff.}(a)][\text{l.coeff.}(b)]^{-1}, \end{aligned}$$

and similarly when elements of $R(\theta;\, \delta)$ are written in the form $c^{-1}d$. On the other hand, if one constructs $R(\theta;\, \delta)$ formally, following Ore's original procedure, then these equations may be used to define orders and leading coefficients for elements of $R(\theta;\, \delta)$, and the common multiple properties (Corollary 5.2) can be applied to show that these are well-defined.

Theorem 5.4. *Let R be a differential field, and set $Q = R(\theta;\, \delta)$. If a is any nonzero element of Q whose order is invertible in R, then $C_Q(a)$ is commutative. In fact, $C_Q(a)$ is a maximal subfield of Q.*

Proof. Set $T = R((\theta^{-1};\, \delta))$. As $C_T(a)$ is commutative by Corollary 3.2, $C_Q(a)$ must be commutative as well.

Clearly the inverse of any nonzero element of $C_Q(a)$ also lies in $C_Q(a)$, hence $C_Q(a)$ is a subfield of Q. If K is any subfield of Q that contains a, then the commutativity of K implies that $K \subseteq C_Q(a)$. Thus $C_Q(a)$ is maximal among subfields of Q.

Theorem 5.5. *Let R be a differential field of characteristic zero, let F be the subfield of constants of R, and set $Q = R(\theta;\, \delta)$. Let a be a nonzero element of Q with order 0 and leading coefficient a_0. If $a_0 \notin F$, then $C_Q(a)$ is isomorphic (as an F-algebra) to a subfield of R that contains F. Thus $C_Q(a)$ is commutative, and is a maximal subfield of Q.*

Proof. Set $T = R((\theta^{-1};\, \delta))$. Since $a_0 \notin F$, we have $\delta(a_0)$ invertible in R, hence Theorem 3.3 provides a ring isomorphism of $C_T(a)$ onto R. We note that this isomorphism must, in the present context, be an F-algebra isomorphism, hence it restricts to an F-algebra embedding of $C_Q(a)$ into R. As $C_Q(a)$ is a division algebra, the image of $C_Q(a)$ under this embedding must be a subfield of R that contains F.

In particular, it follows that $C_Q(a)$ is commutative. As in Theorem 5.4, we conclude that $C_Q(a)$ is a maximal subfield of R.

Combining Theorems 5.4 and 5.5, we obtain the following result, analogous to Theorem 3.5.

THEOREM 5.6. *Let R be a field of characteristic zero with a nonzero derivation, let F be the subfield of constants of R, and set $Q = R(\theta;\ \delta)$. Then the center of Q is F. For any non-central element $a \in Q$, the centralizer $C_Q(a)$ is a maximal subfield of Q. Conversely, all maximal subfields of Q have the form $C_Q(a)$ for non-central elements $a \in Q$.*

PROOF. The proof that F is the center of Q is the same as in Theorem 3.5; namely, it is clear that elements of F are central in Q, while elements of $R - F$ fail to commute with θ, and elements of $Q - R$ fail to commute with elements of $R - F$ (which is nonempty because $\delta \neq 0$).

Given any non-central element $a \in Q$, we may show that $C_Q(a)$ is commutative either as an application of Theorem 3.5, or as an application of Theorems 5.4 and 5.5 using the method of Theorem 3.5. It then follows that $C_Q(a)$ is a maximal subfield of Q.

Finally, consider a maximal subfield K of Q. As F is not a maximal subfield of Q [for instance, $F(\theta)$ is a subfield of Q properly containing F], we may choose an element $a \in K - F$. Then $K \subseteq C_Q(a)$ because K is commutative. On the other hand, $C_Q(a)$ is a subfield of Q by the previous paragraph, hence we conclude from the maximality of K that $K = C_Q(a)$.

By analogy with Theorems 1.2, 1.11, and 3.9, we might expect a result of the following sort. Given a differential field R with subfield F of constants, and an element a with nonzero order n in the division ring $Q = R(\theta;\ \delta)$, the centralizer $C_Q(a)$ should be a finite-dimensional division algebra over the field $F(a)$, of dimension at most n^2 in general, and of dimension dividing n in case n is invertible in F. Such a result does hold when a is an operator in $R[\theta;\ \delta]$, as we show in Theorem 5.8, but in general the situation is not as nice as expected. For instance, the dimension of $C_Q(a)$ over $F(a)$ is not bounded by any function of the order of a, as Example 5.9 shows. It is not clear, in fact, whether in general $C_Q(a)$ must necessarily be finite-dimensional over $F(a)$.

PROPOSITION 5.7. *Let R be a differential field, let F be the subfield of constants of R, and set $S = R[\theta;\ \delta]$ and $Q = R(\theta;\ \delta)$. If $a \in S$ is any nonzero operator, then*

$$C_Q(a) = \{uv^{-1} \mid u \in C_S(a);\ v \in F[a];\ v \neq 0\}.$$

PROOF. Obviously if $u \in C_S(a)$ and $v \in F[a]$ with $v \neq 0$, then uv^{-1} is in $C_Q(a)$.

Conversely, consider a nonzero element $x \in C_Q(a)$, and write $x = bc^{-1}$ for some nonzero operators $b, c \in S$. According to Corollary 5.2, there exist nonzero operators a_1, c_1, d, d_1 in S such that $ac_1 = ca_1$ and $cd_1 = c_1d$. Set $a_2 = a_1d$ and $c_2 = c_1d = cd_1$, and note that $ca_2 = ca_1d = ac_1d = ac_2$, whence $c^{-1}a = a_2c_2^{-1}$. Then

$$abd_1c_2^{-1} = abd_1(cd_1)^{-1} = abc^{-1} = ax = xa = bc^{-1}a = ba_2c_2^{-1}$$

and so $abd_1 = ba_2$. In addition, $acd_1 = ac_2 = ca_2$. In S^2, which is both a left and a right S-module, we thus have $a(b, c)d_1 = (b, c)a_2$.

We define a subset W of S^2 by setting $W = \{w \in S^2 \mid ws \in (b, c)S$ for some nonzero $s \in S\}$. It is an easy exercise, using Corollary 5.2, to show that W is a right S-submodule of S^2. Given any nonzero $w \in W$, we have $ws = (b, c)s'$ for some nonzero $s, s' \in S$. In addition, $s't = d_1t'$ for some nonzero $t, t' \in S$ (Corollary 5.2 again), whence $aw(st) = a(b, c)s't = a(b, c)d_1t' = (b, c)a_2t'$ and so $aw \in W$. Thus $aW \subseteqq W$.

Inasmuch as S is a principal ideal domain (Proposition 5.1), all submodules of free S-modules are free [**11**, Theorem 17, p. 43]. Thus W must be a free right S-module. Passing to the right vector space Q^2 over Q, we see that WQ has the same dimension over Q as the rank of W. On the other hand, we see from the definition of W that $WQ = (b, c)Q$, so that WQ is one-dimensional. Thus W is a free right S-module of rank one.

Now $W = (y, z)S$ for some $y, z \in S$. In particular, as $(b, c) \in W$ we have $(b, c) = (y, z)p$ for some $p \in S$. Consequently, $y, z \neq 0$ and $yz^{-1} = yp(zp)^{-1} = bc^{-1} = x$. Since $aW \subseteqq W$, we must have $a(y, z) = (y, z)q$ for some $q \in S$. In particular, $az = zq$, hence a belongs to the set $J = \{s \in S \mid sz \in zS\}$. Note that J is an F-algebra, and that zS is an ideal of J. According to Corollary 1.10, $\dim_F(J/zS) \leqq \operatorname{ord}(z)^2$, so that J/zS is a finite-dimensional F-algebra.

As a result, the image of a in J/zS must be algebraic over F. Thus there must exist a nonzero operator $v \in F[a]$ which vanishes modulo zS, that is, $v \in zS$. Now $v = zr$ for some nonzero $r \in S$. Setting $u = yr$, we obtain $x = yz^{-1} = yr(zr)^{-1} = uv^{-1}$. Since v is in $F[a]$, it commutes with a, as does x. Thus the operator $u = xv$ also commutes with a.

Therefore $x = uv^{-1}$ with $u \in C_S(a)$ and $v \in F[a]$, as desired.

THEOREM 5.8. *Let R be a differential field, let F be the subfield of constants of R, and set $Q = R(\theta; \delta)$. Let $a \in R[\theta; \delta]$ be an operator with positive order n. Then $C_Q(a)$ is a finite-dimensional division algebra over $F(a)$, of dimension at most n^2. Moreover, if n is invertible in F, then $C_Q(a)$ is a finite-dimensional field extension of $F(a)$, of dimension a divisor of n.*

PROOF. Set $S = R[\theta; \delta]$. According to Theorem 1.11, $C_S(a)$ is a free $F[a]$-module of rank at most n^2. Obverving that any basis for $C_S(a)$ as a free $F[a]$-module is also a basis for the algebra

$$C = \{uv^{-1} \mid u \in C_S(a);\ v \in F[a];\ v \neq 0\}$$

over $F(a)$, we see that C is a finite-dimensional $F(a)$-algebra of dimension at most n^2. As $C_Q(a) = C$ by Proposition 5.7, this proves the first part of the theorem.

If n is invertible in F, then $C_Q(a)$ is a field, by Theorem 5.4. In addition, Theorem 1.2 shows that $C_S(a)$ is a free $F[a]$-module of rank dividing n, hence we conclude that the dimension of $C_Q(a)$ over $F(a)$ must divide n.

Unfortunately, the dimension estimates in Theorem 5.8 do not hold for other centralizers, as the following example shows.

EXAMPLE 5.9. Let R be any differential field, let F be the subfield of constants of R, and set $Q = R(\theta;\ \delta)$. Let n be any positive integer, and set $a = \theta^{n+1}(\theta^n + 1)^{-1}$ in Q. Then a has order 1, but $C_Q(a)$ is an $(n + 1)$-dimensional extension field of $F(a)$.

PROOF. According to Theorem 5.4, $C_Q(a)$ and $C_Q(\theta)$ are both commutative. As a and θ commute, it follows that $C_Q(a) = C_Q(\theta)$. The centralizer of θ in $R[\theta;\ \delta]$ is obviously $F[\theta]$, hence we see by Proposition 5.7 that $C_Q(\theta) = F(\theta)$. Thus $C_Q(a) = F(\theta)$.

Note that $\theta^{n+1} - a\theta^n - a = 0$. Since a is transcendental over F, it is a prime element of the unique factorization domain $F[a]$, hence Eisenstein's Criterion shows that the polynomial $x^{n+1} - ax^n - a$ in $F(a)[x]$ is irreducible. Therefore $F(\theta)$ is an $(n + 1)$-dimensional field extension of $F(a)$.

Taking a cue from this example, we proceed to derive a modified version of Theorem 5.8 that will cover the centralizer of a fractional differential operator whose numerator and denominator commute. We first need some estimates on the dimensions of certain field extensions, as in the following lemma. We are indebted to R. Donagi and S. Katz for the method used.

LEMMA 5.10. *Let R be a differential field, let F be the subfield of constants of R, and set $Q = R(\theta;\ \delta)$. Let $b, c \in R[\theta;\ \delta]$ be operators with distinct positive orders n and k, set $a = bc^{-1}$, and let m be the greatest common divisor of n and k. Assume that $bc = cb$. Then the dimensions of the subfield $F(b, c)$ of Q over the subfields $F(a)$, $F(b)$, $F(c)$ are related by the inequality*

$$\dim_{F(a)}(F(b, c)) \leqq \dim_{F(b)}(F(b, c)) + \dim_{F(c)}(F(b, c)) - \min\{n/m, k/m\}.$$

PROOF. As $F(b, c) \subseteq C_Q(b) \cap C_Q(c)$, the dimensions $s = \dim_{F(b)}(F(b, c))$ and $t = \dim_{F(c)}(F(b, c))$ are both finite, by Theorem 5.8. Let x and y be independent commuting indeterminates. Since c^{-1} lies in $F(b, c)$, it satisfies a nonzero polynomial in $F(b)[y]$ of degree at most s. Viewing $F(b)$ as the quotient field of $F[b^{-1}]$, we may clear denominators from the polynomial just obtained, and get one of the same degree whose coefficients all lie in $F[b^{-1}]$. In other words, there is some nonzero $p \in F[x, y]$ such that $p(b^{-1}, c^{-1}) = 0$ and p has degree at most s in y. Similarly, there is a nonzero $q \in F[x, y]$ such that $q(b^{-1}, c^{-1}) = 0$ and q has degree at most t in x.

Now set $P = \{f \in F[x, y] \mid f(b^{-1}, c^{-1}) = 0\}$, which is an ideal of $F[x, y]$. Since $F[x, y]/P$ is isomorphic to the integral domain $F[b^{-1}, c^{-1}]$, we see that P is a prime ideal. Also, $p, q \in P$, so P is nonzero. On the other hand, as $n, k > 0$ the fractional operators b^{-1}, c^{-1} are transcendental over F, hence the algebra $F[b^{-1}, c^{-1}]$ cannot be finite-dimensional over F. Thus $F[x, y]/P$ is infinite-dimensional over F. One of the standard versions of the Hilbert Nullstellensatz—e.g., [**4**, Corollary 5.24]—says that $F[x, y]$ modulo any maximal ideal must be finite-dimensional over F. Therefore P is not a maximal ideal of $F[x, y]$.

A well-known fact about the ring $F[x, y]$ is that it does not contain a chain $P_0 \subset P_1 \subset P_2 \subset P_3$ of four distinct prime ideals [**16**, Corollary, p. 83]. As we can already build a chain $0 \subset P \subset M$ (where M is any maximal ideal containing P), there cannot be any nonzero prime ideals properly contained in P. Since $F[x, y]$ is a unique factorization domain, it follows that P must be a principal ideal [**12**, Theorem 5]. Let g be a generator for P.

As p and q lie in P, they each must be divisible by g. Consequently, g has degree at most t in x and degree at most s in y, so that

$$g = \sum_{i=0}^{t} \sum_{j=0}^{s} \alpha_{ij} x^i y^j$$

for some $\alpha_{ij} \in F$, not all zero. Thus

$$\sum_{i=0}^{t} \sum_{j=0}^{s} \alpha_{ij} b^{-i} c^{-j} = 0. \tag{5.1}$$

Note that each term $b^{-i}c^{-j}$ has order $-in - jk$. We now check that a number of these terms have distinct orders.

If $in + jk = i'n + j'k$, then $(i - i')n = (j' - j)k$, and so $(i - i')(n/m) = (j' - j)(k/m)$. Since n/m and k/m are relatively prime, this can happen only if n/m divides $j' - j$ and k/m divides $i - i'$. Thus for $i = 0, 1, \ldots, \min\{t, (k/m) - 1\}$ and $j = 0, 1, \ldots, \min\{s, (n/m) - 1\}$, the terms $b^{-i}c^{-j}$ all have distinct orders. In particular, the terms $b^{-i}c^{-j}$ for which $in + jk < nk/m$ all have distinct orders. Now rewrite equation (5.1) in the form

$$\sum_{\substack{i=0 \\ in+jk<nk/m}}^{t} \sum_{j=0}^{s} \alpha_{ij} b^{-i} c^{-j} = - \sum_{\substack{i=0 \\ in+jk\geq nk/m}}^{t} \sum_{j=0}^{s} \alpha_{ij} b^{-i} c^{-j}. \tag{5.2}$$

The nonzero terms $\alpha_{ij}b^{-i}c^{-j}$ on the left-hand side of equation (5.2) all have distinct orders greater than $-nk/m$, whence this left-hand side either vanishes or has order greater than $-nk/m$. However, the right-hand side of (5.2) has order at most $-nk/m$, hence both sides of (5.2) must vanish. Consequently, $\alpha_{ij} = 0$ for all i, j satisfying $in + jk < nk/m$.

Set $\ell = \min\{n, k\}$. For any nonnegative integers i, j satisfying $i + j < \ell/m$ we have

$$in + jk \leqq (i + j)\max\{n, k\} < (\ell/m)\max\{n, k\} = nk/m$$

and so $\alpha_{ij} = 0$. Now multiply equation (5.1) by b^{s+t}, obtaining

$$\sum_{i=0}^{t} \sum_{j=0}^{s} \alpha_{ij} b^{s+t-i-j}(bc^{-1})^j = 0. \tag{5.3}$$

Since $\alpha_{ij} = 0$ whenever $i + j < \ell/m$, the highest power of b appearing in equation (5.3) is $s + t - (\ell/m)$. Thus b is algebraic over the field $F(a)$ of degree at most $s + t - (\ell/m)$. Inasmuch as $F(b, c) = F(a)(b)$, we conclude that $\dim_{F(a)}(F(b, c)) \leqq s + t - (\ell/m)$, as desired.

THEOREM 5.11. *Let R be a differential field, let F be the subfield of constants of R, and set $Q = R(\theta; \delta)$. Let $b, c \in R[\theta; \delta]$ be operators with distinct positive orders n and k, set $a = bc^{-1}$, and set $\ell = \min\{n, k\}/\gcd\{n, k\}$. If $bc = cb$ and $n - k$ is invertible in F, then $C_Q(a)$ is a finite-dimensional extension field of $F(a)$, and*

$$\dim_{F(a)}(C_Q(a)) \leqq \begin{cases} n + k^2 - \ell & \text{if } n^{-1} \in F, \\ n^2 + k - \ell & \text{if } k^{-1} \in F \\ n + k - \ell & \text{if } n^{-1}, k^{-1} \in F \end{cases}$$

PROOF. Recall from Theorem 5.4 that $C_Q(a)$ is a subfield of Q.

Since $n - k$ is invertible in F, at least one of n, k must be invertible in F. Thus, interchanging b and c if necessary [which is harmless because $cb^{-1} = a^{-1}$], we may assume that n is invertible in F. Consequently, a second application of Theorem 5.4 shows that $C_Q(b)$ is a subfield of Q. Now $C_Q(a)$ and $C_Q(b)$ are both commutative, and a and b commute as well, hence $C_Q(a) = C_Q(b)$. As anything in $C_Q(a)$ commutes with both a and b and so commutes with c, we also have $C_Q(a) \subseteqq C_Q(c)$.

Applying Theorem 5.8 to the centralizers $C_Q(b)$ and $C_Q(c)$, we find that $C_Q(a)$ has dimension at most n over $F(b)$, and $C_Q(a)$ has dimension at most k^2 over $F(c)$. Set $r = \dim_{F(b,c)}(C_Q(a))$; $s = \dim_{F(b)}(F(b, c))$; $t = \dim_{F(c)}(F(b, c))$. Thus $rs \leqq n$ and $rt \leqq k^2$. According to Lemma 5.10, $\dim_{F(a)}(F(b, c)) \leqq s + t - \ell$, and therefore

$$\dim_{F(a)}(C_Q(a)) \leqq r(s + t - \ell) \leqq rs + rt - \ell \leqq n + k^2 - \ell.$$

In case n and k are both invertible in F, Theorem 5.8 shows that $rs \leqq n$

and $rt \leqq k$, whence the dimension of $C_Q(a)$ over $F(a)$ is at most $n + k - \ell$.

The sharpest case of Theorem 5.11 occurs when n and k are relatively prime, and are both invertible in F, since then $n + k - \ell$ is just the maximum of n and k. This value is attained, as Example 5.9 shows.

COROLLARY 5.12. *Let R be a differential field of characteristic zero, let F be the subfield of constants of R, and set $Q = R(\theta; \delta)$. Let $b, c \in R[\theta; \delta]$ be nonzero operators with orders n and k, set $a = bc^{-1}$, and assume that $a \notin F$. If $bc = cb$ and $n + k > 0$, then $C_Q(a)$ is a finite-dimensional field extension of $F(a)$, of dimension at most $n + k$.*

PROOF. Interchanging b and c if necessary, we may assume that $k \geqq n$. If $n = 0$, then $b \in R$ and a is the inverse of an operator in $R[\theta; \delta]$ of positive order k, in which case the desired result follows from Theorem 5.8. If $k > n > 0$, then we just apply Theorem 5.11.

Now assume that $k = n > 0$, and let b_n, c_n be the leading coefficients of b and c. Since $bc = cb$, Lemma 1.1 shows that $b_n = \alpha c_n$ for some $\alpha \in F$. Setting $d = b - \alpha c$, we see that d is a nonzero operator (because $a \notin F$) of order less than n. As $a = (dc^{-1}) + \alpha$, we have $C_Q(a) = C_Q(dc^{-1})$ and $F(a) = F(dc^{-1})$. In view of the cases proved above, we conclude that $C_Q(a)$ is a field extension of $F(a)$ of dimension at most $\operatorname{ord}(d) + k$, which is less than $n + k$.

We conjecture that Theorem 5.11 should still hold in some form, perhaps with weaker bounds on the dimension of the centralizer, even when b and c do not commute. The only support we have for this conjecture (beyond Theorem 5.11 itself) stems from the following results of Resco and Resco-Small-Wadsworth. (The first of these results is asserted in [**26**, Théorème 2; **27**, Théorème IV.3.14], but the proofs are incomplete.) If a suitable version of Theorem 5.11 were to hold for centralizers of arbitrary elements of $R(\theta; \delta)$ of nonzero order, we could in turn derive these results from it.

THEOREM 5.13 (Resco, Small, Wadsworth). *Let R be a field of characteristic zero with a nonzero derivation, let F be the subfield of constants of R, and set $Q = R(\theta; \delta)$. Let K be a subfield of Q that contains F.*

(a) *The transcendence degree of K over F is no greater than the transcendence degree of R over F.*

(b) *If R is a finitely generated field extension of F, then so is K.*

PROOF. (a) [**21**, Theorem 4.8]; (b) [**23**, Theorem 5].

COROLLARY 5.14. *Let R be a field of characteristic zero with a nonzero derivation, let F be the subfield of constants of R, and set $Q = R(\theta; \delta)$.*

Let a be any non-central element of Q. If R is a finitely generated field extension of F with transcendence degree one, then $C_Q(a)$ is a finite-dimensional field extension of $F(a)$.

PROOF. By Theorem 5.6, $C_Q(a)$ is a maximal subfield of Q. If a has nonzero order, then it is clear that a is transcendental over F. If a has order 0 and its leading coefficient a_0 is in F, then $a - a_0$ is a nonzero element of Q of negative order, whence $a - a_0$ is transcendental over F and thus a is also. Finally, if a has order 0 and $a_0 \notin F$, then it follows easily from the assumption of characteristic zero that a_0 is transcendental over F, hence a is transcendental over F.

Thus $F(a)$ is a transcendental extension of F in all cases. As R has transcendence degree one over F, so does $C_Q(a)$, by Theorem 5.13(a), whence $C_Q(a)$ must be an algebraic extension of F(a). On the other hand, since R is finitely generated over F, Theorem 5.13(b) shows that $C_Q(a)$ is finitely generated over F and thus also over $F(a)$. Therefore $C_Q(a)$ is finite-dimensional over $F(a)$.

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